Algebra and Logic, Vol. ?, No. ?, 200?

Isomorphism types of Rogers semilattices for families from different levels of the arithmetical hierarchy [1]

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UDC 510.55

Abstract

We investigate differences in the isomorphism types of Rogers semilattices of computable numberings of families of sets lying in different levels of the arithmetical hierarchy.

Among the many possible applications of generalized computable numberings, introduced in [10], a particularly interesting and popular one is the study of *arithmetical numberings*, i.e. numberings of families of arithmetical sets. When considering a family $\mathcal{A}$ of Σ^0_n- sets, generalized computable numberings can be characterized as follows: a numbering α of $\mathcal{A}$ is generalized computable if and only if the set $\{\langle x, i\rangle : x \in \alpha(i)\}$ is Σ^0_n. Such a numbering α will be simply called in the following Σ^0_n*–computable*. We recall that if α and β are numberings of the same family of objects, then one says that α is *reducible to* β (in symbols: $\alpha \leq \beta$) if there exists a computable function f such that $\alpha = \beta \circ f$. We write $\alpha \equiv \beta$ if $\alpha \leq \beta$ and $\beta \leq \alpha$. Since $\leq$ is a preordering relation, it follows that $\equiv$ is an equivalence relation. If $\mathcal{A}$ is a family of Σ^0_n- sets, then $\equiv$ partitions the set $\mathrm{Com}^0_{\mathrm{n}}(\mathcal{A})$ of all Σ^0_n-computable numberings of $\mathcal{A}$ into equivalence classes, thus originating a degree structure, denoted by $\mathcal{R}^0_n(\mathcal{A})$ and called the *Rogers semilattice* of $\mathcal{A}$. The equivalence class of a numbering α (depending of course on the collection of numberings under study) will be denoted by the symbol $\deg(\alpha)$. In this paper we continue the investigation of the isomorphism types of these Rogers semilattices, started in [3]. We are interested in differences between elementary theories and isomorphism types for different arithmetical levels. In [4] and [5] it was shown that for every fixed level of the arithmetical hierarchy there exist infinitely many families with pairwise different elementary theories. In [3] we established that for every n the isomorphism type of the Rogers semilattice of *some* $\Sigma^0_{n+5}-$computable family $\mathcal{B}$ is different from the isomorphism type of the Rogers semilattice $\mathcal{R}^0_{n+1}(\mathcal{A})$ of any arbitrary $\Sigma^0_{n+1}-$ computable family $\mathcal{A}$. In this paper we improve on this result by showing that for every n the isomorphism type of the Rogers semilattice of *any* non trivial $\Sigma^0_{n+4}-$computable family $\mathcal{B}$ is different from the isomorphism type of the Rogers semilattice $\mathcal{R}^0_{n+1}(\mathcal{A})$ of any $\Sigma^0_{n+1}-$computable family $\mathcal{A}$.

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[1]Partially supported by grant INTAS–00–499 *Computability in Hierarchies and Topological Spaces.*

[2]Partially supported by grant RFBR–01–00??

[3]Partially supported by NSFC Grand International Joint Project *New Directions in Theory and Applications of Models of Computation*, No.60310213.

For unexplained terminology and notations relative to computability theory, our main references are the textbooks of A.I. Mal'tsev [12], H. Rogers [13] and R. Soare [14]. For the main concepts and notions of the theory of numberings and computable Boolean algebras we refer to the book of Yu.L. Ershov [7] and the book of S.S. Goncharov [9]. For the basic notions, notations and methods relative to arithmetical numberings and their Rogers semilattices we will refer to [1] and [2]. For the ease of the reader, and to make the paper more self contained, we only recall here the definition of the Lachlan operator for numberings, and some of its properties summarized in Lemma 1 below.

Definition 1. *If β is a numbering of a family $\mathcal{A}$, and C is a nonempty c.e. set, with f a computable function such that* $\mathrm{range}(f) = C$, *then we define $\beta_C \leftrightharpoons \beta \circ f$.*

The definition does not depend on f: If we define β_C starting from any other computable function g such that $\mathrm{range}(g) = C$ then we get a numbering which is equivalent to the one given by f. The assignment $C \mapsto \beta_C$ from c.e. sets to numberings (up to equivalence of numberings) is called *Lachlan operator*.

Lemma 1 ([2], Lemma 2.2). *For every pair A, B of c.e. sets and for every pair of numberings α, β, we have:*

(1) The following are equivalent:

- *(a) $\beta_A \leqslant \beta_B$;*
- *(b) there is a partial computable function φ satisfying* $\mathrm{dom}(\varphi) \supseteq A$, *$\varphi[A] \subseteq B$ and for all $x \in A$, $\beta(x) = \beta(\varphi(x))$;*

(2) if $A \subseteq B$ then $\beta_A \leqslant \beta_B$;

(3) if $\beta_A \leqslant \beta_B$, then $\beta_B \equiv \beta_{A\cup B}$;

(4) if $\alpha \leqslant \beta$ then $\alpha \equiv \beta_C$ for some c.e. set C;

(5) if $\alpha \leqslant \beta$, and $\alpha \equiv \beta_C$, for some c.e. set C, then for every γ such that $\alpha \leqslant \gamma \leqslant \beta$ there exists a c.e. set D with $C \subseteq D$ and $\gamma \equiv \beta_D$;

(6) $\beta_{A\cup B} \equiv \beta_A \oplus \beta_B$;

The following three lemmas (taken from [3]) and the notion of an $\mathbf{X}$– computable Boolean algebra play a key role in establishing our claim. Recall (see [9]) that a Boolean algebra $\mathfrak{A}$ is called $\mathbf{X}$– *computable* if its universe, operations, and relation are $\mathbf{X}$– computable.

In the following lemma, the symbol $[\gamma, \delta]$ denotes the following interval of degrees in $\mathcal{R}^0_{n+1}(\mathcal{A})$:

$$[\gamma, \delta] \leftrightharpoons \{\deg(\beta) \mid \gamma \leq \beta \leq \delta\}.$$

Lemma 2. *Let γ and δ be Σ^0_{n+1}– computable numberings of a family $\mathcal{A}$. If $[\gamma, \delta]$ is a Boolean algebra, then it is $\mathbf{0}^{(n+3)}$– computable.*

Proof. Given n, $\mathcal{A}$, γ, and δ as in the hypothesis of the lemma, we first observe that by (4) and (5) of Lemma 1, there exists a c.e. set C such that $\gamma \equiv \delta_C$ and

$$[\gamma, \delta] = \{\deg(\delta_X) \mid X \text{ is c.e. and } X \supseteq C\}.$$

For every i, let $U_i \leftrightharpoons C \cup W_i$. This gives an effective listing of all c.e. supersets of C. By Lemma 1 (1b), for every i, j, we have $\delta_{U_i} \leqslant \delta_{U_j}$ if and only if

$$\exists p[\forall x(x \in U_i \Rightarrow \exists y(\varphi_p(x) = y \;\&\; y \in U_j)) \\ \& \; \forall x \forall y(x \in U_i \;\&\; \varphi_p(x) = y \Rightarrow \delta(x) = \delta(y))]$$

Since $\delta \in \mathrm{Com}^0_{n+1}(\mathcal{A})$, this implies that $\delta_{U_i} \leqslant \delta_{U_j}$ is a Σ^0_{n+3}– relation in i, j.

Let us consider the equivalence relation η on ω defined by

$$(i, j) \in \eta \Leftrightarrow \delta_{U_i} \leqslant \delta_{U_j} \;\&\; \delta_{U_j} \leqslant \delta_{U_i}.$$

Let $B \leftrightharpoons \{x \mid \forall y(y < x \Rightarrow (x, y) \notin \eta)\}$. Define a bijection $\psi_1 : B \longrightarrow [\gamma, \delta]$, by letting $\psi_1(i) = \deg(\delta_{U_i})$, for all $i \in B$. It is evident that ψ_1 induces in $\mathcal{R}^0_{n+1}(\mathcal{A})$ a partially ordered set $\mathfrak{B}$ which is a Boolean algebra isomorphic to $[\gamma, \delta]$. The interval $\mathfrak{B}$ is a $\mathbf{0}^{(n+3)}$– computable partially ordered set. It follows from [9, Theorem 3.3.4] and [6], that the Boolean algebra $\mathfrak{B}$ relatively to the corresponding Boolean operations is $\mathbf{0}^{(n+3)}$– computable too. □

Lemma 3 (L. Feiner). *Let $\mathfrak{F}$ be a computable atomless Boolean algebra. Then for every $\mathbf{X}$ there is an ideal J such that J is $\mathbf{X}$– c.e. and the quotient $\mathfrak{F}/J$ is not isomorphic to any $\mathbf{X}$– computable Boolean algebra.*

Proof. See [8]. □

Below, we will use the following notations. For a given c.e. set H, $\{V_i \mid i \in \omega\}$ denotes an effective listing of all c.e. supersets of the set H defined, for instance, by $V_i \leftrightharpoons H \cup W_i$, for all i. We will assume for convenience that $V_0 = H$. Let ε_H stand for the distributive lattice of the c.e. supersets of H. For a given c.e. set $V \supseteq H$, let V^* denote the image of V under the canonical homomorphism of ε_H onto ε^*_H (i.e. ε_H modulo the finite sets), and let $\subseteq^*$ denote the partial ordering relation of ε^*_H. Obviously, if J is an ideal in ε_H then $J^* \leftrightharpoons \{V^* \mid V \in J\}$ is an ideal in ε^*_H.

As is known (see, for instance, [9]), if $\mathfrak{A}$ is a Boolean algebra and J is an ideal of $\mathfrak{A}$, then the universe of the quotient Boolean algebra $\mathfrak{A}/J$ is given by the set of equivalence classes $\{[a]_J \mid a \in \mathfrak{A}\}$ under the equivalence relation $\equiv_J$ given by

$$a \equiv_J b \Leftrightarrow \exists c_1, c_2 \in J(a \vee c_1 = b \vee c_2),$$

and the partial ordering relation is given by

$$[a]_J \leq_J [b]_J \Leftrightarrow a - b \in J.$$

where $a - b$ stands for $a \wedge \neg b$.

Lemma 4. *Let $\mathcal{B}$ be a Σ^0_{m+1}–computable family, $\beta \in \mathrm{Com}^0_{m+1}(\mathcal{B})$, and let H be any c.e. set such that $\beta(H) = \mathcal{B}$ and ε^*_H is a Boolean algebra. Let $\psi_2 : \varepsilon_H \longrightarrow [\beta_H, \beta]$ be the mapping given by $\psi_2(V_i) = \deg(\beta_{V_i})$ for all i, and let I be any ideal of ε_H. Then ψ_2 induces an isomorphism of ε^*_H/I^* onto $[\beta_H, \beta]$ if and only if for every i, j*

(1) $V_i \in I \Rightarrow \beta_{V_i} \leqslant \beta_H$;

(2) $V_i - V_j \notin I \Rightarrow \beta_{V_i} \nleqslant \beta_{V_j}$ (where $V_i - V_j \leftrightharpoons (V_i \setminus V_j) \cup H$.)

Proof. Let $H, \mathcal{B}, \beta, \psi_2$ be given. The "only if" direction is immediate. As to show that the conditions stated in the lemma are also sufficient, we can argue as follows. By (4–5) of Lemma 1, we have that every γ with $\beta_H \leqslant \gamma \leqslant \beta$ is of the form $\gamma \equiv \beta_C$ for some c.e. set $C \supseteq H$. Then the mapping induced by ψ_2 is clearly onto.

Suppose now that $[V^*_i]_{I^*} \subseteq^*_{I^*} [V^*_j]_{I^*}$. Then $V^*_i - V^*_j \in I^*$. But $V^*_i - V^*_j = (V_i - V_j)^*$, with $V_i - V_j$ a c.e. superset of H, since ε^*_H is a Boolean algebra. Then $V_i - V_j \in I$. On the other hand,

$$V_i = (V_i - V_j) \cup (V_i \cap V_j).$$

Now, by (1), $\beta_{V_i - V_j} \leqslant \beta_H$, so by (3) of Lemma 1, $\beta_{V_i} \equiv \beta_{V_i \cap V_j}$, hence $\beta_{V_i} \leqslant \beta_{V_j}$ by (2) of Lemma 1, as $V_i \cap V_j \subseteq V_j$.

Finally, if $[V^*_i]_{I^*} \not\subseteq^*_{I^*} [V^*_j]_{I^*}$ then $V_i - V_j \notin I$, and therefore by (2) $\beta_{V_i} \nleqslant \beta_{V_j}$. □

Theorem 1. *For every n, every non-trivial Σ^0_{n+5}–computable family $\mathcal{B}$, and every Σ^0_{n+1}–computable family $\mathcal{A}$, the Rogers semilattices $\mathcal{R}^0_{n+5}(\mathcal{B})$ and $\mathcal{R}^0_{n+1}(\mathcal{A})$ are not isomorphic.*

Proof. Let n be given, let $\mathcal{B}$ be an arbitrary non-trivial Σ^0_{n+5}–computable family, and let $\mathcal{A}$ be any Σ^0_{n+1}–computable family. By Lemma 2, all Boolean intervals of $\mathcal{R}^0_{n+1}(\mathcal{A})$ are $\mathbf{0}^{(n+3)}$–computable Boolean algebras. Therefore, to show the theorem it is sufficient:

(i) to consider a computable atomless Boolean algebra $\mathfrak{F}$ and an ideal J of $\mathfrak{F}$ as in Feiner's Lemma such that J is c.e. in $\mathbf{0}^{(n+3)}$ and $\mathfrak{F}/J$ is not isomorphic to any $\mathbf{0}^{(n+3)}$–computable Boolean algebra,

(ii) to find Σ^0_{n+5}–computable numberings α and β of $\mathcal{B}$ such that the interval $[\alpha, \beta]$ of $\mathcal{R}^0_{n+5}(\mathcal{B})$ is a Boolean algebra isomorphic to $\mathfrak{F}/J$.

First, we consider item (i) above. Let $\mathfrak{F}$ be a computable atomless Boolean algebra. According to a famous result of Lachlan, [11], there exists a hyperhypersimple set H such that ε^*_H is isomorphic to $\mathfrak{F}$. We fix such a set H.

We refer to the textbook of Soare, [14], for the details of a suitable isomorphism χ of ε^*_H onto $\mathfrak{F}$. We only notice that starting from a computable listing $\{b_0, b_1, \ldots\}$ of the elements of $\mathfrak{F}$ one can find a Σ^0_3–computable Friedberg numbering $\{B_0, B_1, \ldots\}$ of a subfamily of the family ε_H such that $\varepsilon^*_H = \{B^*_0, B^*_1, \ldots\}$, and $\chi(B^*_i) = b_i$.

We will use the techniques for embedding posets into intervals of Rogers semilattices which have been developed in [2]. Let J be any $\mathbf{0}^{(n+3)}$ - c.e. ideal of $\mathfrak{F}$ satisfying the conclusions of Lemma 3, and let $\hat{J} = \{j \in \omega \mid b_j \in J\}$. Then $\hat{J}$ is a $\mathbf{0}^{(n+3)}$ - c.e. set, $I^* \leftrightharpoons \{B_j^* \mid j \in \hat{J}\}$ is an ideal of ε_H^* and $\mathfrak{F}/J$ is isomorphic to ε_H^*/I^*. So, instead of the Boolean algebra $\mathfrak{F}/J$ in item (ii) above we can consider ε_H^*/I^*.

Let $I \leftrightharpoons \{V \mid V \in \varepsilon_H \; \& \; V^* \in I^*\}$, and let $\hat{I} = \{i \in \omega \mid V_i^* \in I^*\}$. Obviously, I is an ideal of ε_H.

Lemma 5. *The relations "$V_i \in I$" (equivalently: "$i \in \hat{I}$"), in i, and "$V_i - V_j \in I$", in i, j, are both $\mathbf{0}^{(n+3)}$ - c.e.*

Proof. First of all note that for every sets A, B the relation "$A =^* B$" is c.e. in the relation "$A = B$". Indeed, if $D_0, D_1, \dots$ is the canonical numbering of the family of finite sets (see [12] or [13]) then

$$A =^* B \Leftrightarrow \exists p \exists q (A \cup D_p = B \cup D_q).$$

Since

$$i \in \hat{I} \Leftrightarrow \exists j (V_i =^* B_j \; \& \; j \in \hat{J}) \qquad \text{and}$$

$$V_i - V_j \in I \Leftrightarrow \exists k ((V_i \cap \overline{V_j}) \cup H = V_k \; \& \; k \in \hat{I}),$$

a simple calculation shows that $\hat{I} \in \Sigma^0_{n+4}$ and that the relation "$V_i - V_j \in I$" is also c.e. in $\mathbf{0}^{(n+3)}$. □

Due to Lemma 4, we can now construct a suitable numbering β of $\mathcal{B}$ and consider the corresponding mapping ψ_2 which will give us an isomorphism ε_H^*/I^* onto the interval $[\beta_H, \beta]$.

The requirements. First of all, we need that the numbering β satisfy the requirement:

$$\mathbf{B} : \beta[H] = \mathcal{B}$$

to guarantee that β_H is a numbering of the whole family $\mathcal{B}$. Then in view of Lemma 4 we must satisfy, for every i, j, p, the requirements:

$$\mathbf{P}_i : V_i \in I \Rightarrow \beta_{V_i} \leqslant \beta_H$$
$$\mathbf{R}_{i,j,p} : V_i - V_j \notin I \Rightarrow \beta_{V_i} \nleqslant \beta_{V_j} \text{ via } \varphi_p$$

where by "$\beta_{V_i} \nleqslant \beta_{V_j}$ via φ_p" we mean that φ_p does not reduce β_{V_i} to β_{V_j} in the sense of Lemma 1(1b).

We use the oracle $\mathbf{0}^{(n+4)}$ in our construction to answer questions such as "$V_i \in I$?" or "$V_i - V_j \in I$?" and to verify some properties of c.e. sets and partial computable functions. Since all computations are relative to the oracle $\mathbf{0}^{(n+4)}$, we will get that $\beta \in \mathrm{Com}^0_{n+5}(\mathcal{B})$.

The strategy for B. We fix a numbering $\alpha \in \mathrm{Com}^0_{n+5}(\mathcal{B})$ and build by stages a $\mathbf{0}^{(n+4)}$-computable function $a(x)$ with $\mathrm{range}(a) \subseteq H$. We will "insert" the numbering α into the numbering β by letting $\beta(a(x)) = \alpha(x)$ for all x. Since we will never change the values $\beta(a(x))$ we will have as a consequence the equality $\beta[H] = \mathcal{B}$.

The strategy for $\mathbf{R}_{i,j,p}$. Since $\mathcal{B}$ is nontrivial, fix two different sets $A, B \in \mathcal{B}$. To meet $\mathbf{R}_{i,j,p}$ we will destroy any possible reducibility of the numbering β_{V_i} to the numbering β_{V_j} via any partial computable function φ_p with $V_i \subseteq \mathrm{dom}(\varphi_p)$ and $\varphi_p[V_i] \subseteq V_j$. We choose some β-index $x \in V_i \setminus V_j$ and let $\beta(x) = A$ and $\beta(\varphi_p(x)) = B$ or, conversely, $\beta(x) = B$ and $\beta(\varphi_p(x)) = A$. Note that $x \neq \varphi_p(x)$ since $x \notin V_j$ and $\varphi_p[V_i] \subseteq V_j$.

The strategy for $\mathbf{P}_i$. Fix an infinite computable set $R \subseteq H$, and fix a computable partition of R into disjoint infinite computable sets R_i, $i \in \omega$. Finally fix a computable sequence $\{r_i\}_{i\in\omega}$ of injective unary partial computable functions such that $\mathrm{dom}(r_i) = V_i \setminus R$ and $\mathrm{range}(r_i) = R_i$. If $i \in \hat{I}$ then it is sufficient to keep the equality $\beta(x) = \beta(r_i(x))$ for all $x \in V_i \setminus R$ to meet the requirement $\mathbf{P}_i$. Indeed, by Lemma 1, if so then we have $\beta_{V_i\setminus R} \leqslant \beta_{R_i}$. Since $\beta_{V_i} \equiv \beta_{V_i\setminus R} \oplus \beta_R \leqslant \beta_{R_i} \oplus \beta_R \equiv \beta_R$, it then follows that $\beta_{V_i} \leqslant \beta_H$.

Unfortunately, there could be conflicts of two possible types between $\mathbf{P}$–strategies and $\mathbf{R}$–strategies. More precisely, the strategy for $\mathbf{R}_{i,j,p}$ may want $\beta(x) \neq \beta(\varphi_p(x))$ for some $x \in V_m$ with $m \in \hat{I}$, while the strategy $\mathbf{P}_m$ forces us to have $\beta(x) = \beta(r_m(x))$. Since the functions φ_p, r_m are fixed a-priori, it follows that the equality $\varphi_p(x) = r_m(x)$ is possible and in this case we have a conflict.

Moreover, for every i, j, m, there exist infinitely many numbers p such that $\varphi_p \restriction (V_i \setminus V_j) = r_m \restriction (V_i \setminus V_j)$, and it may seem impossible to prevent conflicts at all. Fortunately, this is not so, and the next lemma and its corollary give us a tool to avoid almost all conflicts between a $\mathbf{P}_m$–strategy for any fixed $m \in \hat{I}$, and all $\mathbf{R}$–strategies. For $t \in \omega$ we will denote by U_t the set $\bigcup\{V_s : s \leq t \;\&\; s \in \hat{I}\}$.

Lemma 6. *Let H and I be chosen as above. Let V' and V be arbitrary sets of the lattice ε_H such that $V' \notin I$ and $V \in I$. Then the following properties hold*

(a) $V' - V \in \varepsilon_H$

(b) $V' - V \notin I$*, and, in particular,* $V' \setminus V$ *is an infinite set.*

Proof. Property *(a)* follows from the trivial equality

$$(V' - V)^* = (V')^* - V^*.$$

Property *(b)* can be easily verified by contradiction using the following equality:

$$V' = (V' - V) \cup (V' \cap V).$$

□

Corollary 1. *For every $i, j, t \in \omega$, if $V_i - V_j \notin I$ then $V_i \setminus (V_j \cup U_t)$ is infinite.*

We give absolute priority to the $\mathbf{R}$– strategies versus the $\mathbf{P}$– strategies. For every i, j, p, we exclude all the conflicts between the strategy for $\mathbf{R}_{i,j,p}$ and the strategies for $\mathbf{P}_m$, for all $m \in \hat{I}$ and $m \leq \langle i, j, p \rangle$, in the following way. We choose a β–index x to satisfy the requirement $\mathbf{R}_{i,j,p}$ from the set $V_i \setminus (V_j \cup U_{\langle i,j,p \rangle})$ instead of the set $V_i \setminus V_j$ as in the above description of the strategy for $\mathbf{R}_{i,j,p}$. And we do not pay attention to the conflicts between the strategy for a fixed $\mathbf{R}_{i,j,p}$ and the $\mathbf{P}_m$– strategies with $m \in \hat{I}$ and $m > \langle i, j, p \rangle$. Thus, for every fixed $m \in \hat{I}$, we will have at most finitely many conflicts of the strategy for $\mathbf{P}_m$ with the $\mathbf{R}$– strategies.

Conflicts of type 2 between $\mathbf{R}_{i,j,p}$– strategies and $\mathbf{P}_m$– strategies may occur even in the case $\varphi_p(x) \neq r_m(x)$, due to the function a which is built by the strategy for the requirement $\mathbf{B}$. Assume that a number $y \in H \setminus R$ has become value of the function a and we have defined $\beta(y) = \alpha(z)$ for some z. If x is chosen to meet the requirement $\mathbf{R}_{i,j,p}$ then we have to let $\beta(x)$ and $\beta(\varphi_p(x))$ be equal to either A or B, and $\beta(x) \neq \beta(\varphi_p(x)$. Suppose also that $\varphi_p(x) \in R$. It may occur that $\varphi_p(x) = r_m(y)$ for some $m \in \hat{I}$. The strategy for $\mathbf{P}_m$ forces us to have $\beta(\varphi_p(x)) = \beta(r_m(y)) = \alpha(z)$. We get a conflict if either $\alpha(z) \notin \{A, B\}$ or $\alpha(z) \in \{A, B\}$ and $\beta(\varphi_p(x)) \neq \alpha(z)$. Unfortunately, we can not see whether $\alpha(z) = A$ or $\alpha(z) = B$ with the oracle $\mathbf{0}^{(n+4)}$. Again, for every $m \in \hat{I}$, we would get only finitely many instances of conflicts of this type if we restricted the choice of x by the additional restraint $m > \langle i, j, p \rangle$.

To describe the construction we still need some auxiliary notions and notations. For every $x \in \overline{R}$ we consider the set $\{x\} \cup \{r_m(x) \mid m \in \omega\}$. This set is called *star* with *center* x. Obviously, stars with different centers are disjoint, and the collection of all stars forms a partition of ω. For every $x \in \omega$, it is easy to compute the center x^+ of the star which contains x: namely, $x^+ = x$ if $x \notin R$ and $x^+ = r_m^{-1}(x)$ if $x \in R$ and R_m is the element of the partition $\{R_i\}_{i \in \omega}$ which contains x.

In terms of stars, our plan to avoid the conflicts between $\mathbf{P}$– strategies and $\mathbf{R}$– strategies mentioned above may be described as follows. First of all, given x, conflicts of the first type do not to appear at all if $x \neq (\varphi_p(x))^+$ as well as $x = (\varphi_p(x))^+$ but $\varphi_p(x) \in R_m$ for $m \notin \hat{I}$. And for every fixed m we allow only finitely many strategies $\mathbf{R}_{i,j,p}$ to injure $\mathbf{P}_m$, i.e. to make $\beta(\varphi_p(x)) \neq \beta(x)$ but $\beta(r_m(x)) \neq \beta(x)$. Stars with centers in range(a) are the unique source of conflicts of the second type. And for every fixed m we allow only finitely many such stars to injure $\mathbf{P}_m$.

Now we will build, by a stage construction, a numbering β of the family $\mathcal{B}$ and an auxiliary function a. If a value $\beta(x)$ or $a(x)$ has not been explicitly modified by the end of stage $t + 1$ then by default $\beta^{t+1}(x) = \beta^t(x)$ or $a^{t+1}(x) = a^t(x)$ respectively. It should be mentioned that in the construction below we never change already defined values of the functions β and a to undefined ones.

The construction

Stage 0. Let $\beta(x)$ and $a(x)$ be undefined for all x. Go to the next stage.

Stage t + 1. Let $t = \langle i, j, p \rangle$. Carry out the following three procedures starting from Procedure $\mathbf{R}_{i,j,p}$ first.

Procedure $\mathbf{R}_{i,j,p}$. Check the following conditions (we can do this relatively to the oracle $\mathbf{0}^{(n+4)}$):

(i) $V_i - V_j \notin I$;

(ii) $V_i \subseteq \mathrm{dom}(\varphi_p)$ and $\varphi_p[V_i] \subseteq V_j$.

If one of (i) or (ii) fails then go to Procedure $\mathbf{P}_i$.

Otherwise, choose the least element x of the set

$$X^t \leftrightharpoons \{x \mid x \in V_i \setminus (V_j \cup U_t) \ \&\ \beta^t(x)\uparrow \ \&\ \beta^t(\varphi_p(x))\uparrow\}$$

such that at least one of the following two conditions holds:

(iii) $\exists y(y \neq x \ \&\ y \in X^t \ \&\ \varphi_p(y) = \varphi_p(x))$;

(iv) $(\varphi_p(x))^+ \in \mathrm{range}(a^t) \Rightarrow \forall m(\varphi_p(x) \in R_m \Rightarrow m > t)$.

(See (7) below for the existence of such an x).

If (iii) holds then pick the least y satisfying (iii) and let $\beta(x) = A, \beta(y) = B$. Go to Procedure $\mathbf{P}_i$.

If (iii) does not hold (but (iv) does) then denote by z the center $(\varphi_p(x))^+$ and carry out the instructions of the following two cases and after that go to Procedure $\mathbf{P}_i$.

Case 1: $\varphi_p(x) \in R$, $z \neq x$, and $z \notin \mathrm{range}(a^t)$. If $\beta^t(z)\uparrow$ then let $\beta(z) = A$. Denote by C the set $\beta(z)$, and denote by D an element of the set $\{A, B\} \setminus \{C\}$. Let $\beta(x) = D$ and $\beta(\varphi_p(x)) = C$.

Case 2: Case 1 does not hold. Let $\beta(x) = A$ and $\beta(\varphi_p(x)) = B$.

Procedure $\mathbf{P}_i$. Choose the least number $x \in R_i$ such that $\beta(x)$ is still undefined. If $\beta(x^+)$ is also undefined then let $\beta(x^+) = \beta(x) = A$. If $\beta(x^+)\downarrow$ then let $\beta(x) = \beta(x^+)$. Go to Procedure $\mathbf{B}$.

Procedure B. Pick the least number $y \in H \setminus R$ such that β is still undefined in all points of the star with center y, and let $a(t) = y$ and $\beta(y) = \alpha(t)$. Go to the next stage.

Obviously, β is a Σ^0_{n+5}−computable numbering.

Properties of the construction The construction satisfies the following properties:

(1) For every t, the functions β^t and a^t have finite domains.

(2) For every x, there exists a stage t starting from which $\beta(x)$ becomes defined forever. Besides, after this stage t the function β never changes its value on that index x.

(3) a is total function with $\mathrm{range}(a) \subseteq H \setminus R$. For every x, $\alpha(x) = \beta(a(x))$.

(4) For every x, $\beta(x) = A$ or $\beta(x) = B$ or $\beta(x) = \alpha(y)$ for some y.

(5) For every x, t, if $\beta^t(x) \downarrow$ then $\beta^t(x^+) \downarrow$.

(6) For every x, t, if $x \in \mathrm{range}(a^t)$ then $\beta^t(x) \downarrow$.

Properties (1)–(6) are evident, (2)–(4) imply that β is a numbering of the family $\mathcal{B}$.

(7) For every i, j, p, t, if conditions (i),(ii) above hold then there exists $x \in X^t$ satisfying (iii) or (iv).

To see this, choose any numbers i, j, p, t such that conditions (i),(ii) hold and assume that both (iii) and (iv) fail. Then φ_p is injective on X^t and

$$(\varphi_p(x))^+ \in \mathrm{range}(a^t) \ \& \ \exists m(\varphi_p(x) \in R_m \ \& \ m \leq t).$$

The set X^t is infinite by Lemma 6 and property (1). By property (1), the set $Y \leftrightharpoons \{y \mid r_m^{-1}(y) \in \mathrm{range}(a^t) \ \& \ m \leq t\}$ is finite since all the functions $r_k, k \in \omega$, are injective. Therefore, φ_p maps in a one-to-one fashion the infinite set X^t into the finite set Y. A contradiction.

(8) For every i, j, p, if conditions (i),(ii) hold then there exists $z \in V_i$ such that $\beta(z) \neq \beta(\varphi_p(z))$.

Let $V_i - V_j \notin I$, $V_i \subseteq \mathrm{dom}(\varphi_p)$, and $\varphi_p[V_i] \subseteq V_j$, and let $t = \langle i, j, p\rangle$. If (iii) holds at stage $t+1$ then φ_p maps two different numbers $x, y \in V_i$ into the same number $\varphi_p(x) = \varphi_p(y)$ and, by the construction, $\beta(x) \neq \beta(y)$. Therefore, at least one among the inequalities $\beta(x) \neq \beta(\varphi_p(x))$ and $\beta(y) \neq \beta(\varphi_p(y))$ holds.

Suppose now that (iii) fails at stage $t+1$ and let $x \in X^t$ be a number chosen by Procedure $\mathbf{R}_{i,j,p}$ at this stage. By the construction, $x \in V_i \setminus (V_j \cup U_t)$ and both $\beta^t(x)$ and $\beta^t(\varphi_p(x))$ are undefined. It implies that $x \notin V_m$ and $r_m(x)$ is undefined for all $m \in \hat{I}, m \leq t$. Note that $\varphi_p(x) \notin \mathrm{range}(a^t)$ because of property (6). We consider the next four possibilities:
(a) $\varphi_p(x) \notin R$;
(b) $\varphi_p(x) \in R$, $(\varphi_p(x))^+ = x$;
(c) $\varphi_p(x) \in R$, $(\varphi_p(x))^+ \neq x$, and $(\varphi_p(x))^+ \in \mathrm{range}(a^t)$;
(d) $\varphi_p(x) \in R$, $(\varphi_p(x))^+ \neq x$, and $(\varphi_p(x))^+ \notin \mathrm{range}(a^t)$.

For (a),(b),(c) we have $\beta(x) = A$ and $\beta(\varphi_p(x)) = B$ by Case 2 of Procedure $\mathbf{R}_{i,j,p}$ at stage $t+1$. For (d), by Case 1, we have $\beta(x) = D$, $\beta(\varphi_p(x)) = C$ with $\{C, D\} = \{A, B\}$.

(9) For every $m \in \hat{I}$ and almost all $x \in V_m \setminus R$, $\beta(x) = \beta(r_m(x))$.

Let $m \in \hat{I}$. Suppose that $v \in V_m \setminus R$ and $\beta(v) \neq \beta(r_m(v))$ and let us prove that this inequality is caused by a conflict between the strategy for $\mathbf{P}_m$ and an $\mathbf{R}_{i,j,p}$– strategy for some i, j, p.

Let $s+1$ and $t+1$ be the stages at which correspondingly $\beta(v)$ and $\beta(r_m(v))$ have been defined. By property (2), $\beta(v) = \beta^{s+1}(v)$ and $\beta(r_m(v)) = \beta^{t+1}(r_m(v))$.

Let us consider the star with center v and denote by $q+1$ the least stage at which β has been defined on a point of this star for the first time. By property (5), $q = s$ and $s \leq t$ since $(r_m(v))^+ = v$. We divide now our consideration into two cases: $s = t$ and $s < t$.

$s = t$. This means that β is defined exactly in two points of the star with the center v by the end of stage $t+1$, and that one of these points is v. Therefore, $\beta(v)$ can not be defined at stage $t+1$ by Procedure $\mathbf{B}$, and (iii) does not hold at stage $t+1$. Thus, we have to examine the possibilities (a)–(d) considered in the proof of property (8). Obviously, $\beta(v)$ and $\beta(r_m(v))$ can not be defined by (a),(c) since in these cases β is defined in the centers of two disjoint stars. Possibility (d) is also impossible because by construction in this case the values of β in the points v and $r_m(v)$ have to be identical.

So we have to check only possibility (b). We will keep the notations in the description of Procedure $\mathbf{R}_{i,j,p}$, and in the proof of property (8). Following these notations, $v = x$, $\varphi_p(x) = r_m(x)$, $t = \langle i, j, p\rangle$. We have a conflict of type 1: $\beta(x) = A$, $\beta(\varphi_p(x)) = \beta(r_m(x)) = B$. Moreover, $x \in V_i \setminus (V_j \cup U_t)$, and, therefore, $m > \langle i, j, p\rangle$. This implies that we can have at most finitely many β–indices $v \in V_m \setminus R$ for which inequality $\beta(v) \neq \beta(\varphi_p(v))$ is caused by a conflict of type 1.

$s < t$. Because of property (5), this means that we get to stage $t+1$, at which β is defined on the point $r_m(v)$, only after β has been defined on the center v. As for the case $s = t$, it is easy to see that, at stage $t+1$, $\beta(r_m(v))$ can not be defined by Procedure $\mathbf{B}$, (iii) does not hold, and possibilities (a),(b),(d) can not cause the inequality $\beta(v) \neq \beta(\varphi_p(v))$.

Let us keep our conventions on notations and consider possibility (c). Then $\varphi_p(x) = r_m(x)$, $(\varphi_p(x))^+ = v$, $v \in \mathrm{range}(a^t)$, and, hence, (iv) implies inequality $m > \langle i, j, p\rangle$. We have $\beta(x) = A$, $\beta(r_m(x)) = B$, and $\beta(v) \neq B$. Thus, we can have at most finitely many β–indices $v \in V_m \setminus R$ for which the inequality $\beta(v) \neq \beta(\varphi_p(v))$ is caused by a conflict of type 2.

Properties (3),(8),(9) imply that all the requirements are satisfied.

□

Theorem 2. *For every n, every non-trivial Σ^0_{n+4}– computable family $\mathcal{B}$, and every Σ^0_{n+1}– computable family $\mathcal{A}$, the Rogers semilattices $\mathcal{R}^0_{n+4}(\mathcal{B})$ and $\mathcal{R}^0_{n+1}(\mathcal{A})$ are not isomorphic.*

Proof. We start with a Σ^0_{n+4}– computable family $\mathcal{B}$ and a numbering $\alpha \in \mathrm{Com}^0_{n+4}(\mathcal{B})$. Instead of working directly with the relation "$V_i - V_j \in I$" (in i, j) and the set $\hat{I}$, we use enumerations relatively to the oracle $\mathbf{0}^{(n+3)}$ of this relation and set, which are $\mathbf{0}^{(n+3)}$ - c.e. by Lemma 5.

Denote the relation "$V_i - V_j \in I$" by $Q(i,j)$ and let by $Q^t(i,j), t \in \omega$, be a suitable approximation to the relation, i.e :

$Q^t(i,j)$ is $\mathbf{0}^{(n+3)}$ –computable relation in i, j, t;

$Q^t(i,j) \to Q^{t+1}(i,j)$ for every i, j, t;

$Q(i,j) \leftrightarrow \exists s \forall t \geq s Q^t(i,j)$ for every i, j.

Let $\hat{I}^t, t \in \omega$, be an enumeration of the set $\hat{I}$ relatively to $\mathbf{0}^{(n+3)}$, and let $\hat{U}_t$ and $\hat{X}^t$ be approximations of the sets U_t and X^t, namely,

$$\hat{U}_t = \bigcup \{V_k \mid k \leq t \ \&\ k \in \hat{I}^t\} \text{ and}$$

$$\hat{X}^t = \{x \mid x \in V_i \setminus (V_j \cup \hat{U}^t) \ \&\ \beta^t(x)\uparrow \ \&\ \beta^t(\varphi_p(x))\uparrow\} \text{ for } t = \langle i, j, p\rangle.$$

Here and in what follows we use the same notations as in the proof of Theorem 1. The construction below is a full approximation of the construction given in Theorem 1.

The construction

Stage 0. Let $\beta(x)$ and $a(x)$ be undefined for all x. Go to the next stage.

Stage t + 1. Let $t = \langle i, j, p\rangle$. Carry out the following three procedures starting from Procedure $\mathbf{R}_{i,j,p}$ first. We execute all instructions at the stage computably in the oracle $\mathbf{0}^{(n+3)}$.

Procedure $\mathbf{R}_{i,j,p}$. Check whether condition

(ii) $V_i \subseteq \mathrm{dom}(\varphi_p)$ and $\varphi_p[V_i] \subseteq V_j$

holds. If (ii) does not hold then go to Procedure $\mathbf{P}_i$. If it does then search for the least $s > t$ such that $Q^s(i, j)$ or there exists $x \in \hat{X}^s$ such that at least one of the following two conditions holds:

(iii)′ $\exists y(y \neq x \; \& \; y \in \hat{X}^s \; \& \; \varphi_p(y) = \varphi_p(x))$;

(iv) $(\varphi_p(x))^+ \in \mathrm{range}(a^t) \Rightarrow \forall m(\varphi_p(x) \in R_m \Rightarrow m > t)$.

If $Q^s(i, j)$ then go to Procedure $\mathbf{P}_i$. Otherwise choose the least $x \in \hat{X}^s$ which satisfies the conditions above. If (iii)′ holds then pick the least y satisfying (iii)′ and let $\beta(x) = A, \beta(y) = B$. Go to Procedure $\mathbf{P}_i$.

If (iii)′ does not hold (but (iv) does) then denote by z the center $(\varphi_p(x))^+$ and carry out the instructions of the following two cases and after that go to Procedure $\mathbf{P}_i$.

Case 1: $\varphi_p(x) \in R$, $z \neq x$, and $z \notin \mathrm{range}(a^t)$. If $\beta^t(z) \uparrow$ then let $\beta(z) = A$. Denote by C the set $\beta(z)$, and denote by D the set $\{A, B\}\backslash C$. Now let $\beta(x) = D$ and $\beta(\varphi_p(x)) = C$.

Case 2: Case 1 does not hold. Let $\beta(x) = A$ and $\beta(\varphi_p(x)) = B$.

Procedure $\mathbf{P}_i$. Choose the least number $x \in R_i$ such that $\beta(x)$ is still undefined. If $\beta(x^+)$ is also undefined then let $\beta(x^+) = \beta(x) = A$. If $\beta(x^+) \downarrow$ then let $\beta(x) = \beta(x^+)$. Go to Procedure **B**.

Procedure B. Pick the least number $y \in H \setminus R$ such that β is still undefined on all points of the star with center y, and let $a(t) = y$ and $\beta(y) = \alpha(t)$. Go to the next stage.

Properties of the construction Obviously, β is a Σ^0_{n+4}−computable numbering. It is easily seen that the construction is just a modification of the one in Theorem 1. This is why we omit unnecessary repetitions in the proofs of the properties of the modified construction. Properties (1)–(6) are exactly the same as in the proof of Theorem 1. They guarantee that β is a Σ^0_{n+4}−computable numbering of the family $\mathcal{B}$. Let now consider the remaining properties.

(7) For every i, j, p, if condition (ii) holds at stage $t+1$ with $t = \langle i, j, p\rangle$ then there exists $s > t$ such that $Q^s(i, j)$ or there exists $x \in \hat{X}^s$ satisfying at least one of (iii)′ or (iv).

If $V_i - V_j \in I$ then evidently there exists s such that $Q^s(i, j)$. Suppose now that $V_i - V_j \notin I$ and let s be the least number such that $s > t$ and

$$\{k \mid k \leq \langle i, j, p\rangle \; \& \; k \in \hat{I}\} = \{k \mid k \leq \langle i, j, p\rangle \; \& \; k \in \hat{I}^s\}.$$

Then $\hat{U}_s = U_t$, $\hat{X}^s = X^t$, and, hence, condition (iii)$'$ becomes identical to condition (iii). It follows that now we can argue as we have done in the proof of property (7) of the previous construction.

(8) For every i, j, p, if $V_i - V_j \notin I$ and condition (ii) holds then there exists $z \in V_i$ such that $\beta(z) \neq \beta(\varphi_p(z))$.

Let $V_i - V_j \notin I$. Note that $Q^s(i, j)$ fails for all s, and so we can repeat the proof of property (8) as in Theorem 1.

(9) For every $m \in \hat{I}$ and almost all $x \in V_m \setminus R$, $\beta(x) = \beta(r_m(x))$.

Let $m \in \hat{I}$ and let t_0 be the least number such that $m \in \hat{I}^{t_0}$. Then $m \in \hat{U}_s$ for all $s \geq t_0$, and, hence, $V_m \cap \hat{X}^t = \emptyset$ for all $t \geq t_0$. Repeating the arguments used in the proof of property (9) of Theorem 1 we can show that for every $t > \max\{t_0, m\}$ there is no $v \in V_m \setminus R$ such that

- $\beta(v) \neq \beta(r_m(v))$;
- at least one of $\beta(v)$ or $\beta(r_m(v))$ is defined at stage t;
- both $\beta(v)$ and $\beta(r_m(v))$ are defined by the end of stage t.

Property (1) implies that the number of v's which satisfy these three conditions at stages $t \leq \max\{t_0, m\}$ is finite.

Again, properties (3),(8),(9) imply that all the requirements are satisfied.

□

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Received December ??, 2004.

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