On a hierarchy of groups of computable automorphisms

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Abstract

A complete description of groups of kind $\bigoplus_{i\in I} \mathbb{Z}_{p_i}$ that can be realized as groups of all computable automorphisms for appropriate computable models is obtained. We introduce a three–level classification of isomorphism types of groups of computable automorphisms by possible arithmetical complexity of their orbits and prove this classification to be nontrivial. Bibliogr. 5.

Basic notions and results on groups of computable automorphisms are contained in [1].

Recall some basic definitions. *A computable model*

$$\mathfrak{M} = \left\langle A, f_0^{n_0}, \ldots; P_0^{m_0}, \ldots \right\rangle$$

is a model in which A is a computable subset of natural numbers ω, the mappings $i \mapsto n_i$ (number of arguments of f_i) and $i \mapsto m_i$ (number of arguments of P_i)are computable, and all operations f_i and predicates P_i are computable uniformly on i. *A computable automorphism* of a computable model $\mathfrak{M}$ is an automorphism of $\mathfrak{M}$ which is a computable function on its universe. All such automorphisms form a group which will be denoted as $\mathrm{Aut}_c\mathfrak{M}$.

One of the basic problems in the study of groups of computable automorphisms is finding a characterization of this class of groups. However any attempt to describe it meets serious difficulties. In particular, this class cannot be described as the class of all groups computable in some oracle [2]. The elementary theory of the class of such groups appears to be computably isomorphic to arithmetic [3]. Also the attempts to find a reasonable description even for finitely generated subgroups fail [4]. The only result that makes things a little bit clearer is

Theorem 1 $[1, 5]$ An arbitrary finitely generated group G is isomorphic to a group of kind $\mathrm{Aut}_c\mathfrak{M}$ for some computable model $\mathfrak{M}$ if and only if its word problem is decidable, (in other words, if and only if it is isomorphic to a computable group).

In this paper, we give one more description of groups of computable automorphisms within a very narrow class of groups which nevertheless enables us to define a natural three-level hierarchy within this class and to prove its nontriviality.

We denote the ith prime by p_i, i. e. $p_0 = 2$, $p_1 = 3, \ldots$.

Theorem 2 A group of kind $\bigoplus_{i \in I} \mathbb{Z}_{p_i}$ is isomorphic to a group of kind $\mathrm{Aut}_c\mathfrak{M}$ for some computable model $\mathfrak{M}$ if and only if $I \in \Sigma^0_3$.

Proof. First we prove this condition to be sufficient. We need the following

Lemma 1 [5] Assume $I \in \Sigma^0_3$. There exists a computable sequence of linear orderings L_j, $j \in \omega$ such that

1. if $j \in I$ then L_j is computably isomorphic to an ordering of type ω^2 with computable set of pairs of neighbors and there exists a monotonic enumeration of the set of all limit elements (here we do not guarantee the homogeneity on j);
2. if $j \notin I$, then L_j is isomorphic to ω.

Note that all computable linear orderings that satisfy the condition 1 of this Lemma are pairwise computably isomorphic.

The basic set of $\mathfrak{M}$ will contain several types of elements. First we consider an ordered set

$$B = \left\{a^0_0 < a^0_1 < a^1_0 < a^1_1 < a^1_2 < \ldots < a^k_0 < \ldots < a^k_{p_k-1} < \ldots\right\}.$$

This set B can be thought of as the union of mutually disjoint blocks of kind $\left\{a^k_0, a^k_1, \ldots, a^k_{p_k-1}\right\}$, $k = 0, 1, \ldots$, situated one after another. We call *j-th block* the p_j element set $\left\{a^j_0, a^j_1, \ldots, a^j_{p_j-1}\right\}$.

Fix a computable linear ordering L of type ω^2 whose set of neighbors is recursive and that possesses a monotonic enumeration of all limit elements.

The basic set of $\mathfrak{M}$ will consist of two disjoint parts: the first one is B and the second one is the set of ordered pairs of kind $\left\langle a^j_i, b\right\rangle$, where b is an element from some ordering L'_j arising in the course of the construction, which almost coincides with L_j from Lemma if $i \neq 0$, and b is an element from L if $i = 0$.

The basic predicates are defined as follows.

1. The predicate U^1 distinguishes the set B.

2. The predicate R^2 is true on a pair of elements $\langle x, y\rangle$ if and only if $U(x), \neg U(y)$ and $y = \langle x, b\rangle$.

3. The predicate P^2 is true on a pair $\langle x, y\rangle$ if $x = a^j_i$, $y = a^j_{i+1}$ and $i < p_j - 1$ or $x = a^j_{p_j-1}$, $y = a^j_0$ (i. e. it forms a cycle of length p_j on the jth block).

4. The predicate $\preceq$ defines a linear ordering on elements of sets $\left\{a^j_k\right\} \times L'_j$, $j \in \omega$, which is obtained by transferring the initial ordering on L'_j by means of the mapping $x \in L'_j \mapsto \left\langle a^j_k, x\right\rangle$. Elements of distinct sets $\left\{a^j_k\right\} \times L'_j$, $j \in \omega$, $0 \leq k \leq p_j - 1$, are pairwise incomparable with respect to $\preceq$.

Idea of the proof. First we hang up the orders L of type ω^2 to elements a^j_0 for all $j \in \omega$ by means of the predicate R. Next we construct the model $\mathfrak{M}$ by steps, hanging rigid linear orderings L_j of type ω or ω^2 from Lemma to all elements except for a_0^j. Each automorphism will permute elements within blocks. To avoid automorphisms that nontrivially permute infinite number of blocks, we add some new elements to the ordering hanged to a_k^j, $k \neq 0$ so that the number of elements added to each such ordering will be finite and no added new element will be maximal there. By this, isomorphism types and algorithmic properties of orders mentioned in Lemma remain the same.

Now let us check that an arbitrary automorphism $\mathfrak{M}$ can move elements within blocks with orderings isomorphic to ω^2 only. Indeed, assume an automorphism f takes elements of one block to elements of another one: $f(a_i^k) = a_j^r$, $k \neq r$. Since f preserves P, we obtain that $k = r$, i. e. the mentioned above mixing is impossible. If we assume that f nontrivially moves elements within blocks with orderings isomorphic ω hanged, then we obtain $f(a_i^j) = a_0^j$, $i \neq 0$. This means that f takes some ordering of type ω onto an ordering of type ω^2, which is impossible. If all hanged orderings within the jth block have type ω^2 then after adding a finite number of new non-maximal elements will preserve recursiveness of the set of neighbors and the existence of monotonic enumeration of all limit elements. All such orderings will be pairwise computably isomorphic. In view of this, there is a computable automorphism that cyclically permutes elements within jth block and does not move elements in other blocks.

Formal description of the construction. We start with the set B defined at the beginning. Define P as above.

In what follows, we understand the words "we build an ordering S over an element a_i^j" as adding new elements of kind $\langle a_i^j, b\rangle$, $b \in S$ to our model on which the ordering $\preceq$ is defined as image of the ordering on S with respect to the mapping $b \mapsto \langle a_i^j, b\rangle$; we put $R\big(a_i^j, \langle a_i^j, b\rangle\big) \quad b \in S$.

We consider our construction to be executed over natural numbers by natural identification of elements of the model we construct with natural numbers.

Fix some Kleene's computable numbering of all partial recursive functions φ_n, $n \in \omega$. Let φ_n^t be a finite part of φ_n computed at first t steps.

We will use the enumeration of orderings L_j $(j \in \omega)$ from Lemma 1 that comes from the process of their generation:

$$L_j^0 \subseteq L_j^1 \subseteq \ldots \subseteq L_j^t \subseteq \ldots \subseteq \bigcup_s L_j^s = L_j.$$

We suppose that at each step only one element is added to the ordering, i. e., $\big|L_j^{t+1} \backslash L_j^t\big| = 1$ for all $t \in \omega$. Fix the same enumeration for the ordering L.

At each step t, we we build an ordering L_j' over an element a_i^j, $i \neq 0$, by putting there new elements enumerated at L_j up to the step t elements and,possibly new elements arising in the course of the construction. We build an ordering L over each a_0^j.

If at step t the elements n, $j \leq t$ $n < j$ and $b_0, b_1, b_0', b_1' \leq t$, $k \neq 0$, $k < p_j$ appeared such that

$$\varphi_n^t\big(\langle a_0^j, b_0\rangle\big) = \langle a_k^j, b_0'\rangle, \quad \varphi_n^t\big(\langle a_0^j, b_1\rangle\big) = \langle a_k^j, b_1'\rangle,$$

and b_0 and b_1 are neighbors in L, and b_0 is not greater than b_1 L, n was not considered before, b'_0 is less than b'_1 in the ordering that is build over a^j_k, then we take the minimal such n and minimal j and add a new element c to the ordering built over a^j_k placing it between b'_0 and b'_1; so that b'_0 and b'_1 will be not neighbors anymore We put also $R\big(a^j_k, \langle a^j_k, c\rangle\big)$. After this we think of n as already considered and never consider it again. Note that if we need to add a new element that corresponds to an element in L_j to the ordering we build over a^j_k, some indeterminacy about the place for it may occur. In this case we put it to the leftmost possible position.

THE CONSTRUCTION IS COMPLETE.

It follows from the construction that the so constructed model is computable.

Note also that for each m in each ordering we build over an element in mth block only a finite number of elements can be added, sine each time we add a new element we consider some $n \leq m$ which could be considered only once.

As is already noted, each automorphism of this model cyclically permutes elements within blocks whose all orderings built over elements of these blocks are isomorphic to ω^2. Since orderings we build over elements of the block are rigid, the way an automorphism acts is completely defined by its action on blocks. Now if a function φ_n defines an automorphism of our model then it cannot permute elements within blocks whose numbers are greater than n nontrivially, since otherwise at some step it will be fixed that φ_n takes a pair of neighbors with respect to $\preceq$ to a pair of non-neighbors, namely it takes a pair of neighbors of some order we build over a^j_0 to a pair of non-neighbors of the ordering we build over a^j_k. A contradiction.

Thus, if we denote a computable automorphism that cyclically permutes elements in ith block for $i \in I$, then each automorphism of the obtained model will be a product of finite number of automorphisms γ_i. This yields an isomorphism between $\bigoplus\limits_{i\in I} \mathbb{Z}_{p_i}$ and $\mathrm{Aut}\,_c\mathfrak{M}$.

Prove necessity. Assume the group $\bigoplus\limits_{i\in I} \mathbb{Z}_{p_i}$ is isomorphic to the group of all automorphisms of some computable model. One can easily ascertain that

$$i \in I \iff \exists x_0 \dots x_{p_i-1}\Big(\bigwedge_{0\leq k<j<p_i} (x_k \neq x_j) \vee (\langle x_0 \dots x_{p_i-1}\rangle \cong_c \langle x_1 \dots x_{p_i-1}, x_0\rangle)\Big). \quad (1)$$

We will do it a little bit later.

As it was noted in [5], the relation $\cong_c$ is always in Σ^0_3, which could be checked immediately. From this equivalence it follows that $I \in \Sigma^3_0$.

Check the equivalence (1). Suppose $i \in I$. It means that our model possesses a computable automorphism of order p_i which yields the existence of the cycle $x_0 \dots x_{p_i-1}$ whose length is the prime p_i. Assume now the right hand condition holds. Take a computable automorphism f that cyclically permutes $x_0 \dots x_{p_i-1}$. Since the automorphism group is isomorphic to the direct sum of cyclic groups of different prime orders, the order of this automorphism is finite and is divisible by p_i By this the group contains a generator of order p_i, i. e., $i \in I$. If it is not true, denote the isomorphic image of f by $\hat{f}$; then $\hat{f} = a^{i_1}_{p_{j_1}} \dots a^{i_q}_{p_{j_q}}$, where $a_{p_{j_s}}$ is a generator of direct summand $\mathbb{Z}_{p_{j_s}}$, $s = 1, \dots, q$, $p_{j_s} \neq p_i$, which implies $f^{p_{j_1} \cdots p_{j_q}}(x_0) \neq x_0$ and $1 \neq (\hat{f})^{p_{j_1} \cdots p_{j_q}} = \left(a_{p_{j_1}}^{i_1} \dots a_{p_{j_q}}^{i_q}\right)^{p_{j_1} \cdots p_{j_q}} = 1$; a contradiction. □

Now we pass to the definition of three-level hierarchy on the class of groups of computable automorphisms and prove it is nontrivial.

Denote by Γ_i the class of all groups isomorphic to groups of all computable automorphisms of computable models M whose relation $\cong_c$ is in Σ_i^0, $i = 1, 2, 3$. As it was noted before, this relation is always in Σ_3^0 and introducing higher levels makes no sense.

Theorem 3 The following relations hold:

$$\Gamma_1 \subsetneqq \Gamma_2 \subsetneqq \Gamma_3.$$

Proof. Both inclusions are obvious. We should prove them to be nontrivial.

Lemma 2. Let $G_I = \bigoplus_{i \in I} \mathbb{Z}_{p_i}$ and let G_I be isomorphic to the group of all computable automorphisms of some computable model M. Assume $k \in \{1, 2, 3\}$. Then if the relation $\cong_c$ on M is in Σ_k^0, then $I \in \Sigma_k^0$.

Proof follows immediately from the equivalence (1).

Resume the proof of the Theorem. Take an arbitrary set $I \in \Sigma_3^0 \setminus \Sigma_2^0$. By Lemma and the fact that the relation $\cong_c$ is always in Σ_3^0 we obtain $G_I \in \Gamma_3 \setminus \Gamma_2$.

It remains to check that the inclusion $\Gamma_1 \subsetneqq \Gamma_2$ is nontrivial. Take an arbitrary immune set $I \in \Pi_1^0 \setminus \Sigma_1^0$ with enumerable complement and construct a model $\mathfrak{M}$ as follows. The signature of $\mathfrak{M}$ will contain a unique binary predicate P. We start the construction from the set which is the union of nontrivial disjoint directed cycles formed by means of the predicate P so that for each prime p_i, $i < \omega$ it will contain exactly one cycle of length p_i. Then we will enumerate the complement I of without repetitions and each time when some new element i will be enumerated into the complements of I, we will add a new element by means of P to the unique cycle of length p_i a new element so that to spoil all its nontrivial symmetries.

It is clear from the construction that the group of all computable automorphisms of the so constructed model will be isomorphic to G_I since by the immunity of I no computable automorphism can move elements in infinitely many cycles.

Moreover, we can see that the relation $\cong_c$ on our model is $0'$-computable; henceforth it is in Σ_2^0 (and even in Δ_2^0). Thus, $G_I \in \Gamma_2$ and by Lemma we have $G_I \notin \Gamma_1$. □

Question: Does there exist a group which is isomorphic to the group of all computable automorphisms of an appropriate computable model with enumerable $\cong_c$ but is not isomorphic to a group of all computable automorphisms in which this relation is computable?

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