Automorphism groups of computably enumerable predicates

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Abstract

In this paper we study the automorphism groups of two very important predicates in computability theory: the predicate $x \in W_y$ and the graph of the universal partial recursive function.

We prove that the automorphisms of those predicates are always computable. We also study the way those groups act on several index sets and obtain some results about the structure of the groups.

1 Introduction

We fix an acceptable system of indices of the partial computable functions (as done, for instance, in [3] or in [6]) and denote φ_n the n-th such function and W_n its domain. We will study the groups of automorphisms of the predicates

$$P(e, x, y) \iff \varphi_e(x) = y$$

and

$$W(x, y) \iff x \in W_y$$

We will denote by $AutP$ and $AutW$ the group of automorphisms of P and W, while $Aut_r P$ and $Aut_r W$ will denote the groups of recursive automorphisms of those predicates. That is, a member of $AutP$ is permutation f of the set of natural numbers (which we will denote as ω) such that for all $e, x, y \in \omega$

$$P(e, x, y) \iff P(f(e), f(x), f(y))$$

and a member of $AutW$ is permutation g of ω such that for all $x, y \in \omega$

$$W(x, y) \iff W(g(x), g(y))$$

If f or g are computable then they will be, respectively, in $Aut_r P$ or in $Aut_r W$. It is clear that all they are groups under the operation of composition of functions.

Of course, the actual members of these groups depend on the chosen coding of partial computable functions. However, the results by Blum (see [2] and [6]) show that, for acceptable systems of indices, the corresponding groups are recursively isomorphic.

Notice that

$$x \in W_y \iff \exists z\,(\varphi_x(y) = z)$$

so $AutP \leq AutW$ and $Aut_rP \leq Aut_rW$.

If f is a computable function we will denote I_f the set of its indices, i.e.

$$I_f = \{n \in \omega : f = \varphi_n\}$$

In particular we will denote I_{id} the set of indices of the identity function and $I_\uparrow$ the set of indices of the nowhere defined function. If $g = \varphi_n$ then we will simply denote $I_n = I_g$.

If A is a r.e. set we will denote

$$H_A = \{x \in \omega : W_x = A\}$$

If $A = W_n$ we will simply denote $H_n = H_A$.

If A is a set we will denote its cardinality by $|A|$.

2 Complexity of the automorphisms

In this section we study the degree of computability of the automorphisms of P and W. First, we show that the groups $AutP$ and $AutW$ are countable and then we prove that, in fact, $AutP = Aut_rP$ and $AutW = Aut_rW$.

First, we state some lemmas that will be used once and again in the proofs of our main results.

Lemma 1 *Let f be an automorphism of W. Then for every natural number n*

$$f(W_n) = W_{f(n)}$$

and

$$f(H_n) = H_{f(n)}$$

Proof. First note that if $x, y \in \omega$ then

$$x \in W_y \iff f(x) \in W_{f(y)}$$

implies, since f is one-to-one and onto, that

$$f(W_y) = W_{f(y)}$$

If $W_a = W_n$ we will have

$$W_{f(n)} = f(W_n) = f(W_a) = W_{f(a)}$$

so

$$f(H_n) \subseteq H_{f(n)}$$

The other inclusion follows when we apply this property to f^{-1} and $H_{f(n)}$. ■

Lemma 2 *Let f be an automorphism of P. Then for every natural number n*

$$f \cdot \varphi_n \cdot f^{-1} = \varphi_{f(n)}$$

and

$$f(I_n) = I_{f(n)}$$

Proof. Notice that for $e, x, y \in \omega$

$$\varphi_e(x) = y \iff \varphi_{f(e)}(f(x)) = f(y)$$

implies, since f is bijective, that

$$f \cdot \varphi_e \cdot f^{-1} = \varphi_{f(e)}$$

The second part of the lemma can be proved in way similar to what we did in the previous one. ■

Remark 1 *Notice that a permutation f satisfying*

$$f(W_n) = W_{f(n)}$$

for every natural number n is, in fact, an automorphism of W. Also, if f satisfies

$$f \cdot \varphi_n \cdot f^{-1} = \varphi_{f(n)}$$

for all n, then it is in $AutP$.

Theorem 1 *There exists $e < \omega$ such that for all $h \in AutW$*

$$W_{h(e)} = W_e \iff h = id$$

Proof.

Consider a total and computable function g such that

$$\varphi_{g(n)}(x) = \begin{cases} 0 & \text{if } x \in \{0, \ldots, n\} \\ \uparrow & \text{otherwise} \end{cases}$$

Then, $W_{g(n)} = \{0, \ldots, n\}$.

Choose e so that $W_e = Range\ g = \{g(0), g(1), \ldots\}$. Let h be an automorphism of W such that $h(e) = e$. Then, from lemma 1

$$\{h(g(0)), h(g(1)), \ldots\} = h(W_e) = W_{h(e)} = W_e = \{g(0), g(1), \ldots\}$$

It is clear that if $i \neq j$ then

$$|W_{h(g(i))}| = |h(W_{g(i)})| = |W_{g(i)}| = i + 1 \neq j + 1 = |W_{g(j)}|$$

so $h(g(i)) \neq g(j)$ if $i \neq j$. It follows that $h(g(i)) = g(i)$ for every i in ω. Then

$$\{h(0), \ldots, h(i)\} = h(W_{g(i)}) = W_{h(g(i))} = W_{g(i)} = \{0, \ldots, i\}$$

and it follows (by induction on i) that $h(i) = i$ for every i in ω. ∎

Remark 2 *Notice that the values of any automorphism f of W are determined by the value $f(e)$ (in fact, by $H_{f(e)}$).*

The previous theorem is important in the view of the results of Krueker (see [1] and [5]). These results make trivial the proof of the following corollary.

Corollary 1 *There is only a countable number of automorphisms of W (and, therefore, of P).*

Theorem 2 *If f is an automorphism of W then it is computable.*

Proof. Let e_0 be an index of the empty set, e_1 and index of $\{e_1\}$ and e_2 an index of $\{e_0, e_1\}$. We consider the set

$$S = \{n_0, \ldots, n_k, \ldots\}$$

where

$$\begin{array}{rcl} W_{n_k} & = & \{b_1^k, b_2^k\} \\ W_{b_2^k} & = & \{k\} \\ W_{b_1^k} & = & \{m_0^k, e_0\} \\ W_{m_0^k} & = & \{m_1^k\} \\ & \vdots & \\ W_{m_k^k} & = & \{e_2\} \end{array}$$

It is clear that S is r.e. since, given k we can compute, in turn, $m_k^k, \ldots, m_0^k, b_1^k, b_2^k$ and n_k. Then, we can fix an index a of S as r.e. set.

Consider the r.e. set

$$W_{f(a)} = \{f(n_0), \ldots, f(n_k), \ldots\}$$

It is clear that

$$\begin{array}{rcl} W_{f(n_k)} & = & \{f(b_1^k), f(b_2^k)\} \\ W_{f(b_2^k)} & = & \{f(k)\} \\ W_{f(b_1^k)} & = & \{f(m_0^k), f(e_0)\} \\ W_{f(m_0^k)} & = & \{f(m_1^k)\} \\ & \vdots & \\ W_{f(m_k^k)} & = & \{f(e_2)\} \end{array}$$

and that

$$\begin{array}{rcl} W_{f(e_0)} & = & \emptyset \\ W_{f(e_1)} & = & \{f(e_0)\} \\ W_{f(e_2)} & = & \{f(e_0), f(e_1)\} \end{array}$$

Note that $|W_{f(e_0)}| = 0$ but $|W_{f(e_1)}| = 1$. Then we have $f(e_0) \neq f(e_1)$ and $|W_{f(e_2)}| = 2$. Note also that $f(n_i) \neq f(n_j)$ if $i \neq j$.

To enumerate the set $A = \{(x, y) : f(x) = y\}$ we repeat the following steps:

- Enumerate a new element m of the set $W_{f(a)}$.
- Enumerate W_m till you obtain two values r and s.
- Enumerate W_r and W_s. One of them will have two members and the other only one. Suppose that

$$\begin{array}{rcl} W_r & = & \{i, j\} \\ W_s & = & \{y\} \end{array}$$

- Enumerate W_i and W_j. One of them will be empty and the other will have one member. Suppose that

$$\begin{array}{rcl} W_i & = & \{c_1\} \\ W_s & = & \emptyset \end{array}$$

- Enumerate W_{c_1} till you obtain a value, say c_2; dovetail the enumeration of W_{c_1} with the enumeration of W_{c_2}; when you find that $c_3 \in W_{c_2}$, dovetail the enumerations of W_{c_1}, W_{c_2} and W_{c_3}; ... and so on. Eventually in one of these sets, say W_{c_n}, there will appear two elements. From the structure of the set W_a (and of $W_{f(a)}$) it follows that

$$y = f(n-1)$$

  Then we add to A the value $(n-1, y)$.

It is clear that the process enumerates the set A, so it is r.e. and, then, f is computable. ■

Remark 3 *Note that then $Aut_r W = Aut W$ and $Aut_r P = Aut P$.*

Remark 4 *Though, for sake of clarity, we won't include the results in this paper, Theorem 2 can be generalized to prove that all functions verifying*

$$f(W_n) = W_{f(n)}$$

for every natural number n*, are computable. It is also possible to prove that an isomorphic embedding of* W *is computable if its range is r.e.*

3 Action of the automorphisms

In this section we show that, despite of the fact of all their members being computable, the groups $AutP$ and $AutW$ are rather complicated. The natural question after theorem 2 is if the groups $AutW$ and $AutP$ form a computable family. Near the end of this section we prove that it is not the case. We study, in turn, the actions of those groups over various sets of indices of computable functions and r.e. sets respectively. For definitions and group-theoretical notions, consult [8].

First, we show that the action on ω of any automorphism of W (respectively, of P) is determined by its action on the sets of indices H_n (respectively on the sets I_n).

Lemma 3 *If* f *is an automorphism of* W *such that for every* n

$$f(H_n) = H_n$$

then f *is the identity function.*

Proof. Suppose that f is not the identity. Then, there exists $n \in \omega$ such that $f(n) \neq n$. Let m such that $W_m = \{n\}$. Then

$$W_{f(m)} = f(W_m) = \{f(n)\} \neq \{n\} = W_m$$

so $f(H_m) \neq H_m$. ■

Lemma 4 *If* f *is an automorphism of* P *such that*

$$f(I_n) = I_n$$

for all n *in* ω *then* f *is the identity function.*

Proof. Every automorphism of P is an automorphism of W. Each set H_n is union of sets I_m. The result follows from previous lemma. ■

To prove the main results of this section, we will use the well-known notion of *recursive inseparability* of sets (see [7] for details).

Definition 1 *If* A *and* B *are subsets of* ω *we say that they are recursively separable if there exist recursive sets* U *and* V *such that*

- $A \subseteq U$
- $B \subseteq V$
- $U \cup V = \omega$
- $U \cap V = \emptyset$

The proof of the following result can be found in [4].

Lemma 5 *If f and g are two different computable functions then I_f and I_g are not recursively separable. If $W_n \neq W_m$ then H_n and H_m are not recursively separable.*

Now we prove the main results of this section.

Theorem 3 *For each $n < \omega$ the group $AutW$ acts n-transitively and faithfully on the set H_ω and on the set $H_\emptyset$.*

Proof. The action is faithfull, because if f is an automorphism of W such that $f(x) = x$ for all $x \in H_\omega$ then $H_\omega \subseteq U = \{x \in \omega : x = f(x)\}$, which is recursive (we proved that f is computable in theorem 2). From lemma 5 we see that f fixes at least one element of each set H_n. From lemmas 1 and 3 it follows that f is the identity function. The proof for $H_\emptyset$ is similar.

Now we prove that the action is also n-transitive. Take $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ two finite subsets of H_ω with n elements each. We will construct an automorphism of W taking a_i into b_i for $i = 1, \ldots, n$.

Given a computable function φ_z we will denote by $\varphi_{\overline{z}}$ the function which is computed in the following way:

Given a natural number x dovetail computations of φ_z. Then $\varphi_{\overline{z}}(x)$ is the first input which gives x as output.

Note that if φ_z is a permutation, then $\varphi_{\overline{z}}$ is its inverse.

We will show how to construct a total and computable function h such that for every $z \in \omega$

- The function $\varphi_{h(z)}$ is a recursive permutation
- $\varphi_{h(z)}(a_i) = b_i$ for $i = 1, \ldots, n$.
- For every $m \notin A$ we have either

  $$\varphi_z(W_m) = W_{\varphi_{h(z)}(m)}$$

  or

  $$W_m = \varphi_{\overline{z}}(W_{\varphi_{h(z)}(m)})$$

Given z the function $\varphi_{h(z)}$ is computed as follows:

Step 0 Let $\varphi_{h(z)}(a_i) = b_i$ for $i = 1, \ldots, n$.

Step $2t+1$ We choose m, the minimum element on which $\varphi_{h(z)}$ is not yet defined. We effectively generate indices (using padding lemma, see [7]) of the set

$$\varphi_z(W_m)$$

until we find a which is not yet in the range of $\varphi_{h(z)}$. Then, we let

$$\varphi_{h(z)}(m) = a$$

Step $2t+2$ We choose b the minimum element which is not yet in the range of $\varphi_{h(z)}$. We effectively generate indices of the set

$$\varphi_{\overline{z}}(W_b)$$

until we find m on which $\varphi_{h(z)}$ in not yet defined and let

$$\varphi_{h(z)}(m) = b$$

Let z a natural number such that $\varphi_z = \varphi_{h(z)}$ (recursion theorem assures that such a number exists). Then we will have

- φ_z is a recursive permutation
- $\varphi_z(a_i) = b_i$ for $i = 1, \ldots, n$.
- For every $m \notin A$ we have either

$$\varphi_z(W_m) = W_{\varphi_z(m)}$$

or

$$W_m = \varphi_{\overline{z}}(W_{\varphi_z(m)})$$

Since φ_z is a permutation we have $\varphi_{\overline{z}} = \varphi_z^{-1}$ and then for every $m \notin A$

$$\varphi_z(W_m) = W_{\varphi_z}(m)$$

On the other hand, since φ_z is a permutation and $A, B \subseteq H_\omega$, for every $a_i \in A$ it is true that

$$\varphi_z(W_{a_i}) = \varphi_z(\omega) = \omega = W_{b_i} = W_{\varphi_z(a_i)}$$

Then φ_z is the desired automorphism (see remark 1).

Note that in we have only used that A and B are subsets of H_ω in this last step. If we take A and B to be contained in $H_\emptyset$ we could replace the argument there with the following:

For every a_i in A

$$\varphi_z(W_{a_i}) = \varphi_z(\emptyset) = \emptyset = W_{b_i} = W_{\varphi_z(a_i)}$$

and the proof of the theorem is complete ■

Theorem 4 *For each $n < \omega$ the group $AutP$ acts faithfully and n-transitively and on the set I_{id} and on the set $I_{\uparrow}$.*

Proof. The proof is similar to the one of the previous theorem. We can replace steps $2t+1$ and $2t+2$ with the following

Step $2t+1$ We choose e, the minimum element on which $\varphi_{h(z)}$ is not yet defined. We effectively generate indices of the function

$$\varphi_z \cdot \varphi_e \cdot \varphi_{\overline{z}}$$

until we find a which is not yet in the range of $\varphi_{h(z)}$. Then, we let

$$\varphi_{h(z)}(e) = a$$

Step $2t+2$ We choose b the minimum element which is not yet in the range of $\varphi_{h(z)}$. We effectively generate indices of the function

$$\varphi_{\overline{z}} \cdot \varphi_b \cdot \varphi_z$$

until we find e on which $\varphi_{h(z)}$ in not yet defined and let

$$\varphi_{h(z)}(e) = b$$

Reasoning as we did in the previous theorem, we obtain an automorphism of P with the required action. ■

Remark 5 *Notice that, both in theorem 3 and 4, we can obtain an infinite number of automorphisms with the desired action over $a_1, \ldots, a_n$. Just choose another element a_{n+1} and apply the theorem to $a_1, \ldots, a_{n+1}$ varying b_{n+1}.*

Now, we can easily prove the following

Corollary 2 *There is no computable family of functions including all and only automorphisms of W.*

Proof. Consider that $\{f_n\}_{n\in\omega}$ is such a family. Then, from theorem 3 we conclude

$$H_\omega = \{f_n(x) : n \in \omega\}$$

but the set on the right is r.e. while H_ω is not. ■

Similarly we obtain

Corollary 3 *There is no computable family of functions including all and only automorphisms of P.*

After these results is obviously interesting to study the sets

$$\begin{aligned} C_W &= \{n \in \omega : \varphi_n \in AutW\} \\ C_P &= \{n \in \omega : \varphi_n \in AutP\} \end{aligned}$$

Adding a kind of diagonalization to the proofs of theorems 3 and 4 we can prove the following

Proposition 1 *Both C_W and C_P are productive.*

Proof. We prove it only for C_W, the case of C_P being similar. Suppose $W_b \subseteq C_W$. In the proof of theorem 3 subsitute step $2t+1$ with the following

- Step $2t+1$ We choose m, the minimum element on which $\varphi_{h(z)}$ is not yet defined. We enumerate a new element k of W_b. We effectively generate indices of the set
$$\varphi_z(W_m)$$
until we find a which is not yet in the range of $\varphi_{h(z)}$ and such that $\varphi_b(m) \neq a$. Then, we let
$$\varphi_{h(z)}(m) = a$$

Clearly, the obtained automorphism will be different from all with indices in W_b and, since, the process is uniform, we can effectively generate and index of it. ■

We can still say something else about the complexity of those sets.

Theorem 5 *Both C_W and C_P are Π^0_2-complete.*

Proof. Again both cases are similar, so we concentrate only in C_W.
First we show that $C_W \in \Pi^0_2$:

$$\begin{aligned} m \in C_W &\iff \varphi_m \in AutP \\ &\iff \forall n \forall x \ \left(x \in W_n \leftrightarrow \varphi_m(x) \in W_{\varphi_m(n)}\right) \end{aligned}$$

Applying Tarski-Kuratowski algorithm (see [7]) we get the result. Now, consider $B \in \Pi^0_2$. It is a well-known fact (see [7]) that in this conditions, there exists a recursive predicate $A(x,t)$ such that

$$B = \{x \in \omega : \text{There exist infinitely many } t\text{'s such that } A(x,t)\}$$

We have already proved that C_W is productive, so it contains an infinite r.e. subset D (see [6]). Let f be a total and computable function enumerating D without repetitions.

We construct a computable function g, such that given n, the value $g(n)$ is an index of the function computed in the following way:

Given k, look for elements t (beginning from 0) such that $A(n,t)$. When and only when k such elements are found, return $\varphi_{f(n)}(k)$.

We will have

$$\begin{aligned} g(n) \in C_W &\iff \varphi_{g(n)} \in AutW \\ &\iff \text{There are infinitely many } t\text{'s s.t. } A(x,t) \\ &\iff n \in B \end{aligned}$$

with g clearly 1-1. Then, $B \leq_1 C_W$ and the theorem is proved. ■

4 Properties of the groups AutW and AutP

In this section we modify the proof of theorems 3 and 4 in several ways to obtain results about the structure of the groups $AutP$ and $AutW$. Most of the results and proved only for $AutP$ and automatically inherited by $AutW$.

Lemma 6 *For each $m > 0$ there is an automorphism f of P such that it has exactly one finite cycle, and that cycle has length m.*

Proof. In the proof theorem 4 choose $n = m$ and $b_i = a_{i-1}$ for $i = 2, \dots, n$ and $b_1 = a_n$ (it gives the finite cycle of the desired length). Replace steps $2t + 1$ and $2t + 2$ with the following:

Step $2t + 1$ We choose e, the minimum element on which $\varphi_{h(z)}$ is not yet defined. We effectively generate indices of the function

$$\varphi_z \cdot \varphi_e \cdot \varphi_{\overline{z}}$$

until we find a which is not yet in the range of $\varphi_{h(z)}$, neither in its domain. Then, we let

$$\varphi_{h(z)}(e) = a$$

Step $2t + 2$ We choose b the minimum element which is not yet in the range of $\varphi_{h(z)}$. We effectively generate indices of the function

$$\varphi_{\overline{z}} \cdot \varphi_b \cdot \varphi_z$$

until we find e on which $\varphi_{h(z)}$ is not yet defined and is not yet in its range and let

$$\varphi_{h(z)}(e) = b$$

In this way, no more cycles will close and an automorphism with the announced properties is obtained. ■

Next results follow now easily:

Proposition 2 *The group $AutP$ (and so $AutW$) has an infinite number of elements of infinite order*

Proposition 3 *The group AutP (and so AutW) has an infinite number of conjugacy classes*

Proof. Clearly if we vary m in the previous lemma the automorphisms that we obtain are not conjugated ■

A more complicated modification of the proof of theorem 4 is needed to prove the following embedding theorem.

Theorem 6 *The group AutP (and so AutW) has a free subgroup of infinite rank.*

Proof. Since $AutP$ has elements of infinite order, it is enough to prove that, given $f_1, \dots, f_n \in AutP$ with only trivial relations we can effectively find an automorphism f of P such that $f_1, \dots, f_n$ and f have only trivial relations.

Modify the proof of theorem 4 in the following way:

Before defining $\varphi_{h(z)}$ for a given z fix an effective enumeration $\{w_t\}_{t\in\omega}$ of all the reduced words in $f_1, \dots, f_n, \varphi_{h(z)}$ and their inverses with at least one occurrence of $\varphi_{h(z)}$ or its inverse.

Eliminate step 0. Rename steps $2t+1$ and $2t+2$ to $3t$ and $3t+1$ respectively. Add the following

Step $3t+2$ At this step we care for warranting that $w_t \neq 1$. Suppose that

$$w_t = u_0\varphi_{h(z)}^{\epsilon_1}u_1\varphi_{h(z)}^{\epsilon_2}\cdots\varphi_{h(z)}^{\epsilon_n}u_n$$

where the u_i are in $f_1, \dots, f_n$ and their inverses only and $\epsilon_i \in \{1, -1\}$ for $i = 1, \dots, n$.

We choose m such that $W_m = W_e$ with e as in theorem 1 and $u_n(m)$ is yet neither in the domain of $\varphi_{h(z)}$ nor in its range. Clearly this is possible since we can effectively generate an infinite number indices of W_e and, at this step, $\varphi_{h(z)}$ is only defined on a finite number of elements.

Now, we proceed to define $\varphi_{h(z)}^{\epsilon_n}$ on $u_n(m)$ as we would do in steps $3k$ (if $\epsilon_n = 1$) or steps $3k+1$ (if $\epsilon_n = -1$) taking care that

1. $u_{n-1}\varphi_{h(z)}^{\epsilon_n}u_n(m) \neq m$
2. $u_{n-1}\varphi_{h(z)}^{\epsilon_n}u_n(m) \neq \varphi_{h(z)}^{\epsilon_n}(m)$
3. If $n > 1$ and $\epsilon_{n-1} = 1$ then $u_{n-1}\varphi_{h(z)}^{\epsilon_n}u_n(m)$ is not yet in the domain of $\varphi_{h(z)}$
4. If $n > 1$ and $\epsilon_{n-1} = -1$ then $u_{n-1}\varphi_{h(z)}^{\epsilon_n}u_n(m)$ is not yet in the range of $\varphi_{h(z)}$

It is clear that we can always satisfy conditions $1, 3$ and 4, since for the definition of $\varphi_{h(z)}^{\epsilon_n}$ on $u_n(m)$ we can choose from an infinite number of values and each condition only eliminates a finite number of elements.

Condition number 2 is different. If, while generating indices to define $\varphi_{h(z)}^{\epsilon_n}$ on $u_n(m)$, we find one of them, say b, such that $u_{n-1}(b) = b$ then we stop the process.

If we are able to satisfy the four conditions, $\varphi_{h(z)}^{\epsilon_{n-1}}$ will not be defined on $u_{n-1}\varphi_{h(z)}^{\epsilon_n}u_n(m)$ and we proceed to define it satisfying conditions similar to 1-4 above.

We iterate this process till the value $w_t(m)$ is defined (and, thus, satisfying $w_t(m) \neq m$) or till the it must be stopped because of the impossibility of satisfying one of the conditions with number 2.

Now, as in theorem 4 we use recursion theorem to get z so that $\varphi_z = \varphi_{h(z)}$. We have already proved that φ_z is an automorphism of P. Suppose there exists t such that $w_t = 1$. It is only possible if at step $3t+2$ of the definition of φ_z one of the conditions with number 2 could not be satisfied. Then, we have found an index m of W_e and a number k such that

$$u_{k-1}\varphi_{h(z)}^{\epsilon_k}u_k \cdots \varphi_{h(z)}^{\epsilon_n}u_n(m) = \varphi_{h(z)}^{\epsilon_k}u_k \cdots \varphi_{h(z)}^{\epsilon_n}u_n(m)$$

Denote $g = \varphi_{h(z)}^{\epsilon_k}u_k \cdots \varphi_{h(z)}^{\epsilon_n}u_n$ which is an automorphism of P (and so of W). Then

$$u_{k-1}g(m) = g(m)$$

and

$$g^{-1}u_{k-1}g(m) = m$$

We can apply theorem 1 to conclude $g^{-1}u_{k-1}g = 1$ and, consequently, $u_{k-1} = 1$. But u_{k-1} is a reduced word on $f_1, \ldots, f_n$ and their inverses and they generate a free group. We have reached a contradiction and, then, $w_t \neq 1$. ■

Finally, we turn to the study of the centers of the groups $AutP$ and $AutW$. Again we use a modification of the main construction in section 3. In the next result $Symm_r\omega$ will denote the group of computable permutations of ω.

Lemma 7 *Any group G such that $AutP \leq G \leq Symm_r\omega$ is centerless.*

Proof. Take $f \neq id$ in G. Find (effectively) x such that $f(x) \neq x$. Modify steps $2t+1$ and $2t+2$ in the proof theorem 4 so they include the following:

Step $2t+1$ We don't allow to define $\varphi_{h(z)}$ on $f(x)$ till it is defined on x. When we define $\varphi_{h(z)}(f(x))$ we choose a value different from $f(\varphi_{h(z)}(x))$ (which is already defined).

Step $2t+2$ We don't allow $f(x)$ to be included in the domain of $\varphi_{h(z)}$ in this step.

Clealy we will have

$$\varphi_{h(z)}(f(x)) \neq f(\varphi_{h(z)}(x))$$

for every z and f will not commute with the automorphism of P given by the recursion theorem. ■

In particular we have

Theorem 7 *The groups $AutP$ and $AutW$ are centerless.*

References

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