ON ELEMENTARY THEORIES OF ROGERS SEMILATTICES [1]

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Describing the elementary theories of Rogers semilattices is one of the main problems of the theory of numberings. For the classical case of computable families of computably enumerable sets, V.V. V'jugin showed in [1] the existence of infinitely many families with pairwise elementarily different Rogers semilattices. Nevertheless, the study of Rogers semilattices undertaken in recent years (see [2]–[5]) for families of sets at levels greater than one in the Kleene-Mostowsky hierarchy has shown

- significative differences as regards algebraic and elementary properties of these semilattices compared with the classical case;
- homogeneity of the structure of their ideals and intervals.

This initially seemed to lead to conjecture that the Rogers semilattices for infinite families of sets of any fixed high level of the arithmetical hierarchy have similar properties. But very soon it was noted that, for every level of the hierarchy, there are at least two families of this level such that their Rogers semilattices are not elementarily equivalent, [6]. Our aim is to prove that, for every level of the arithmetical hierarchy, there exist infinitely many families with pairwise non-elementarily equivalent Rogers semilattices.

We recall some necessary definitions and notations. A numbering α of a family $\mathcal{A} \subseteq \Sigma^0_{n+1}$ is called $\Sigma^0_{n+1}-$ *computable* if

$$\{\langle x, m\rangle \mid x \in \alpha(m)\} \in \Sigma^0_{n+1}.$$

The set of all $\Sigma^0_{n+1}-$ computable numberings of a family $\mathcal{A}$ is denoted by $\mathrm{Com}^0_{n+1}(\mathcal{A})$. A family $\mathcal{A}$ is called Σ^0_{n+1}- *computable* if $\mathrm{Com}^0_{n+1}(\mathcal{A}) \neq \emptyset$. A numbering α is *reducible* to a numbering β if $\alpha = \beta \circ f$ for some computable function f. Reducibility of numberings is a pre-ordering relation on $\mathrm{Com}^0_{n+1}(\mathcal{A})$ which induces in the usual way a quotient structure $\mathcal{R}^0_{n+1}(\mathcal{A})$ which is an upper semilattice called *Rogers semilattice* of $\Sigma^0_{n+1}-$ computable numberings of the family $\mathcal{A}$. We refer to [2] and [4] for details on $\Sigma^0_{n+1}-$ computable numberings and related topics.

THEOREM 1. For every $k \in \omega$, there exist infinitely many infinite $\Sigma^0_{k+1}-$ computable families with pairwise elementarily different Rogers semilattices.

Proof. Let k be an arbitrary natural number. We will construct a sequence $\{\mathcal{B}_e\}_{e\geq 1}$ of infinite $\Sigma^0_{k+1}-$ computable families such that

$$\mathrm{Th}(\mathcal{R}^0_{k+1}(\mathcal{B}_{e'})) \neq \mathrm{Th}(\mathcal{R}^0_{k+1}(\mathcal{B}_{e''}))$$

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for all $e' \neq e''$.

In fact we will construct a sequence $\{\mathcal{A}_n\}_{n\geq 1}$ of families of sets of which the desired sequence $\{\mathcal{B}_e\}_{e\geq 1}$ is a subsequence.

Let M stand for any arbitrary $\mathbf{0}^{(k)}-$maximal set, and let n be a positive natural number.

Let $E_n^1, E_n^2, \ldots, E_n^n$ be a computable partition of ω into infinite computable sets. Let $f_n^i, i \in [1,n]$, denote some computable bijection of ω onto E_n^i. Clearly, $M_i \rightleftharpoons \overline{E}_n^i \bigcup f_n^i(M)$ is a $\mathbf{0}^{(k)}-$maximal set for each i.

Let $\mathcal{A}_n^i$, $i \in [1,n]$, stand for the family $\{M_i \bigcup \{x\} \mid x \in \overline{M}_i\}$. The families $\mathcal{A}_n^i$ are evidently Σ_{k+1}^0-computable. Finally define $\mathcal{A}_n \rightleftharpoons \bigcup\limits_{i\in[1,n]} \mathcal{A}_n^i$.

Lemma 1. For all numbers $n > 0$ and $i \in [1,n]$, for every numbering $\alpha \in \mathrm{Com}_{k+1}^0(\mathcal{A}_n)$, the index set w.r.t. α of the family $\mathcal{A}_n^i$ is Σ_{k+1}^0-set.

Proof. For every $i \in [1,n]$, denote by b_i^0 and b_i^1 any two distinct numbers of $\overline{M}_i$. Since, for every x and $i \in [1,n]$,

$$\alpha(x) \notin \mathcal{A}_n^i \Longleftrightarrow b_i^0 \in \alpha(x) \,\&\, b_i^1 \in \alpha(x),$$

it follows that the index set of the family $\mathcal{A}_n \setminus \mathcal{A}_n^i$ w.r.t. α is a $\mathbf{0}^{(k)}-$computably enumerable set. This immediately implies that, for every $i \in [1,n]$, the index set of $\mathcal{A}_n^i$ is a Σ_{k+1}^0-set since the subfamilies $\mathcal{A}_n^1, \mathcal{A}_n^2, \ldots, \mathcal{A}_n^n$ are pairwise disjoint. Lemma 1 is proved.

Lemma 2. For all numbers $n \geq 1$ and $i \in [1,n]$, and for every numbering $\beta \in \mathrm{Com}_{k+1}^0(\mathcal{A}_n)$, if β is the join $\beta = \beta_0 \oplus \beta_1$ of some numberings β_0 and β_1 then all but finitely many sets of $\mathcal{A}_n^i$ are contained in at least one of families $\beta_0(\omega)$, $\beta_1(\omega)$.

Proof. Let $n, i, \beta, \beta_0, \beta_1$ satisfy the hypothesis of lemma 2. By lemma 1, the index set $Q \rightleftharpoons \{x \mid \beta(x) \in \mathcal{A}_n^i\}$ is a Σ_{k+1}^0-set. The sets $Q_0 \rightleftharpoons \{x \mid 2x \in Q\}$ and $Q_1 \rightleftharpoons \{x \mid 2x+1 \in Q\}$ are obviously also Σ_{k+1}^0-sets. Note that $Q_j, j \leq 1$, is exactly the set of β_j-indices x such that $\beta_j(x) \in \mathcal{A}_n^i$.

For $j \leq 1$, let $\mathcal{B}_j \rightleftharpoons \beta_j(Q_j)$ and let $B_j \rightleftharpoons \bigcup\limits_{A\in\mathcal{B}_j} A$. Then B_j is superset of M_i and

$$\mathcal{B}_0 \bigcup \mathcal{B}_1 = \mathcal{A}_n^i \quad \text{and} \quad B_0 \cup B_1 = \omega.$$

The numberings β_0 and β_1 are evidently Σ_{k+1}^0-computable. This implies that B_0, B_1 are infinite Σ_{k+1}^0-supersets of the $\mathbf{0}^{(k)}-$maximal set M_i. Therefore, at least one of $\mathcal{B}_0 \setminus \mathcal{B}_1$ and $\mathcal{B}_1 \setminus \mathcal{B}_0$ is finite. Otherwise, the sets $B_i \setminus M_i$ and $\omega \setminus B_i$, $i \leq 1$, are both infinite, and this is in contradiction with $\mathbf{0}^{(k)}-$maximality of M_i. Thus, at least one of the families $\beta_j(Q_j) = \beta_j(\omega) \cap \mathcal{A}_n^i, j \leq 1$, contains all but finitely many elements of $\mathcal{A}_n^i$. Lemma 2 is proved.

Lemma 3. For all numbers $n > 0$ and $m \geq n$ and for all numberings $\gamma_1^0, \gamma_1^1, \gamma_2^0, \gamma_2^1, \ldots, \gamma_{m+1}^0, \gamma_{m+1}^1 \in \mathrm{Com}_{k+1}^0(\mathcal{A}_n)$, if $\gamma_1^0 \oplus \gamma_1^1 \equiv \gamma_2^0 \oplus \gamma_2^1 \equiv \cdots \equiv \gamma_{m+1}^0 \oplus \gamma_{m+1}^1$ then there exist a numbering $\delta \in \mathrm{Com}_{k+1}^0(\mathcal{A}_n)$ and a binary string $\varepsilon_1\varepsilon_2\ldots\varepsilon_m$ such that $\delta \leqslant \gamma_{m+1}^0$ and $\delta \leqslant \gamma_1^{\varepsilon_1} \oplus \gamma_2^{\varepsilon_2} \oplus \cdots \oplus \gamma_m^{\varepsilon_m}$.

Proof. Let the numbers n, m and the numberings $\gamma_1^0, \gamma_1^1, \ldots, \gamma_{m+1}^0, \gamma_{m+1}^1$ satisfy the hypothesis of lemma 3. For every $i, 1 \leq i \leq m$, we construct numbering $\gamma_i^{\varepsilon_i}$ of some subfamily of the family $\mathcal{A}_n$ such that $\delta_i^{\varepsilon_i} \leqslant \gamma_{m+1}^0$, $\delta_i^{\varepsilon_i} \leqslant \gamma_i^{\varepsilon_i}$, and $\mathcal{A}_n^i \setminus \delta_i^{\varepsilon_i}(N)$ is finite (possibly, empty) as follows. Since $\gamma_{m+1}^0 \leqslant \gamma_i^0 \oplus \gamma_i^1$ it follows that $\gamma_{m+1}^0 = \delta_i^0 \oplus \delta_i^1$ some numberings $\delta_i^0 \leqslant \gamma_i^0, \quad \delta_i^1 \leqslant \gamma_i^1$.

By lemma 2, the family $\mathcal{A}_n^i$ is almost entirely included in $\delta_i^0(\omega)$ or $\delta_i^1(\omega)$. Let $\varepsilon_i \in \{0, 1\}$ be the least number such that the family $\mathcal{A}_i^n \setminus \delta_i^{\varepsilon_i}(\omega)$ is finite. Then

$$\delta_1^{\varepsilon_1} \oplus \delta_2^{\varepsilon_2} \oplus \cdots \oplus \delta_n^{\varepsilon_n} \leqslant \gamma_{m+1}^0,$$

$$\delta_1^{\varepsilon_1} \oplus \delta_2^{\varepsilon_2} \oplus \cdots \oplus \delta_n^{\varepsilon_n} \leqslant \gamma_1^{\varepsilon_1} \oplus \gamma_2^{\varepsilon_2} \oplus \cdots \oplus \gamma_n^{\varepsilon_n},$$

and $\delta_1^{\varepsilon_1} \oplus \delta_2^{\varepsilon_2} \oplus \cdots \oplus \delta_n^{\varepsilon_n}(\omega)$ contains all the sets of $\mathcal{A}_n$ except the ones belonging to some finite subfamily $\mathcal{B}$ of the family $\mathcal{A}_n$.

Let β be some solvable Σ_{k+1}^0-computable numbering of the family $\mathcal{B}$, and let $\varepsilon_{n+1}\varepsilon_{n+2} \ldots \varepsilon_m$ be an arbitrary binary string. Then the numbering $\delta \rightleftharpoons \delta_1^{\varepsilon_1} \oplus \delta_2^{\varepsilon_2} \oplus \cdots \oplus \delta_n^{\varepsilon_n} \oplus \beta$ and the binary string $\varepsilon_1\varepsilon_2 \ldots \varepsilon_m$ satisfy the conclusion of lemma 3.

Definition. We will say that two Σ_{k+1}^0-computable numberings β_0 and β_1 of a family $\mathcal{B}$ induce a *minimal pair* in the Rogers semilattice $\mathcal{R}_{k+1}^0(\mathcal{B})$ if there is no numbering $\beta \in \mathrm{Com}_{k+1}^0(\mathcal{B})$ such that $\beta \leqslant \beta_0$ and $\beta \leqslant \beta_1$.

The numberings γ_{m+1}^0 and $\gamma_1^{\varepsilon_1} \oplus \gamma_2^{\varepsilon_2} \oplus \cdots \oplus \gamma_m^{\varepsilon_m}$ built above in the proof of lemma 3 do not induce a minimal pair in $\mathcal{R}_{k+1}^0(\mathcal{A}_n)$. We propose now a regular way of constructing numberings which induce minimal pairs in the Rogers semilattice $\mathcal{R}_{k+1}^0(\mathcal{A}_n^i)$.

Let us fix two different numbers $a_0, a_1 \in \overline{M}$ and, for every $i \in [1, n]$, define the numberings α_i^s, $s \leq 1$, as follows: for every x,

$$\alpha_i^s(x) \rightleftharpoons \begin{cases} M_i \bigcup \{f_n^i(a_s)\}, \text{ if } x \in M, \\ M_i \bigcup \{f_n^i(x)\}, \text{ otherwise.} \end{cases}$$

It is obvious that $\alpha_i^s \in \mathrm{Com}_{k+1}^0(\mathcal{A}_n^i)$.

Lemma 4. The numberings α_i^0 and α_i^1 induce a minimal pair in $\mathcal{R}_{k+1}^0(\mathcal{A}_n^i)$.

Proof. By contradiction. Assume that some numbering $\gamma \in \mathrm{Com}_{k+1}^0(\mathcal{A}_n^i)$ is reducible to the numberings α_i^0, α_i^1 by means of computable functions g_0, g_1, respectively.

First of all, note that, by definition of the numberings α_i^0, α_i^1, for every y,

$$y \in \overline{M} \Longleftrightarrow \alpha_i^0(y) = \alpha_i^1(y).$$

Next, for every $z \notin \{a_0, a_1\}$ we prove the equivalence

$$\alpha_i^0(z) = \alpha_i^1(z) \Longleftrightarrow \exists x(g_0(x) = z = g_1(x)).$$

Let $\alpha_i^0(z) = \alpha_i^1(z)$ and $z \notin \{a_0, a_1\}$. Then the set $\alpha_i^0(z)$ has a unique index w.r.t. the numbering α_i^0. Therefore there exists x such that $g_0(x) = z$. Equality

$\alpha_i^0(z) = \alpha_i^1(z)$ implies that $g_1(x) = z$ since $\alpha_i^1(z)$ has also a unique index w.r.t. the numbering α_i^1.

Conversely, if $g_0(x) = z = g_1(x)$ for some x then $\alpha_i^0(z) = \alpha_i^0(g_0(x))$ and $\alpha_i^1(g_1(x)) = \alpha_i^1(z)$. Therefore, $\alpha_i^0(z) = \gamma(x) = \alpha_i^1(z)$.

Thus we have that, for every z,

$$z \in \overline{M} \setminus \{a_0, a_1\} \Longleftrightarrow z \notin \{a_0, a_1\} \,\&\, \exists x(g_0(x) = z = g_1(x))$$

and, consequently, $\overline{M}$ is computably enumerable. This is a contradiction with the fact that M is $\mathbf{0}^{(k)}$– maximal. Lemma 4 is thus proved.

Lemma 5. For every $m > 0$ and $n \geq 2^{2^{m+1}}$, there exist numberings $\beta_1^0, \beta_1^1, \beta_2^0, \beta_2^1, \ldots, \beta_{2^m}^0, \beta_{2^m}^1 \in \mathrm{Com}_{k+1}^0(\mathcal{A}_n)$ such that:

(1) $\beta_1^0 \oplus \beta_1^1 \equiv \beta_2^0 \oplus \beta_2^1 \equiv \cdots \equiv \beta_{2^m}^0 \oplus \beta_{2^m}^1$;

(2) for every $i \in [1, 2^m]$, the numberings β_i^0 and β_i^1 induce a minimal pair in the Rogers semilattice $\mathcal{R}_{k+1}^0(\mathcal{A}_n)$;

(3) for every number $l \leq m$, every set $I = \{i_1 < i_2 < \ldots < i_l\} \subseteq [1, 2^m]$, every binary string $\sigma_1\sigma_2\ldots\sigma_l$, and for all numbers $\varepsilon \in \{0, 1\}$ and $i \in [1, 2^m] \setminus I$, the numberings β_i^ε and $\beta_{i_1}^{\sigma_1} \oplus \beta_{i_2}^{\sigma_2} \oplus \cdots \oplus \beta_{i_l}^{\sigma_l}$ induce a minimal pair in $\mathcal{R}_{k+1}^0(\mathcal{A}_n)$.

Proof. Let m be any positive number, let $l \leq m$, and let n be an arbitrary number which satisfies the inequality $n \geq 2^{2^{m+1}}$. Fix any sequence of subsets $X_1, X_2, \ldots, X_{2^m}$ of the set $[1, n]$ such that the inequality

$$|X_1^{\varepsilon_1} \bigcap X_2^{\varepsilon_2} \bigcap \cdots \bigcap X_{2^m}^{\varepsilon_{2^m}}| \geq 2^{m+1}$$

holds for every binary string $\varepsilon_1\varepsilon_2\ldots\varepsilon_{2^m}$. Here, for any set X, the symbol $|X|$ denotes the cardinality of X, and $X_i^1 \leftrightharpoons X_i$, $X_i^0 \leftrightharpoons [1, n] \setminus X_i$, $i \in [1, 2^m]$.

For every $i \in [1, 2^m]$, we define two numberings β_i^0 and β_i^1 by letting

$$\beta_i^s \rightleftharpoons \sum_{p \in X_i^0} \alpha_p^{1-s} \oplus \sum_{q \in X_i^1} \alpha_q^s, \quad s \in \{0, 1\}.$$

Statement (1) holds trivially since the following equivalence

$$\beta_i^0 \oplus \beta_i^1 \equiv \sum_{j=1}^{n} \alpha_j^0 \oplus \sum_{j=1}^{n} \alpha_j^1$$

holds for every $i \in [1, 2^m]$.

Let us prove statement (2) by contradiction. Assume that some numbering $\gamma \in \mathrm{Com}_{k+1}^0(\mathcal{A}_n)$ is reducible to both β_i^0 and β_i^1 for some $i \in [1, 2^m]$.

Since the families $\mathcal{A}_n^1, \mathcal{A}_n^2, \ldots, \mathcal{A}_n^n$ are pairwise disjoint and

$$\beta_i^0 = \sum_{p \in X_i^0} \alpha_p^1 \oplus \sum_{q \in X_i^1} \alpha_q^0,$$

the reducibility $\gamma \leqslant \beta_i^0$ implies that there exist $\Sigma^0_{k+1}-$computable numberings $\gamma_j, j \in [1,n]$, of the families $\mathcal{A}_n^j$ such that $\gamma \equiv \gamma_1 \oplus \gamma_2 \oplus \cdots \oplus \gamma_n$ and $\gamma_p \leqslant \alpha_p^0$ for all $p \in X_i^1$ and $\gamma_q \leqslant \alpha_q^1$ for all $q \in X_i^0$. The reducibility $\gamma \leqslant \beta_i^1$ implies that, for every $j \in [1,n]$,

$$\gamma_j \leqslant \sum_{p \in X_i^0} \alpha_p^0 \oplus \sum_{q \in X_i^1} \alpha_q^1.$$

And we have $\gamma_p \leqslant \alpha_p^1$ for all $p \in X_i^1$ and $\gamma_q \leqslant \alpha_q^0$ for all $q \in X_i^0$ since the families $\mathcal{A}_n^1, \mathcal{A}_n^2, \ldots, \mathcal{A}_n^n$ are pairwise disjoint. Thus, $\gamma_j \leqslant \alpha_j^0$ and $\gamma_j \leqslant \alpha_j^1$ for every $j \in [1,n]$. This is in contradiction with lemma 4.

We prove statement (3) by contradiction too. Suppose that some numbering γ of the family $\mathcal{A}_n$ is reducible to the numberings β_i^ε and $\beta_{i_1}^{\sigma_1} \oplus \beta_{i_2}^{\sigma_2} \oplus \cdots \oplus \beta_{i_l}^{\sigma_l}$. Here, for all $j \in [1,l]$, the numbers ε, i and σ_j, i_j are chosen as in statement (3). Similarly to the proof of statement (2), we use a decomposition $\gamma \equiv \gamma_1 \oplus \gamma_2 \oplus \cdots \oplus \gamma_n$, where $\gamma_j \in \mathrm{Com}^0_{k+1}(\mathcal{A}_n^j)$ for every $j \in [1,n]$.

Due to the choice of the sets $X_1, X_2, \ldots, X_{2^m}$ and to the inequality $l+1 < 2^{m+1}$, there exists at least one number

$$j \in X_i^\varepsilon \bigcap X_{i_1}^{1-\sigma_1} \bigcap X_{i_2}^{1-\sigma_2} \bigcap \cdots \bigcap X_{i_l}^{1-\sigma_l}.$$

Since $j \in X_i^\varepsilon$, using the decomposition $\beta_i^\varepsilon = \sum_{p \in X_i^0} \alpha_p^{1-\varepsilon} \oplus \sum_{q \in X_i^1} \alpha_q^\varepsilon$ it is easy to check that, in both cases $\varepsilon = 0$ and $\varepsilon = 1$, α_j^1 is an element of this decomposition, but at the same time α_j^0 is not. Consequently, $\gamma_j \leqslant \alpha_j^1$.

Analogously, one can check that α_j^0 does enter each decomposition of the numberings $\beta_{i_1}^{\sigma_1}, \beta_{i_2}^{\sigma_2}, \ldots, \beta_{i_l}^{\sigma_l}$ whereas α_j^1 does not. This implies that $\gamma_j \leqslant \alpha_j^0$.

Thus, α_j^0 and α_j^1 induce a minimal pair in $\mathcal{R}^0_{k+1}(\mathcal{A}_n^j)$, in contradiction with lemma 4. Therefore lemma 5 is proved.

Using lemmas 1–5 we can now deduce the statement of the theorem as follows. Define a computable function h by letting $h(1) = 16$ and $h(e+1) = 2^{2^{h(e)+1}}$ for every $e \geq 1$. Let $\mathcal{B}_e \rightleftharpoons \mathcal{A}_{h(e)}$ for every $e \geq 1$. Lemmas 3, 5 imply that $\mathrm{Th}(\mathcal{R}^0_{k+1}(\mathcal{B}_{e'})) \neq \mathrm{Th}(\mathcal{R}^0_{k+1}(\mathcal{B}_{e''}))$ for every $e' \neq e''$.

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