ISOMORPHISM TYPES AND THEORIES OF ROGERS SEMILATTICES OF ARITHMETICAL NUMBERINGS

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Abstract We investigate differences in isomorphism types and elementary theories of Rogers semilattices of arithmetical numberings, depending on different levels of the arithmetical hierarchy. It is proved that new types of isomorphism appear as the arithmetical level increases. It is also proved the incompleteness of the theory of the class of all Rogers semilattices of any fixed level. Finally, no Rogers semilattice of any infinite family at arithmetical level $n \geq 2$ is weakly distributive, whereas Rogers semilattices of finite families are always distributive.

Keywords: Numberings; arithmetical hierarchy; computable numberings; Rogers semilattice; elementary theory; isomorphism type; Boolean algebras; hyperhypersimple sets; Feiner's hierarchy; computable algebra.

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For the basic notions, notations and methods relative to arithmetical numberings and their Rogers semilattices we will refer to [2] and [3]. In this paper we investigate Rogers semilattices of arithmetical numberings from the point of view of computable algebra. We continue the investigation of the isomorphism types and the elementary types of Rogers semilattices both for finite and infinite arithmetical families. In this direction we are interested in the differences between elementary theories and isomorphism types for different arithmetical levels. For unexplained terminology and notations relative to computability theory, our main references are the textbooks of A.I. Mal'tsev [10], H. Rogers [13] and R. Soare [14]. For the main concepts and notions of the theory of numberings and computable Boolean algebras we refer to the book of Yu.L. Ershov [5] and the book of S.S. Goncharov [7].

1. Distinguishing Rogers semilattices of finite families

Given arithmetical families $\mathcal{A}$ and $\mathcal{B}$, the corresponding Rogers semilattices may look very different from each other. The results illustrated in [3] already enable us to make some comments on this matter. First of all, we already know that if $\mathcal{A}$ and $\mathcal{B}$ are finite families such that $\mathcal{A}$ has least element with respect to inclusion and $\mathcal{B}$ does not, then $\mathcal{R}^0_{n+1}(\mathcal{A})$ and $\mathcal{R}^0_{n+1}(\mathcal{B})$ are not elementarily equivalent, since $\mathcal{R}^0_{n+1}(\mathcal{A})$ possesses a greatest element, whereas $\mathcal{R}^0_{n+1}(\mathcal{B})$ does not: See Theorem 3.2, [2].

2. Distinguishing Rogers semilattices of finite families from Rogers semilattices of infinite families

It is not difficult to point out elementary differences between Rogers semilattices of finite families and Rogers semilattices of infinite families. For instance, we may observe:

Theorem 2.1. *If $\mathcal{A}$ is a finite family of Σ^0_{n+1} sets, then $\mathcal{R}^0_{n+1}(\mathcal{A})$ has least element.*

Proof. Let $\mathcal{A} \leftrightharpoons \{A_0, \dots, A_k\}$. Let $\{X_0, \dots, X_k\}$ be a computable partition of $\mathbb{N}$ into computable sets. For every $x \in X_i$ let $\alpha(x) \leftrightharpoons A_i$. It is easy to see that $\alpha \leqslant \beta$, for all $\beta \in \mathrm{Com}^0_{n+1}(\mathcal{A})$. □

On the other hand, there exist infinite families of c.e. sets $\mathcal{B}$ such that $\mathcal{R}^0_1(\mathcal{B})$ has no minimal elements, [1] and [15]. Moreover, if $\mathcal{B}$ is an infinite Σ^0_{n+2}−computable family of sets, then there exists a numbering $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{B})$ such that no minimal numbering of $\mathcal{B}$ is reducible to α, [12], [3].

Thus $\mathcal{R}^0_{n+2}(\mathcal{A})$ is always elementarily different from $\mathcal{R}^0_{n+2}(\mathcal{B})$, if $\mathcal{A}$ is finite and $\mathcal{B}$ is infinite.

3. Weak distributivity

In this section we show another interesting and natural elementary difference between $\mathcal{R}^0_{n+2}(\mathcal{A})$, with $\mathcal{A}$ finite, and $\mathcal{R}^0_{n+2}(\mathcal{B})$, with $\mathcal{B}$ infinite. In the former case we always get a distributive semilattice, whereas in the latter case we always get a nondistributive, not even weakly distributive, semilattice. First we give some definitions.

Definition 3.1. An upper semilattice $\langle L, \vee, \leqslant\rangle$ is called *distributive* if for every $a_1, a_2, b \in L$, if $b \leq a_1 \vee a_2$ then there exist $b_1, b_2 \in L$ such that $b_1 \leq a_1, b_2 \leq a_2$ and $b = b_1 \vee b_2$.

The following proposition states a well known and general fact of the theory of numberings.

Proposition 3.1. *For every numberings $\alpha_0, \alpha_1, \beta$, if $\beta \leqslant \alpha_0 \oplus \alpha_1$ then there exist numberings β_0, β_1 such that $\beta_0 \leqslant \alpha_0, \beta_1 \leqslant \alpha_1$ and $\beta \equiv \beta_0 \oplus \beta_1$.*

Proof. Let f be a computable function such that range(f) contains both even and odd numbers and $\beta(x) = (\alpha_0 \oplus \alpha_1)(f(x))$ for all x. Let $R_0 \risingdotseq \{x \mid f(x) \text{ even}\}$ and $R_1 \risingdotseq \{x \mid f(x) \text{ odd}\}$. The sets R_0, R_1 are clearly computable and, therefore, there exist computable functions g_i such that $R_i = \text{range}(g_i)$, $i \leq 1$. For $i \leq 1$, define the numbering β_i by $\beta_i(x) \risingdotseq \beta(g_i(x))$ for all x. It is now easy to check that $\beta_0 \leqslant \alpha_0, \beta_1 \leqslant \alpha_1$ and $\beta \equiv \beta_0 \oplus \beta_1$. □

Remark 3.1. Note that even if α_0 and α_1 are both numberings of the same family $\mathcal{A}$ then neither β_0 nor β_1 need be mappings from $\mathbb{N}$ *onto* $\mathcal{A}$.

Theorem 3.1. *For every n and for every finite family $\mathcal{A} \subseteq \Sigma^0_{n+1}$, $\mathcal{R}^0_{n+1}(\mathcal{A})$ is a distributive upper semilattice.*

Proof. Let $\alpha_0, \alpha_1, \beta$, be arbitrary $\Sigma^0_{n+1}-$computable numberings of $\mathcal{A}$ such that $\beta \leqslant \alpha_0 \oplus \alpha_1$. By Proposition 3.1, there exist numberings $\beta_0 \leqslant \alpha_0, \beta_1 \leqslant \alpha_1$ such that $\beta \equiv \beta_0 \oplus \beta_1$. Let $\mathcal{A} = \{A_0, A_1, \ldots, A_k\}$. For $i \leq k$ and for all x define

$$\beta'_i(x) \risingdotseq \begin{cases} A_x & \text{if } x \leq k, \\ \beta_i(x-k-1) & \text{if } x > k. \end{cases}$$

It is easy to verify that $\beta'_0 \leqslant \alpha_0, \beta'_1 \leqslant \alpha_1$ and that $\beta \equiv \beta'_0 \oplus \beta'_1$. On the other hand, each β'_i is a numbering of the whole family $\mathcal{A}$. Therefore, $\mathcal{R}^0_{n+1}(\mathcal{A})$ is a distributive upper semilattice. □

The situation is different if we consider infinite families. First of all, we notice:

Remark 3.2. It is easy to see that the three element upper semilattice $L_0 = \{a, b, c\}$, where a and b are incomparable and $c = a \vee b$, is not distributive. There exist many Rogers semilattices which contain L_0 as an ideal, [5], and, therefore, are not distributive. However, if we add $\perp$ to L_0, we do obtain a distributive lattice.

This remark motivates our next definition.

Definition 3.2. An upper semilattice $\mathfrak{L} = \langle L, \leqslant \rangle$ is *weakly distributive* if $\mathfrak{L}_\perp = \langle L \cup \{\perp\}, \leqslant_\perp \rangle$ is distributive, where $\perp \notin \mathfrak{L}$ and

$$\leqslant_\perp \leftrightharpoons \leqslant \cup \{(\perp, a) \mid a \in L \cup \{\perp\}\}.$$

Proposition 3.2. *An upper semilattice $\langle L, \vee, \leqslant \rangle$ is weakly distributive if and only if for every $a_1, a_2, b \in L$, if $b \leq a_1 \vee a_2$ and $b \nleq a_1, b \nleq a_2$ then there exist $b_1, b_2 \in L$ such that $b_1 \leq a_1, b_2 \leq a_2$ and $b = b_1 \vee b_2$.*

Proof. Immediate. □

Lemma 3.1. *Let $\mathcal{A}$ be an infinite Σ^0_{n+2}–computable family. Then there exists a finite subfamily $\mathcal{A}_0$ of $\mathcal{A}$ and numberings $\alpha, \beta \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ and $\gamma \in \mathrm{Com}^0_{n+2}(\mathcal{A}_0)$ such that*

(1) the ideal $\hat{\alpha}$ has no minimal elements in $\mathcal{R}^0_{n+2}(\mathcal{A})$;

(2) β is a minimal numbering;

(3) $\gamma \nleq \beta$.

Proof. Let $\mathcal{A}$ be an infinite Σ^0_{n+2}–computable family. Then by [12] (see also Corollary 1.5.1, [3]) there exists a numbering $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ such that no minimal numbering of $\mathcal{A}$ is reducible to α. Given α, numberings β and γ with the desired properties can be constructed as follows. Let M be any maximal set, and let A, B, C be three different sets of $\mathcal{A}$. By the construction of Theorem 1.3, [2], consider the numbering $\beta \leftrightharpoons \alpha_{M,A}$. Now, let $\mathcal{A}_0 \leftrightharpoons \{B, C\}$, and let γ be any Σ^0_{n+2}–computable non-solvable numbering of $\mathcal{A}_0$. The existence of such a γ follows for instance from Theorem 1.3, [3]. Recall that a numbering ν is called *solvable* if the predicate $\nu(x) = \nu(y)$ is computable in x, y.

We claim that $\gamma \nleq \alpha_{M,A}$. Indeed, if this were not the case, then there would be a computable function f such that $\gamma = \alpha_{M,A} \circ f$. From the way $\mathcal{A}_0$ and $\alpha_{M,A}$ are defined, we would have $\mathrm{range}(f) \subseteq \overline{M}$; but by maximality of M this implies that $\mathrm{range}(f)$ is finite. Therefore γ would be a decidable numbering, a contradiction. □

Theorem 3.2. *For every n, the Rogers semilattice of any infinite Σ^0_{n+2}–computable family is not weakly distributive.*

Proof. Let $\mathcal{A}$ be an infinite Σ^0_{n+2}–computable family. Let the numberings α, β, γ and let a finite subfamily $\mathcal{A}_0 \leftrightharpoons \{A_0, A_1, \ldots, A_k\}$ of the family $\mathcal{A}$ satisfy properties (1–3) of Lemma 3.1.

Assume for a contradiction that $\mathcal{R}^0_{n+2}(\mathcal{A})$ is weakly distributive. Let ν_0, ν_1 stand for the numberings $\alpha \oplus \gamma$ and β, respectively. Then, the numbering $\mu = \gamma \oplus \beta$ is reducible to $\nu_0 \oplus \nu_1$, and clearly $\mu \nleqslant \nu_1$. On the other hand we can also argue that $\mu \nleqslant \nu_0$, as follows. Assume for a contradiction that $\gamma \oplus \beta \leqslant \alpha \oplus \gamma$; then $\beta \leqslant \alpha \oplus \gamma$ and therefore by Proposition 3.1 there exist $\beta_0 \leqslant \alpha$ and $\beta_1 \leqslant \gamma$ such that $\beta \equiv \beta_0 \oplus \beta_1$. Note that range$(\beta_0) \supseteq \mathcal{A} \setminus \mathcal{A}_0$. We may in fact assume that range$(\beta_0) = \mathcal{A}$, otherwise define

$$\beta_0'(i)(x) \leftrightharpoons \begin{cases} A_x & \text{if } x \leq k, \\ \beta_0(x-k-1) & \text{otherwise.} \end{cases}$$

It is now easy to see that range$(\beta_0') = \mathcal{A}$, $\beta \equiv \beta_0' \oplus \beta_1$ and $\beta_0' \leqslant \alpha$. But $\beta_0 \equiv \beta$ since β is minimal, contradicting that α does not bound any minimal numbering of $\mathrm{Com}^0_{n+2}(\mathcal{A})$.

By the weak distributivity assumption, there exist numberings μ_0, μ_1 of $\mathcal{A}$ such that $\mu_0 \leqslant \nu_0, \mu_1 \leqslant \nu_1$, and $\mu \equiv \mu_0 \oplus \mu_1$. Since β is minimal, it follows that μ_1 is equivalent to β.

It follows from $\mu_0 \leqslant \alpha \oplus \gamma$ that $\mu_0 \equiv \alpha_0 \oplus \gamma_0$ for some $\alpha_0 \leqslant \alpha$ and $\gamma_0 \leqslant \gamma$. Note that range$(\alpha_0) \supseteq \mathcal{A} \setminus \mathcal{A}_0$. As for β_0, we may suppose that range$(\alpha_0) = \mathcal{A}$.

On the other hand, $\mu_0 \leqslant \gamma \oplus \beta$ and, hence, $\alpha_0 \leqslant \gamma \oplus \beta$. Again, by Proposition 3.1 there exist numberings α_1, γ_1 (where, again, we may suppose that α_1 is a mapping of $\mathbb{N}$ onto $\mathcal{A}$) such that $\gamma_1 \leqslant \gamma, \alpha_1 \leqslant \beta$, and $\alpha_0 \equiv \gamma_1 \oplus \alpha_1$.

So the numbering α_1 of $\mathcal{A}$ is reducible to the minimal numbering β. Therefore, $\alpha_1 \equiv \beta$. It implies $\alpha_0 \equiv \gamma_1 \oplus \beta$ and, hence, $\gamma_1 \oplus \beta \leqslant \alpha$. Therefore, the minimal numbering β is reducible to α, and this again is in contradiction with the choice of α. The theorem is proved. □

Modulo the existence of suitable α, β and γ the argument above holds also for Σ^0_1–computable families.

Corollary 3.2.1. *Let $\mathcal{A}$ be a family of c.e. sets such that the Rogers semilattice $\mathcal{R}^0_1(\mathcal{A})$ has at least one minimal element* deg(β)*, contains a principal ideal $\hat{\alpha}$ without minimal elements, and there is a numbering γ of a finite subfamily such that $\gamma \nleqslant \beta$. Then $\mathcal{R}^0_1(\mathcal{A})$ is not weakly distribu-*

tive. In particular, the Rogers semilattice of the computable numberings of the family of all c.e. sets is not weakly distributive.

The hypotheses of the previous corollary are not trivial. We recall in fact that for every computable infinite family of computable functions, every ideal of the corresponding Rogers semilattice contains at least one minimal element, [5]. On the other hand, there exist families of c.e. sets $\mathcal{A}$ such that $\mathcal{R}^0_1(\mathcal{A})$ has no minimal elements, [1] and [15] and there exist also infinite families of c.e. sets with trivial Rogers semilattice. In view of these well known facts of the classical theory of numberings, the following question seems to be of some interest.

Question 1. *Does there exist a computable infinite family $\mathcal{A}$ of c.e. sets such that $\mathcal{R}^0_1(\mathcal{A})$ is non-trivial and distributive? Does there exist a computable infinite family $\mathcal{A}$ of c.e. sets such that $\mathcal{R}^0_1(\mathcal{A})$ is non-trivial and weakly distributive?*

4. Distinguishing Rogers semilattices of infinite families

Having seen that the Rogers semilattice $\mathcal{R}^0_{n+2}(\mathcal{A})$ is always elementarily different from $\mathcal{R}^0_{n+2}(\mathcal{B})$ when $\mathcal{A}$ is finite, and $\mathcal{B}$ is infinite, what about elementary differences between Rogers semilattices of distinct infinite families?

Theorem 4.1. *For every n there exist Σ^0_{n+1}–computable infinite families $\mathcal{A}$ and $\mathcal{B}$ such that their Rogers semilattices $\mathcal{R}^0_{n+1}(\mathcal{A})$ and $\mathcal{R}^0_{n+1}(\mathcal{B})$ are not elementary equivalent.*

Proof. It is immediate to see that for every n there are infinite Σ^0_{n+1}–computable families with universal numberings. It suffices to take for every n the family of all Σ^0_{n+1} sets. Then the standard numbering $\rho(e) = W_e^{\mathbf{0}^{(n)}}$ is universal in $\mathrm{Com}^0_{n+1}(\Sigma^0_{n+1})$.

On the other hand, there are infinite Σ^0_{n+1}–computable families without universal numberings. This fact follows easily from straightforward relativizations of known result of S.Marchenkov [11] that every non-trivial Rogers semilattice of a family of total computable functions has no greatest element. □

In Theorem 4.1, instead of the quoted result by Marchenkov, one could use also the following fact of the theory of numberings which is an immediate consequence of the Lachlan's paper [8]: The family $\mathcal{F}$ of all finite sets has no computable principal numbering.

This family allows also to exhibit Rogers semilattices which are not elementarily equivalent with respect to $\leqslant_{\mathbf{0}^{(i)}}$, for every $i \leq n$.

Theorem 4.2. *For every n there exist $\Sigma^0_{n+1}-$computable infinite families $\mathcal{A}$ and $\mathcal{B}$ such that their Rogers semilattices $\mathcal{R}^{0,\mathbf{0}^{(i)}}_{n+1}(\mathcal{A})$ and $\mathcal{R}^{0,\mathbf{0}^{(i)}}_{n+1}(\mathcal{B})$ are not elementary equivalent for every $i \leq n$.*

Proof. Fix any $i \leq n$. Let $\mathcal{F} \leftrightharpoons \{F \mid F \text{ is a finite set}\}$. It is immediate to see that $\mathcal{F}$ is $\Sigma^0_{n+1}-$computable for every n. Assume that $\alpha \in \mathrm{Com}^0_{n+1}(\mathcal{F})$. We will show that there exists $\beta \in \mathrm{Com}^0_{n+1}(\mathcal{F})$ such that $\beta \nleqslant_{\mathbf{0}^{(i)}} \alpha$. We begin with building a numbering γ such that $\mathrm{range}(\gamma) \subseteq \mathcal{F}$ and $\gamma \nleqslant_{\mathbf{0}^{(i)}} \alpha$, by stages. For every e, the requirements for building γ are

$$R_e : \varphi_e^{\mathbf{0}^{(i)}} \text{ total } \Rightarrow (\exists x)(\gamma(x) \neq \alpha(\varphi_e^{\mathbf{0}^{(i)}}(x))).$$

We will work with a given $\mathbf{0}^{(n)}-$computable approximation $\{\alpha^s(x)\}_{x,s\in\mathbb{N}}$ to the numbering α.

If at stage $s+1$ we do not explicitly define $\gamma^{s+1}(x)$ for some x then $\gamma^{s+1}(x) = \gamma^s(x)$.

Stage 0) For all e define $\gamma^0(e) \leftrightharpoons \emptyset$.

Stage $s+1$) Consider all $e \leq s$ such that

- $\varphi^{\mathbf{0}^{(i)}}_{e,s}(e)\downarrow$ (say $\varphi^{\mathbf{0}^{(i)}}_{e,s}(e) = y$);
- $\gamma^s(e)) \subseteq \alpha^s(y)$.

For each such e, choose $z \notin \alpha^s(y)$, and define $\gamma^{s+1}(e) \leftrightharpoons \gamma^s(e) \cup \{z\}$.

Finally let $\gamma(x) \leftrightharpoons \bigcup_s \gamma^s(x)$. It is now easy to see that the numbering $\beta \leftrightharpoons \alpha \oplus \gamma$ has the desired properties. First of all β is $\Sigma^0_{n+1}-$computable, since γ is $\Sigma^0_{n+1}-$computable by construction; $\mathrm{range}(\beta) = \mathcal{F}$ since $\alpha \leqslant \beta$; and finally $\beta \nleqslant_{\mathbf{0}^{(i)}} \alpha$ since $\gamma \nleqslant_{\mathbf{0}^{(i)}} \alpha$.

Thus, as suitable families $\mathcal{A}$ and $\mathcal{B}$, we could consider the family $\mathcal{F}$ and the family Σ^0_{n+1} of all $\Sigma^0_{n+1}-$sets, respectively. $\square$

Remark 4.1. The Rogers semilattice $\mathcal{R}^{0,\mathbf{0}^{(i)}}_{n+1}(\mathcal{A})$ of any finite family $\mathcal{A} \subseteq \Sigma^0_{n+1}$ consists of a single element for every $i > n$ and, therefore, all such semilattices are elementary equivalent.

5. Isomorphism types

Finally we show that the isomorphism type of Rogers semilattice $\mathcal{R}^0_{n+1}(\mathcal{A})$ of an arbitrary $\Sigma^0_{n+1}-$computable family $\mathcal{A}$ may be very different from that of $\mathcal{R}^0_{m+1}(\mathcal{B})$ of some $\Sigma^0_{m+1}-$computable family $\mathcal{B}$, for distinct n and m.

The following three lemmas and the notion of an $\mathbf{X}$–computable Boolean algebra play a key role in establishing this claim. Recall (see [7]) that a Boolean algebra $\mathfrak{A}$ is called $\mathbf{X}$–*computable* if its universe, operations, and relation are $\mathbf{X}$–computable.

Lemma 5.1. *Let γ and δ be Σ^0_{n+1}–computable numberings of a family $\mathcal{A}$. If $[\gamma, \delta]$ is a Boolean algebra, then it is $\mathbf{0}^{(n+3)}$–computable.*

Proof. Given n, $\mathcal{A}$, γ, and δ as in the hypothesis of the lemma, we first observe that by (4) and (5) of Lemma 2.2, [3], there exists a c.e. set C such that $\gamma \equiv \delta_C$ and

$$[\gamma, \delta] = \{\deg(\delta_X) \mid X \text{ c.e. and } X \supseteq C\}.$$

For every i, let $U_i \leftrightharpoons C \cup W_i$. This gives an effective listing of all c.e. supersets of C. By Lemma 2.2, [3], (1b), for every i, j, we have $\delta_{U_i} \leqslant \delta_{U_j}$ if and only if

$$\exists p[\forall x(x \in U_i \Rightarrow \exists y(\varphi_p(x) = y \;\&\; y \in U_j)) \\ \& \;\forall x \forall y(x \in U_i \;\&\; \varphi_p(x) = y \Rightarrow \delta(x) = \delta(y))]$$

Since $\delta \in \mathrm{Com}^0_{n+1}(\mathcal{A})$, this implies that $\delta_{U_i} \leqslant \delta_{U_j}$ is a Σ^0_{n+3}–relation in i, j.

Let us consider the equivalence η on $\mathbb{N}$ defined by

$$(i, j) \in \eta \Leftrightarrow \delta_{U_i} \leqslant \delta_{U_j} \;\&\; \delta_{U_j} \leqslant \delta_{U_i}.$$

Let $B \leftrightharpoons \{x \mid \forall y(y < x \Rightarrow (x, y) \notin \eta)\}$. Define a bijection $\psi_1 : B \longrightarrow [\gamma, \delta]$, by letting $\psi_1(i) = \deg(\delta_{U_i})$, for all $i \in B$. It is evident that ψ_1 induces in $\mathcal{R}^0_{n+1}(\mathcal{A})$ a partially ordered set $\mathfrak{B}$ which is a Boolean algebra isomorphic to $[\gamma, \delta]$. The interval $\mathfrak{B}$ is a $\mathbf{0}^{(n+3)}$–computable partial ordered set. It follows from [7, Theorem 3.3.4] and [4], that the Boolean algebra $\mathfrak{B}$ relatively to the Boolean operations is $\mathbf{0}^{(n+3)}$–computable too. □

Lemma 5.2 (L. Feiner). *Let $\mathfrak{F}$ be a computable atomless Boolean algebra. Then for every $\mathbf{X}$ there is an ideal J such that J is $\mathbf{X}$–c.e. and the quotient $\mathfrak{F}/J$ is not isomorphic to any $\mathbf{X}$–computable Boolean algebra.*

Proof. See [6]. □

Below, we will use the following notations. For a given c.e. set A, $\{V_i \mid i \in \mathbb{N}\}$ denotes an effective listing of all c.e. supersets of the set A defined, for instance, by $V_i \leftrightharpoons A \cup W_i$, for all i. We will assume for convenience that $V_0 = A$. As in [3], ε_A stands for the distributive lattice of the c.e. supersets of A. For a given c.e. set $V \supseteq A$, let V^* denote the image of V under the canonical homomorphism of ε_A onto ε_A^* (i.e. ε_A modulo the finite sets), and let $\subseteq^*$ denote the partial ordering relation of ε_A^*. Obviously, if J is an ideal in ε_A then $J^* \leftrightharpoons \{V^* \mid V \in J\}$ is an ideal in ε_A^*.

As is known (see, for instance, [7]), if $\mathfrak{A}$ is a Boolean algebra and J is an ideal of $\mathfrak{A}$, then the universe of the quotient Boolean algebra $\mathfrak{A}/J$ is given by the set of equivalence classes $\{[a]_J \mid a \in \mathfrak{A}\}$ under the equivalence relation $\equiv_J$ given by

$$a \equiv_J b \Leftrightarrow \exists c_1, c_2 \in J(a \vee c_1 = b \vee c_2),$$

and the partial ordering relation is given by

$$[a]_J \leq_J [b]_J \Leftrightarrow a - b \in J.$$

where $a - b$ stands for $a \wedge \neg b$.

Lemma 5.3. *Let $\mathcal{B}$ be a Σ^0_{m+1}–computable family, $\beta \in \mathrm{Com}^0_{m+1}(\mathcal{B})$, and let A be any c.e. set such that ε_A^* is a Boolean algebra. Let $\psi_2 : \varepsilon_A \longrightarrow [\beta_A, \beta]$ be the mapping given by $\psi_2(V_i) = \deg(\beta_{V_i})$ for all i, and let I be any ideal of ε_A. Then ψ_2 induces an isomorphism of ε_A^*/I^* onto $[\beta_A, \beta]$ if and only if for every i, j*

(1) $V_i \in I \Rightarrow \beta_{V_i} \leqslant \beta_A$;

(2) $V_i - V_j \notin I \Rightarrow \beta_{V_i} \nleqslant \beta_{V_j}$ (where $V_i - V_j \leftrightharpoons (V_i \setminus V_j) \cup A$.)

Proof. Let $A, \mathcal{B}, \beta, \psi_2$ be given. The "only if" direction is immediate. As to show that the conditions stated in the lemma are also sufficient, we can argue as follows. By (4–5) of Lemma 2.2, [3], we have that every γ with $\beta_A \leqslant \gamma \leqslant \beta$ is of the form $\gamma \equiv \beta_C$ for some c.e. set $C \supseteq A$. Then the mapping induced by ψ_2 is clearly onto.

Suppose now that $[V_i^*]_{I^*} \subseteq^*_{I^*} [V_j^*]_{I^*}$. Then $V_i^* - V_j^* \in I^*$. But $V_i^* - V_j^* = (V_i - V_j)^*$, with $V_i - V_j$ a c.e. superset of A, since ε_A^* is a Boolean algebra. Then $V_i - V_j \in I$. On the other hand,

$$V_i = (V_i - V_j) \cup (V_i \cap V_j).$$

Now, by (1), $\beta_{V_i - V_j} \leqslant \beta_A$, so by (3) of Lemma 2.2, [3], $\beta_{V_i} \equiv \beta_{V_i \cap V_j}$, hence $\beta_{V_i} \leqslant \beta_{V_j}$ by (2) of Lemma 2.2, [3], as $V_i \cap V_j \subseteq V_j$.

Finally, if $[V_i^*]_{I^*} \not\subseteq^*_{I^*} [V_j^*]_{I^*}$ then $V_i - V_j \notin I$, and therefore by (2) $\beta_{V_i} \nleqslant \beta_{V_j}$. □

Theorem 5.1. *For every n there exist $m \geq n$ and a Σ^0_{m+2}–computable family $\mathcal{B}$ such that no Rogers semilattice $\mathcal{R}^0_{n+1}(\mathcal{A})$ of any Σ^0_{n+1}–computable family $\mathcal{A}$ is isomorphic to $\mathcal{R}^0_{m+2}(\mathcal{B})$.*

Proof. Let n be given. By Lemma 5.1, all Boolean intervals of $\mathcal{R}^0_{n+1}(\mathcal{A})$ are $\mathbf{0}^{(n+3)}$– computable Boolean algebras. Therefore, to show the theorem it is sufficient:

(1) to find a suitable number $m \geq n+3$;

(2) to consider a computable atomless Boolean algebra $\mathfrak{F}$ and an ideal J of $\mathfrak{F}$ as in Feiner's Lemma such that J is c.e. in $\mathbf{0}^{(m)}$ and $\mathfrak{F}/J$ is not isomorphic to any $\mathbf{0}^{(m)}$– computable Boolean algebra,

(3) to find a Σ^0_{m+2}– computable family $\mathcal{B}$ and Σ^0_{m+2}– computable numberings α and β of $\mathcal{B}$ such that the interval $[\alpha, \beta]$ of $\mathcal{R}^0_{m+2}(\mathcal{B})$ is a Boolean algebra isomorphic to $\mathfrak{F}/J$.

We will determine the number m later. First, we consider item (2) above. Let $\mathfrak{F}$ be a computable atomless Boolean algebra. According to a famous result of Lachlan, [9], there exists a hyperhypersimple set A such that ε^*_A is isomorphic to $\mathfrak{F}$. We fix such a set A.

We refer to the textbook of Soare, [14], for the details of a suitable isomorphism χ of ε^*_A onto $\mathfrak{F}$. We only notice that starting from a computable listing $\{b_0, b_1, \ldots\}$ of the elements of $\mathfrak{F}$ one can find a listing $\{B_0, B_1, \ldots\}$ of a subfamily of the family ε_A such that $\varepsilon^*_A = \{B^*_0, B^*_1, \ldots\}$, $\chi(B^*_i) = b_i$, and the relation "$x \in B_i$" is Σ^0_3.

We will use the techniques for embedding posets into intervals of Rogers semilattices which have been developed in [3]. Let J be any $\mathbf{0}^{(m)}$ - c.e. ideal of $\mathfrak{F}$. Then $I^* \leftrightharpoons \chi^{-1}[J]$ is an ideal of ε^*_A and $\mathfrak{F}/J$ is isomorphic to ε^*_A/I^*. So, instead of the Boolean algebra $\mathfrak{F}/J$ we can just embed ε^*_A/I^*.

Let now $I \leftrightharpoons \{V_i \mid V^*_i \in I^*\}$. Obviously, I is an ideal of ε_A, and

$$I = \{V_j \mid \exists i (V_j =^* B_i \ \& \ b_i \in J)\}.$$

A simple calculation shows that the relation "$V_j =^* B_i$", in i, j, is Σ^0_5.

Now, if we take $m \leftrightharpoons \max\{4, n+3\}$ then I is Σ^0_{m+1}, as J is $\mathbf{0}^{(m)}$ - c.e. Finally, we choose J to be an ideal of $\mathfrak{F}$ satisfying the conclusions of Lemma 5.2, and we let $I^* \leftrightharpoons \chi^{-1}[J]$ as above.

It should be mentioned a very useful property of the Boolean algebra ε^*_A which we will use in our construction of $\mathcal{B}$: if $V_i \notin I$ then $V_i - V \notin I$ for every $V \in I$ and, in particular, $V_i \setminus V$ is an infinite set. This property can be easily verified by contradiction using the following equality:

$$V_i = (V_i - V) \cup (V_i \cap V).$$

We will now construct a Σ^0_{m+2}– computable family $\mathcal{B}$, and a numbering $\beta \in \mathrm{Com}^0_{m+2}(\mathcal{B})$ such that the interval $[\beta_A, \beta]$ is isomorphic to ε^*_A/I^*.

The requirements. First of all, we will construct $\mathcal{B}$ and β so that $\beta[A] = \mathcal{B}$ to guarantee that β_A is a numbering of the whole family $\mathcal{B}$. Then in view of Lemma 5.3 we must satisfy, for every p, i, j, the requirements:

$$P_i : V_i \in I \Rightarrow \beta_{V_i} \leqslant \beta_A$$
$$R_{i,j,p} : V_i - V_j \notin I \Rightarrow \beta_{V_i} \nleqslant \beta_{V_j} \text{ via } \varphi_p$$

where by "$\beta_{V_i} \nleqslant \beta_{V_j}$ via φ_p" we mean that φ_p does not reduce β_{V_i} to β_{V_j} in the sense of Lemma 2.2(1b), [3].

The construction. We use the oracle $\mathbf{0}^{(m+1)}$ in our construction to answer questions such as "$V_i \in I$?" and to verify some properties of c.e. sets and functions. We fix an infinite computable subset R of the set A and a computable partition of R into disjoint infinite computable sets R_i, $i \in \mathbb{N}$.

Initially we define an auxiliary $\Sigma^0_{m+2}-$ computable numbering β_1 of $\mathcal{B}$ and an auxiliary $\mathbf{0}^{(m+1)}-$ computable function $r(i,t)$. If at the end of any stage $t+1$, $\beta_1^{t+1}(x)$ or $r(x, t+1)$ have not been explicitly modified then they are understood to retain the same value as in the previous stage.

<u>Stage 0</u>) Let $\beta_1^0(x) = \emptyset$, $r(i, 0) = 0$ for all x, i.

<u>Stage $t+1$</u>) Let $t = \langle i, j, p\rangle$. Find k such that $V_k = V_i - V_j$ (we can do it with oracle). Check the following conditions:

(i) $V_k \notin I$;

(ii) $V_i \subseteq \operatorname{dom}(\varphi_p)$ and $\varphi_p[V_i] \subseteq V_j$.

If one of (i) or (ii) fails then do nothing. Otherwise let $U_t \leftrightharpoons \bigcup\{V_s : s \leq t \;\&\; V_s \in I\}$. Notice that $A \subseteq U_t$ since we have chosen $V_0 = A$. Choose the least element x in the set

$$\{y \mid V_i \setminus U_t \;\&\; \varphi_p(y) \neq y\} \setminus \{y \mid \beta_1^t(y) \neq \emptyset\}.$$

(See (3) below for the existence of such an x). Take a new number a and define $\beta_1^{t+1}(x) = \beta_1^t(x) \cup \{a\}$. If $\varphi_p(x) \in R_m$ and $m > t$ then define $r(m, t+1) = \max\{r(m,t), \varphi_p(x)\}$. Go to the next stage.

Obviously, β_1 is a $\Sigma^0_{m+2}-$computable numbering of the family $\mathcal{B} \leftrightharpoons \{\beta_1(x) \mid x \in \mathbb{N}\}$.

Properties of the construction. The construction satisfies the following properties:

(1) For every $x \in R$, $\beta_1(x) = \emptyset$. Each non-empty set of $\mathcal{B}$ has exactly one index relative to β_1.

(2) For every m, i, if $i \geq m$ then $r(m, i) = r(m, m+1)$.

Properties (1),(2) are evident.

(3) For every i, j, k, p, t, if $V_k = V_i - V_j$ and conditions (i),(ii) above hold then the set $\{x \mid V_i \setminus U_t \ \&\ \varphi_p(x) \neq x\}$ is infinite.

Indeed, $V_i \setminus V_j$ is infinite since $V_k = (V_i \setminus V_j) \cup A$ and $V_k \notin I$ by (i). Condition (ii) implies that $\varphi_p(x) \neq x$ for all $V_i \setminus V_j$.

So, if we assume that the set $\{x \mid V_i \setminus U_t \ \&\ \varphi_p(x) \neq x\}$ is finite then we obtain that $V_i \setminus V_j \subseteq^* U_t$ and, hence, $V_k \subseteq^* U_t$. This is a contradiction with (i) since $U_t \in I$.

Let us now define a numbering $\beta \in \mathrm{Com}^0_{m+2}(\mathcal{B})$. Let $\beta(x) \leftrightharpoons \beta_1(x)$ for all $x \in \overline{R}$. For every $i \in \mathbb{N}$, let ψ_i be a partial computable one-to-one function from $\{x \in R_i : x > r(i, i+1)\}$ onto $V_i \setminus R$. For every i and every $x \in R_i$, let

$$\beta(x) \leftrightharpoons \begin{cases} \beta_1(\psi_i(x)) & \text{if } x > r(i, i+1) \text{ and } V_i \in I\ , \\ \emptyset & \text{otherwise.} \end{cases}$$

For every i if $V_i \in I$ then $\beta_{V_i} \leqslant \beta_R$ via the partial computable function φ such that for every $x \in V_i$

$$\varphi(x) \leftrightharpoons \begin{cases} \psi_i^{-1}(x) & \text{if } x \in V_i \setminus R, \\ x & \text{if } x \in R. \end{cases}$$

We are using again (1b) of Lemma 2.2, [3], so all requirements P_i are satisfied.

We have $\beta[A] = \beta[\mathbb{N}]$ since for every $x \notin A$, one can find i_0 such that $V_{i_0} = A \cup \{x\} \in I$. The requirement P_{i_0} is satisfied, therefore, $\beta_{V_{i_0}} \leqslant \beta_A$, and, in particular, $\beta(x) \in \beta[A]$.

Finally, let us check the requirements $R_{i,j,p}$. Let $i, j \in \mathbb{N}$ and $V_i - V_j \notin I$: we want to show that $\beta_{V_i} \not\leqslant \beta_{V_j}$ via φ_p. Let $t \leftrightharpoons \langle i, j, p\rangle$. Consider the number x chosen at stage $t+1$. Notice that $x \notin A$, $\varphi_p(x) \neq x$ and $\beta(x) = \beta_1(x) \neq \emptyset$.

If $\varphi_p(x) \notin R$ then $\beta(x) \neq \beta(\varphi_p(x))$ by property (1) since β and β_1 coincide on $\overline{R}$.

Let now $\varphi_p(x)$ be in R_m for some m. If $V_m \notin I$, or $V_m \in I$ but $\varphi_p(x) \leq r(m, m+1)$, then $\beta(\varphi_p(x)) = \emptyset$, and, hence, $\beta(x) \neq \beta(\varphi_p(x))$.

If $m > t$ then by construction $\varphi_p(x) \leq r(m, m+1)$ and, again we have $\beta(x) \neq \beta(\varphi_p(x))$.

It remains to consider the case when $m \leq t$, $V_m \in I$, and $\varphi_p(x) > r(m, m+1)$. In this case we have $\varphi_p(x) \in \mathrm{dom}(\psi_m)$ and, by construction of β, $\beta(\varphi_p(x)) = \beta_1(\psi_m(\varphi_p(x)))$. Since range$(\psi_m) \subseteq V_m$, $x \in V_i \setminus U_t$, and $V_m \subseteq U_t$, it follows that $x \neq \psi_m(\varphi_p(x))$. Now (1) implies inequality $\beta(x) \neq \beta(\varphi_p(x))$. □

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