ELEMENTARY PROPERTIES OF ROGERS SEMILATTICES OF ARITHMETICAL NUMBERINGS *

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We investigate differences in the elementary theories of Rogers semilattices of arithmetical numberings, depending on structural invariants of the given families of arithmetical sets. It is shown that at any fixed level of the arithmetical hierarchy there exist infinitely many families with pairwise elementary different Rogers semilattices.

1. Preliminaries and Background

For unexplained terminology and notations relative to computability theory, our main references are the textbooks of A.I. Mal'tsev [1], H. Rogers [2] and R. Soare [3]. For the main concepts and notions of the theory of numberings we refer to the book of Yu.L. Ershov [4].

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Definition 1.1. Any surjective mapping α of the set ω of natural numbers onto a nonempty set A is called a *numbering* of A. Let α and β be numberings of A. We say that a numbering α is *reducible* to a numbering β (in symbols, $\alpha \leqslant \beta$) if there exists a computable function f such that $\alpha(n) = \beta(f(n))$ for any $n \in \omega$. We say that the numberings α and β are *equivalent* (in symbols, $\alpha \equiv \beta$) if $\alpha \leqslant \beta$ and $\beta \leqslant \alpha$.

S. S. Goncharov and A. Sorbi suggested in [5] a general approach for studying classes of objects which admit constructive descriptions in formal languages. This approach allows to unify in a very natural way various notions of computability and relative computability for different classes of constructive objects. Throughout this paper we will confine ourselves to families of arithmetical subsets of ω. We take in this case a Gdel numbering $\{\Phi_i\}_{i\in\omega}$ of the first-order arithmetical formulas, and apply this approach as follows, see [5]:

Definition 1.2. A numbering α of a family $\mathcal{A}$ of $\Sigma^0_{n+1}-$sets, with $n \geq 0$, is called $\Sigma^0_{n+1}-$ *computable* if there exists a computable function f such that, for every m, $\Phi_{f(m)}$ is a $\Sigma_{n+1}-$formula of Peano arithmetic and $\alpha(m) = \{x \in \omega \mid \mathfrak{N} \models \Phi_{f(m)}(\overline{x})\}$ (where the symbol $\overline{x}$ stands for the numeral for x and $\mathfrak{N}$ denotes the standard model of Peano arithmetic). The set of $\Sigma^0_{n+1}-$computable numberings of $\mathcal{A}$ will be denoted by $\mathrm{Com}^0_{n+1}(\mathcal{A})$.

Computable numberings of families of sets which are first-order definable in the standard model of Peano arithmetic are called *arithmetical numberings.* A family $\mathcal{A}$ for which $\mathrm{Com}^0_{n+1}(\mathcal{A}) \neq \emptyset$ will be called $\Sigma^0_{n+1}-$*computable.* If $n = 0$ then Σ^0_1- computable numberings and classical computable numberings of families of c.e. sets coincide.

The relation $\equiv$ is an equivalence relation on $\mathrm{Com}^0_{n+1}(\mathcal{A})$ and the reducibility $\leqslant$ induces a partial order on the equivalence classes of this relation. The equivalence class of a numbering α is called the *degree* of α, denoted by $\deg(\alpha)$. The partially ordered set $\langle \mathrm{Com}^0_{n+1}(\mathcal{A})/{\equiv}, \leqslant\rangle$ of the degrees of $\Sigma^0_{n+1}-$computable numberings of $\mathcal{A}$ will be denoted by $\mathcal{R}^0_{n+1}(\mathcal{A})$. If α and β are in $\mathrm{Com}^0_{n+1}(\mathcal{A})$, then a new numbering $\alpha \oplus \beta$ of $\mathcal{A}$ is defined as follows: $\alpha \oplus \beta(2n) = \alpha(n)$ and $\alpha \oplus \beta(2n+1) = \beta(n)$, and $\deg(\alpha \oplus \beta)$ determines the least upper bound of the pair $\deg(\alpha)$, $\deg(\beta)$ in $\mathcal{R}^0_{n+1}(\mathcal{A})$. Thus, $\mathcal{R}^0_{n+1}(\mathcal{A})$ can be regarded as an upper semilattice.

Definition 1.3. The upper semilattice $\mathcal{R}^0_{n+1}(\mathcal{A})$ is called the *Rogers semilattice* of the class of arithmetical numberings of $\mathcal{A}$.

The Rogers semilattice $\mathcal{R}^0_{n+1}(\mathcal{A})$ can be viewed as a tool for measuring the algorithmic complexity of computations of the family $\mathcal{A}$ as a whole, and the problems of the theory of computable numberings concern mainly the algebraic and elementary properties of the Rogers semilattices.

We continue the investigation of the elementary types of Rogers semilattices for infinite arithmetical families started in [6], [7] and [8]. We are interested in differences between the elementary theories of Rogers semilattices of families of any fixed level of the arithmetical hierarchy.

Everyone who has ever dealt with the classical theory of computable numberings is well aware that general facts about Rogers semilattices of families of c.e. sets are very rare, and at the same time it is very difficult to establish elementary properties that distinguish given structures. Opposite to the classical case, the elementary theories of Rogers semilattices of arithmetical numberings for the level two and higher seem more exciting. In what follows, we briefly examine some algebraic and elementary properties of the Rogers semilattices $\mathcal{R}^0_{n+2}(\mathcal{A})$ for various $\mathcal{A}$.

1.1. *Cardinality, Lattice Properties, Undecidability*

The following two theorems are well-known facts of the theory of computable numberings in the classical case.

Theorem 1.1. (A. B. Khutoretsky, [9]**)** *For every family $\mathcal{A}$ of c.e. sets, if the Rogers semilattice $\mathcal{R}^0_1(\mathcal{A})$ contains at least two elements then it is infinite.*

Theorem 1.2. (V. L. Selivanov, [10]**)** *For every family $\mathcal{A}$ of c.e. sets, if the Rogers semilattice $\mathcal{R}^0_1(\mathcal{A})$ contains at least two elements then it is not a lattice.*

For Σ^0_1-families, Theorems 1.1,1.2 answer questions posed by Yu. L. Ershov. The corresponding questions for the case of $\Sigma^0_{n+2}-$computable families have been answered by Goncharov and Sorbi, [5].

Theorem 1.3. *If a $\Sigma^0_{n+2}-$computable family $\mathcal{A}$ contains at least two elements then the Rogers semilattice $\mathcal{R}^0_{n+2}(\mathcal{A})$ is infinite and is not a lattice.*

Theorem 1.3 gives us a complete description of the families $\mathcal{A} \subseteq \Sigma^0_{n+2}$ since if $\mathcal{A}$ consists of a single element then all numberings of $\mathcal{A}$ are evidently equivalent. For details relative to the classical case, we refer to [11], and we recall the following well-known problem raised by Yu. L. Ershov.

Question 1.1. Under what conditions the Rogers semilattice $\mathcal{R}^0_1(\mathcal{A})$ of a family of c.e. sets is non-trivial?

It should be noted that the elementary theory of $\mathcal{R}^0_{n+2}(\mathcal{A})$ of every non-trivial family $\mathcal{A}$ is a quite complicated. We give evidence to this statement as follows.

Let ε^* denote the bounded distributive lattice obtained by dividing the lattice ε of all c.e. subsets of ω modulo the ideal of all finite sets. We will denote by $\hat{\beta}$ the principal ideal of $\mathcal{R}^0_{n+1}(\mathcal{A})$,

$$\hat{\beta} \risingdotseq \{\deg(\gamma) \mid \deg(\gamma) \leqslant \deg(\beta)\}.$$

Theorem 1.4. (S. Yu. Podzorov, [12], see also [7]) *Let $\mathcal{A}$ be any Σ^0_{n+2}–computable family. There exists a numbering $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ such that*

(1) $\hat{\alpha}$ is isomorphic to $\varepsilon^*\setminus\{\perp\}$ if the family $\mathcal{A}$ is infinite;
(2) $\hat{\alpha}$ is isomorphic to ε^* if the family $\mathcal{A}$ is finite.

Theorem 1.4 and the fact that the elementary theory of ε^* is hereditarily undecidable, [13], immediately yield:

Corollary 1.1. *The elementary theory of every non-trivial Rogers semilattice $\mathcal{R}^0_{n+2}(\mathcal{A})$ is hereditarily undecidable.*

Theorem 1.4 and Corollary 1.1 give us a deep insight into the complexity of Rogers semilattices of Σ^0_{n+2}– computable families. The case of Σ^0_1– computable families is still open:

Question 1.2. Is the elementary theory of any non-trivial Rogers semilattice of a Σ^0_1 – computable family hereditarily undecidable, or at least undecidable?

1.2. *Extremal Elements*

What kind of computable numberings should be thought of as the most natural ones? A partial answer to this question, as well as a motivation for introducing the notion of a universal numbering, is given by next proposition [6].

Proposition 1.1. *Let α be a numbering of a family $\mathcal{A} \subseteq \Sigma^0_{n+1}$. Then the following statements are equivalent:*
(i) α is Σ^0_{n+1} – computable;
(ii) α is reducible to the numbering $W^{\mathbf{0}^{(n)}}$ of the family of all Σ^0_{n+1} – sets;
(iii) α is $\mathbf{0}^{(n)}$ – reducible to $W^{\mathbf{0}^{(n)}}$.

Definition 1.4. A numbering α of $\mathcal{A} \subseteq \Sigma^0_{n+1}$ is called *principal* or *universal in* $\mathrm{Com}^0_{n+1}(\mathcal{A})$ if
(i) $\alpha \in \mathrm{Com}^0_{n+1}(\mathcal{A})$,
(ii) $\beta \leqslant \alpha$ for all numberings $\beta \in \mathrm{Com}^0_{n+1}(\mathcal{A})$.

It is obvious that the greatest element, if any, of the Rogers semilattice of any family $\mathcal{A}$ is exactly the degree of some universal numbering of $\mathcal{A}$. Proposition 1.1 implies also that many essential facts and notions relative to universal numberings are easily lifted from principal computable numberings of families of c.e. sets to arithmetical numberings. For instance,

- Ershov's classification of principal subsets, see [4];
- the closure condition of Lachlan [14] for families of sets to have computable principal numberings;
- existence of universal numberings in $\mathrm{Com}^0_{n+1}(\mathcal{A})$ with respect to $\mathbf{0}^{(n)}-$reducibility, for every finite family $\mathcal{A} \subseteq \Sigma^0_{n+1}$.

In particular, we should mention the following two examples which show a difference between the Rogers semilattices of some infinite families.

Example 1.1. The family $\mathbf{\Sigma}^0_{n+1}$ of all $\Sigma^0_{n+1}-$subsets of ω has a universal numbering in $\mathrm{Com}^0_{n+1}(\mathbf{\Sigma}^0_{n+1})$, namely the relativization $W^{\mathbf{0}^{(n)}}$ of the classical Post numbering W of the family of all c.e. sets.

Example 1.2. For every n, the set $\mathcal{F}$ of all finite sets is obviously $\Sigma^0_{n+1}-$computable and has no universal numbering in $\mathrm{Com}^0_{n+1}(\mathcal{F})$. The latter holds by the relativized version of Lachlan's condition, [14]: if any $\Sigma^0_{n+1}-$computable family has a universal numbering then it is closed under unions of increasing $\Sigma^0_{n+1}-$computable sequences of its members.

These examples show an elementary difference between $\mathcal{R}^0_{n+1}(\mathbf{\Sigma}^0_{n+1})$ and $\mathcal{R}^0_{n+1}(\mathcal{F})$ with regard to the existence/non-existence of the greatest element in these semilattices.

As regards finite families, examples of elementary differences between Rogers semilattices of finite families are provided by the following result of S. A. Badaev, S. S. Goncharov, and A. Sorbi, [6].

Theorem 1.5. *Let $\mathcal{A} \subseteq \Sigma^0_{n+2}$ be a finite family. Then $\mathcal{A}$ has an universal numbering in $\mathrm{Com}^0_{n+2}(\mathcal{A})$ if and only if $\mathcal{A}$ contains a least element under inclusion.*

Again, as in the examples above, existence/non-existence of the greatest element provides an elementary property which allows us to distinguish some Rogers semilattices.

To compare elementary properties of Rogers semilattices of finite families versus Rogers semilattices of infinite families, we can use a different type of extremal elements, namely minimal elements of the semilattices. It is a well-known fact of the theory of numberings that any finite family has a numbering which is reducible to all the numberings of that family, see [4]. And this fact does not depend on either the nature of the family or the computability of the considered numberings. Thus, the Rogers semilattice $\mathcal{R}^0_{n+1}(\mathcal{A})$ of any finite family $\mathcal{A}$ of $\Sigma^0_{n+1}-$sets has a least element. On the other hand, we have the following theorem of S. A. Badaev and S. S. Goncharov, [15].

Theorem 1.6. *For every n, if $\mathcal{A}$ is an infinite $\Sigma^0_{n+2}-$computable family, then $\mathcal{R}^0_{n+2}(\mathcal{A})$ has infinitely many minimal elements.*

Remark 1.1. Theorem 1.6 does not hold for some infinite families of c.e. sets and does hold for other infinite families of c.e. set. Furthermore, the following question is a problem of Yu. L. Ershov known since the 60's. We refer to [11] for details on this problem.

Question 1.3. What is the possible number of minimal elements in the Rogers semilattice $\mathcal{R}^0_1(\mathcal{A})$ of a family of c.e. sets?

1.3. *The Weak Distributivity Property*

In this subsection we are concerned with an interesting and natural elementary property of Rogers semilattices which establishes one more difference between $\mathcal{R}^0_{n+2}(\mathcal{A})$, with $\mathcal{A}$ finite, and $\mathcal{R}^0_{n+2}(\mathcal{B})$, with $\mathcal{B}$ infinite. We refer to [8] for details and proofs. First we recall some definitions.

Definition 1.5. An upper semilattice $\langle L, \vee, \leqslant\rangle$ is called *distributive* if for every $a_1, a_2, b \in L$, if $b \leq a_1 \vee a_2$ then there exist $b_1, b_2 \in L$ such that $b_1 \leq a_1, b_2 \leq a_2$ and $b = b_1 \vee b_2$.

Theorem 1.7. *For every n and for every finite family $\mathcal{A} \subseteq \Sigma^0_{n+1}$, $\mathcal{R}^0_{n+1}(\mathcal{A})$ is a distributive upper semilattice.*

The situation is different if we consider infinite families. First of all, we notice:

Remark 1.2. It is easy to see that the three element upper semilattice $L_0 = \{a, b, c\}$, where a and b are incomparable and $c = a \vee b$, is not distributive. There exist many Rogers semilattices which contain L_0 as an ideal, [4], and, therefore, are not distributive. However, if we add $\perp$ to L_0, we do obtain a distributive lattice.

This remark motivates our next definition.

Definition 1.6. An upper semilattice $\mathfrak{L} = \langle L, \leqslant\rangle$ is *weakly distributive* if $\mathfrak{L}_\perp = \langle L \cup \{\perp\}, \leqslant_\perp\rangle$ is distributive, where $\perp \notin \mathfrak{L}$ and

$$\leqslant_\perp \leftrightharpoons \leqslant \cup \{(\perp, a) \mid a \in L \cup \{\perp\}\}.$$

Proposition 1.2. *An upper semilattice $\langle L, \vee, \leqslant\rangle$ is weakly distributive if and only if for every $a_1, a_2, b \in L$, if $b \leq a_1 \vee a_2$ and $b \nleq a_1, b \nleq a_2$ then there exist $b_1, b_2 \in L$ such that $b_1 \leq a_1, b_2 \leq a_2$ and $b = b_1 \vee b_2$.*

Theorem 1.8. *For every n, the Rogers semilattice of any infinite Σ^0_{n+2}-computable family is not weakly distributive.*

Question 1.4. Does there exist a computable infinite family $\mathcal{A}$ of c.e. sets such that $\mathcal{R}^0_1(\mathcal{A})$ is distributive? Does there exist a computable infinite family $\mathcal{A}$ of c.e. sets such that $\mathcal{R}^0_1(\mathcal{A})$ is weakly distributive?

2. The Main Result

It should be noted that Rogers semilattices of families from different levels of arithmetical hierarchy can be surprisingly different, as can be seen from the following theorem of S. A. Badaev, S. S. Goncharov and A. Sorbi, [8].

Theorem 2.1. *For every n there exist $m \geq n$ and a Σ^0_{m+2}-computable family $\mathcal{B}$ such that no Rogers semilattice $\mathcal{R}^0_{n+1}(\mathcal{A})$ of any Σ^0_{n+1}-computable family $\mathcal{A}$ is isomorphic to $\mathcal{R}^0_{m+2}(\mathcal{B})$.*

The differences between Rogers semilattices established in Theorem 2.1 are based on the fact that ideals of Rogers semilattices of families chosen from different levels of the arithmetical hierarchy have different algorithmic complexities. Unfortunately, these differences are not elementary. So Theorem 2.1 provides a natural motivation for searching elementary properties between Rogers semilattices of families lying in the same level of the arithmetical hierarchy. Some fruitful ideas from the paper of V. V. V'jugin, [16] were very useful for our research.

Theorem 2.2. *For every $k \in \omega$, there exist infinitely many Σ^0_{k+1}– computable families with elementary pairwise different Rogers semilattices.*

Sketch of proof. Let k be an arbitrary natural number. We will construct a sequence $\{\mathcal{B}_e\}_{e \geq 1}$ of infinite Σ^0_{k+1}–computable families such that

$$\mathrm{Th}(\mathcal{R}^0_{k+1}(\mathcal{B}_{e'})) \neq \mathrm{Th}(\mathcal{R}^0_{k+1}(\mathcal{B}_{e''}))$$

for all $e' \neq e''$.

Indeed we will construct a sequence $\{\mathcal{A}_n\}_{n \geq 1}$ of families of sets of which $\{\mathcal{B}^e\}_{e \geq 1}$ is subsequence.

Let M stands for any $\mathbf{0}^{(k)}$– maximal set, and let n be a natural number.

Let $E^1_n, E^2_n, \ldots, E^n_n$ be a computable partition of ω into infinite computable sets. Let f^i_n denote some arbitrary computable bijection of ω onto $E^i_n, i \in [1, n]$. Clearly, $M_i \rightleftharpoons \overline{E}^i_n \bigcup f^i_n(M)$ is a $\mathbf{0}^{(k)}$– maximal set for each $i \in [1, n]$.

Let $\mathcal{A}^i_n$, $i \in [1, n]$, stand for the family $\{M_i \bigcup \{x\} \mid x \in \overline{M}_i\}$. The families $\mathcal{A}^i_n$ are evidently Σ^0_{k+1}– computable. Define $\mathcal{A}_n \rightleftharpoons \bigcup\limits_{i \in [1,n]} \mathcal{A}^i_n$.

Lemma 2.1. *For all numbers $n > 0$ and $i \in [1, n]$, every numbering $\alpha \in \mathrm{Com}^0_{k+1}(\mathcal{A}_n)$, and every set $A \in \mathcal{A}^i_n$, the index sets with respect to α of the subfamilies $\mathcal{A}^i_n$ and $\{A\}$ are Σ^0_{k+1}– sets.*

Lemma 2.2. *For all numbers $n > 1$, $i \in [1, n]$ and every numbering $\nu \in \mathrm{Com}^0_{k+1}(\mathcal{A}_n)$, if ν is join $\nu = \nu_0 \oplus \nu_1$ of some numberings ν_0, ν_1 then all but finitely many sets of $\mathcal{A}^i_n$ are contained either in $\nu_0(\omega)$ or $\nu_1(\omega)$.*

Lemma 2.3. *For all numbers $n > 0$ and $m \geq n$ and all numberings $\gamma^0_1, \gamma^1_1, \gamma^0_2, \gamma^1_2, \ldots, \gamma^0_{m+1}, \gamma^1_{m+1} \in \mathrm{Com}^0_{k+1}(\mathcal{A}_n)$, if $\gamma^0_1 \oplus \gamma^1_1 \equiv \gamma^0_2 \oplus \gamma^1_2 \equiv \ldots \equiv \gamma^0_{m+1} \oplus \gamma^1_{m+1}$ then there exist a numbering $\delta \in \mathrm{Com}^0_{k+1}(\mathcal{A}_n)$ and a binary sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ such that $\delta \leqslant \gamma^0_{m+1}$ and $\delta \leqslant \gamma^{\varepsilon_1}_1 \oplus \gamma^{\varepsilon_2}_2 \oplus \ldots \oplus \gamma^{\varepsilon_m}_m$.*

Definition 2.1. We will say that any two Σ^0_{k+1}–computable numberings ν_0, ν_1 of a family $\mathcal{A}$ induce a *minimal pair* in the Rogers semilattice $\mathcal{R}^0_{k+1}(\mathcal{A})$ if there is no numbering $\nu \in \mathrm{Com}^0_{k+1}(\mathcal{A})$ such that $\nu \leqslant \nu_0$ and $\nu \leqslant \nu_1$.

In the proof of Lemma 2.3 we construct two numberings which do not induce a minimal pair in $\mathcal{R}^0_{k+1}(\mathcal{A}_n)$. On the other hand we consider now some regular way of constructing numberings which induce minimal pairs in Rogers semilattice $\mathcal{R}^0_{k+1}(\mathcal{A}_n)$.

For every $i \in [1,n]$, we fix two different numbers $a_i^0, a_i^1 \in \overline{M}_i$ and define numberings α_i^s, $s \leq 1$ as follows: for every x, we let

$$\alpha_i^s(x) \rightleftharpoons \begin{cases} M_i \bigcup \{a_i^s\}, & \text{if } x \in M, \\ M_i \bigcup \{f_n^i(x)\} & \text{otherwise.} \end{cases}$$

It is obvious that $\alpha_i^s \in \mathrm{Com}_{k+1}^0(\mathcal{A}_n^i)$.

Lemma 2.4. *The numberings α_i^0 and α_i^1 induce a minimal pair in $\mathcal{R}_{k+1}^0(\mathcal{A}_n^i)$.*

Lemma 2.5. *For every $m > 0$ and $n \geq 2^{2^{m+1}}$, there exist numberings $\beta_1^0, \beta_1^1, \beta_2^0, \beta_2^1, \ldots, \beta_{2^m}^0, \beta_{2^m}^1 \in \mathrm{Com}_{k+1}^0(\mathcal{A}_n)$ such that*

- $\beta_1^0 \oplus \beta_1^1 \equiv \beta_2^0 \oplus \beta_2^1 \equiv \ldots \equiv \beta_{2^m}^0 \oplus \beta_{2^m}^1$;
- *for every $i \in [1, 2^m]$, the numberings β_i^0 and β_i^1 induce a minimal pair in $\mathcal{R}_{k+1}^0(\mathcal{A}_n)$;*
- *for every $l \leq m$, every set $I = \{i_1 < i_2 < \ldots < i_l\} \subseteq [1, 2^m]$, every binary sequence $\sigma_1, \sigma_2, \ldots, \sigma_l$, and every $\varepsilon \in \{0,1\}$ and $i \in [1, 2^m] \setminus I$, the numberings β_i^ε and $\beta_{i_1}^{\sigma_1} \oplus \beta_{i_2}^{\sigma_2} \oplus, \ldots, \oplus \beta_{i_l}^{\sigma_l}$ induce a minimal pair in $\mathcal{R}_{k+1}^0(\mathcal{A}_n)$.*

Using Lemmas 2.1–2.5 we can now deduce the statement of the theorem as follows. Define a computable function h by letting $h(1) = 16$ and $h(e+1) = 2^{2^{h(e)+1}}$ for every $e \geq 1$. Let $\mathcal{B}_e \rightleftharpoons \mathcal{A}_{h(e)}$ for every $e \geq 1$. Lemmas 2.3, 2.5 imply that $Th(\mathcal{R}_{k+1}^0(\mathcal{B}_{e'})) \neq Th(\mathcal{R}_{k+1}^0(\mathcal{B}_{e''}))$ for every $e' \neq e''$.

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