ALGEBRAIC PROPERTIES OF ROGERS SEMILATTICES OF ARITHMETICAL NUMBERINGS *

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Abstract We investigate initial segments and intervals of Rogers semilattices of arithmetical families. We prove that there exist intervals with different algebraic properties; the elementary theory of any Rogers semilattice at arithmetical level $n \geq 2$ is hereditarily undecidable; the class of all Rogers semilattices of a fixed level $n \geq 2$ has an incomplete theory.

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One of the main tasks of the theory of numberings is the study and the characterization of algebraic and elementary properties of the Rogers semilattices of the families of numberings under investigation. The study of Rogers semilattices was started by H. Rogers, [20]. In the theory of numberings, see [9], the notion of a Rogers semilattice plays a fundamental role. Rogers semilattices are used to classify properties of computable numberings for different families.

For the basic notions and notations relative to arithmetical numberings the reader is referred to [4]. For unexplained terminology and notations relative to computability theory, our main references are the textbooks of A.I. Mal'tsev [18], H. Rogers [21] and R. Soare [23]. For the main concepts and notions of the theory of numberings we refer to the book of Yu.L. Ershov [9].

This paper intends to continue the program of research initiated by S. Badaev and S. Goncharov in [2]. In particular we address some of the possible questions on how complicated the Rogers semilattice $\mathcal{R}^0_{n+1}(A)$, of a given $\Sigma^0_{n+1}-$computable family, can be.

1. The cardinality of Rogers semilattices

The question of the cardinality of the Rogers semilattices of Σ^0_1-families was settled by Khutoretsky, [14].

Theorem 1.1 (A. Khutoretsky). *Let $\mathcal{A}$ be a Σ^0_1-computable family. If the Rogers semilattice $\mathcal{R}^0_1(\mathcal{A})$ contains at least two distinct elements then it is infinite.*

Proof. See [14]. □

Indeed, Khutoretsky shows that it is possible to embed a linear ordering of type ω into $\mathcal{R}^0_1(\mathcal{A})$ above any non-greatest element of $\mathcal{R}^0_1(\mathcal{A})$. Subsequently Badaev, [1], was able to prove that in every nontrivial Rogers semilattice $\mathcal{R}^0_1(\mathcal{A})$, for every non-greatest element one can embed a chain with no endpoints containing that element, and for every non-minimal element one can embed a chain with no endpoints containing that element.

The following result, [22], shows that $\mathcal{R}^0_1(\mathcal{A})$ is never a lattice, if $\mathcal{R}^0_1(\mathcal{A})$ is not trivial.

Theorem 1.2 (V. Selivanov). *Let $\mathcal{A}$ be a Σ^0_1-computable family of c.e. sets. If $\mathcal{R}^0_1(\mathcal{A})$ contains at least two elements then it is not a lattice.*

Proof. See [22]. □

For Σ^0_1– families, both Theorem 1.1 and Theorem 1.2 answer questions posed by Ershov, [6]. The corresponding questions for Σ^0_{n+2}– computable families have been answered by Goncharov and Sorbi, [12].

Theorem 1.3 (S. Goncharov, A. Sorbi). *If a Σ^0_{n+2}– computable family $\mathcal{A}$ contains at least two elements then the Rogers semilattice $\mathcal{R}^0_{n+2}(\mathcal{A})$ is infinite and is not a lattice.*

Proof. See [12]. The proof consists of two cases, distinguishing whether we deal with a finite family or an infinite Σ^0_{n+2}– computable family $\mathcal{A}$. We notice that if $\mathcal{A}$ is infinite, then the claim follows also from Badaev and Goncharov's result [3] and from Theorem 1.3, [4], which reads that for every infinite Σ^0_{n+2}– computable family $\mathcal{A}$ the corresponding Rogers semilattice $\mathcal{R}^0_{n+2}(\mathcal{A})$ has infinitely many minimal elements. Thus $\mathcal{R}^0_{n+2}(\mathcal{A})$ is obviously infinite and not a lattice since no pair of distinct minimal elements can have greatest lower bound.

The original proof of this claim given in [12] is however quite different. We wish to sketch this proof here since it employs a method for constructing numberings with some desired property which promises to have applications also in different contexts.

So, let $\mathcal{A}$ be a Σ^0_{n+2}– computable infinite family. Let $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$, and let a, b be such that $\alpha(a) \neq \alpha(b)$. We will construct a family $\{\alpha_i\}_{i \in \mathbb{N}}$ of Σ^0_{n+2}– computable numberings of $\mathcal{A}$, such that for no $\beta \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ do we have $\beta \leqslant \alpha_i, \alpha_j$ if $i \neq j$. Notice that this implies that $\mathcal{R}^0_{n+2}(\mathcal{A})$ is infinite and not a lattice.

By Lemma 1.1, [4], let $\{B_{k,t}\}_{k,t \in \mathbb{N}}$ be a $\mathbf{0}^{(n)}$– computable sequence of finite sets (given according to their canonical indices) such that for every k, $\alpha(k) = \underline{\lim}\,_t B_{k,t}$. In constructing the numberings α_i, $i \in \mathbb{N}$, first of all we will ensure that each α_i is a numbering of a subfamily of $\mathcal{A}$ by arranging that for every x there exists k such that $\alpha_i(x) = \underline{\lim}\,_t B_{k,t}$. Then we aim to satisfy the following requirements, for every i, j, k, l, y, with $i \neq j$:

$$P_{i,j,k,l} : \varphi_k, \varphi_l \text{ total } \Rightarrow (\exists x)(\alpha_i(\varphi_k(x)) \neq \alpha_j(\varphi_l(x))$$
$$Q_{i,y} : (\exists x)(\alpha(y) = \alpha_i(x)).$$

We observe that satisfaction of all $Q_{i,y}$'s ensures that each α_i is a numbering of the whole family $\mathcal{A}$.

In the course of the construction below, at step t, for every i, k we define a finite set $D^i_{k,t}$ so that eventually

$$\alpha_i(k) \leftrightharpoons \underline{\lim}\,_t D^i_{k,t}.$$

We will arrange that the sequence $\{D^i_{k,t}\}_{k,t\in\mathbb{N}}$ is $\mathbf{0}^{(n)}$–computable. Then by Lemma 1.1, [4], each α_i is Σ^0_{n+2}–computable.

The construction is by stages. Suppose that $D^i_{k,t} = B_{x,t}$, and at stage $t+1$, we define $D^i_{k,t+1} = B_{x',t+1}$ for $x' \neq x$. Then we say that at $t+1$ we *redefine* D^i_k; if we do not redefine D^i_k at stage $t+1$ then it is understood that $D^i_{k,t+1} = B_{x,t+1}$.

We arrange the requirements in an effective priority listing

$$R_0 < R_1 < \dots,$$

in which each $P_{i,j,k,l}$, $i \neq j$, and each $Q_{i,y}$ appears exactly once.

At stage t we define also the value of a restraint function $r(h,t)$ for every requirement R_h, and we declare some requirements temporarily satisfied or temporarily unsatisfied.

Stage 0) Define $D^i_{k,0} \leftrightharpoons B_{k,0}$, for every $i,k \in \mathbb{N}$. Let $r(h,0) \leftrightharpoons -1$, for every $h \in \mathbb{N}$. All requirements are *temporarily unsatisfied.*

Stage $t+1$) We say that a requirement R_h *requires attention at* $t+1$, if R_h is temporarily unsatisfied at the end of stage t, and there is a number x such that $\varphi_{k,t}(x), \varphi_{l,t}(x) > r(h,t)$ if $R_h = P_{i,j,k,l}$.

Choose the least h such that R_h requires attention.

- if $R_h = P_{i,j,k,l}$, then choose some such x and define

$$D^i_{\varphi_k(x),t+1} \leftrightharpoons B_{a,t+1}$$
$$D^j_{\varphi_l(x),t+1} \leftrightharpoons B_{b,t+1}.$$

  (Notice that if we never redefine $D^i_{\varphi_k(x)}$ and $D^j_{\varphi_l(x)}$ at any later stage then

$$\alpha_i(\varphi_k(x)) = \underline{\lim}\,_t D^i_{\varphi_k(x),t} = \underline{\lim}\,_t B_{a,t} = \alpha(a),$$
$$\alpha_j(\varphi_l(x)) = \underline{\lim}\,_t D^j_{\varphi_l(x),t} = \underline{\lim}\,_t B_{b,t} = \alpha(b),$$

  hence $\alpha_i(\varphi_k(x)) \neq \alpha_j(\varphi_l(x))$.) Set

$$r(h',t+1) = \max\{\varphi_k(x), \varphi_l(x)\}$$

  for all $h' > h$.

- If $R_h = Q_{i,y}$ then choose $x = r(h,t)+1$ and define

$$D^i_{x,t+1} \leftrightharpoons B_{y,t+1}.$$

  Set $r(h',t+1) = x$, for all $h' > h$.

Whatever the case, declare R_h *temporarily satisfied* at the end of stage $t+1$, declare $R_{h'}$ *temporarily unsatisfied* at the end of stage $t+1$, for all $h' > h$.

This ends the construction. Notice that $r(h,t)$ and the function assigning to each triple i,k,t a number x such that $D^i_{k,t} = B_{x,t}$ are in fact computable, so $\{D^i_{k,t}\}_{k,t\in\mathbb{N}}$ is $\mathbf{0}^{(n)}-$computable. For the verification, a standard inductive argument shows that each requirement requires attention only finitely many times, and thus for every i,k we redefine D^i_k only finitely often. We pause only to point out where the assumption that $\mathcal{A}$ is infinite is needed in this argument. Consider $R_h = P_{i,j,k,l}$, and suppose that t_0 is the least stage such that no $R_{h'}$ with $h' < h$ requires attention at any $t \geq t_0$. Thus $r(h) = \lim_t r(h,t)$ exists, being $r(h) = r(h,t_0)$, and, starting from stage t_0, R_h is temporarily unsatisfied until it eventually requires attention. Now, if $\beta \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ is such that $\beta = \alpha_i \circ \varphi_k$ and $\beta = \alpha_j \circ \varphi_l$ (with φ_k, φ_l total) then since $\mathcal{A}$ is infinite we have that both φ_k and φ_l have infinite ranges, so that eventually we find x such that $\varphi_k(x), \varphi_l(x) > r(h)$. At this point R_h requires attention (for the last time), we act accordingly on R_h, $D^i_{\varphi_k(x)}$ and $D^j_{\varphi_l(x)}$ are never again redefined, and thus, as already remarked, our action ensures that $\beta \neq \alpha_i \circ \varphi_k$ or $\beta \neq \alpha_j \circ \varphi_l$, and therefore the requirement R_h is eventually met. (Notice that a requirement of the form $P_{i,j,k,l}$ can be met even if it is eventually always temporarily unsatisfied, for instance if φ_k and φ_l are not both total.)

Remark 1.1. Notice that the relation in i,x,y "$x \in \alpha_i(y)$" is itself Σ^0_{n+2}.

Let us consider the case in which $\mathcal{A}$ is finite, say $\mathcal{A} = \{A_0, \dots, A_k\}$. If C is a proper c.e. subset of $\mathbb{N}$ then define

$$\alpha^C(x) \leftrightharpoons \begin{cases} A_{x+2} & \text{if } x < k-1, \\ A_0 & \text{if } x \geq k-1 \text{ and } x+1-k \in C \\ A_1 & \text{if } x \geq k-1 \text{ and } x+1-k \notin C. \end{cases}$$

If now $\beta = \alpha^C \circ f$, where f is computable, then one easily shows that $\beta \equiv \alpha^D$, where $D = f^{-1}[C]$. Hence every principal ideal of the c.e. $m-$degrees is isomorphic to an ideal of $R^0_{n+2}(\mathcal{A})$. The result then follows in this case from the fact that the semilattice of the c.e. $m-$degrees is not a lattice, and that it has infinite ideals, see [7] and [17]. □

Apparently, classical computability may seem very close to Σ^{-1}_n-computability (the classes Σ^{-1}_n refer to the finite levels of the Ershov hierarchy of the Δ^0_2 sets), since classical computability is definable by means of Σ-formulas of first order arithmetic, while Σ^{-1}_n–computability is definable by means of Boolean combinations of these formulas.

Unfortunately, the methods employed in Theorems 3.1 – 3.3 do not seem to apply to the case of Σ^{-1}_n–computability. In this regard, both Ershov's questions – on cardinality of Rogers semilattices and on whether or not a Rogers semilattice is always a lattice – are still open. It looks appropriate to raise these questions here (see also [2]). Consider computable numberings of families of Σ^{-1}_{n+1} sets in the sense of Definition 1.1, [4]:

Question 1. *Let $\mathcal{A} \subseteq \Sigma^{-1}_{n+1}$ be such that $\mathcal{R}^{-1}_{n+1}(\mathcal{A})$ contains at least two elements. Is $\mathcal{R}^{-1}_{n+1}(\mathcal{A})$ infinite?*

Question 2. *Let $\mathcal{A} \subseteq \Sigma^{-1}_{n+1}$ be such that $\mathcal{R}^{-1}_{n+1}(\mathcal{A})$ contains at least two elements. Is $\mathcal{R}^{-1}_{n+1}(\mathcal{A})$ a lattice?*

2. Intervals in Rogers semilattices

We first introduce some preliminary notations and remarks. If α and β are Σ^0_{n+1}–computable numberings of a family $\mathcal{A}$, then we denote by $[\alpha, \beta]$ the closed interval of $\mathcal{R}^0_{n+1}(\mathcal{A})$,

$$[\alpha, \beta] \leftrightharpoons \{\deg(\gamma) \mid \deg(\alpha) \leqslant \deg(\gamma) \leqslant \deg(\beta)\}.$$

We also let

$$(\alpha, \beta) \leftrightharpoons \{\deg(\gamma) \mid \deg(\alpha) < \deg(\gamma) < \deg(\beta)\}.$$

Moreover, we will denote by $\hat{\beta}$ the principal ideal of $\mathcal{R}^0_{n+1}(\mathcal{A})$,

$$\hat{\beta} \leftrightharpoons \{\deg(\gamma) \mid \deg(\gamma) \leqslant \deg(\beta)\}.$$

Finally, we let $\hat{\hat{\beta}} \leftrightharpoons \hat{\beta} \setminus \{\deg(\beta)\}$.

In the rest of the paper, the following Definition 2.1 and Lemma 2.2 will play fundamental role.

Definition 2.1. Let β be a numbering of any family $\mathcal{A}$, and let C be any nonempty $\mathbf{X}$–c.e. set. For every $\mathbf{X}$–computable function f such that range$(f) = C$, define $\beta_{C,\mathbf{X}} \leftrightharpoons \beta \circ f$. If C is c.e. we simply write β_C instead of $\beta_{C,\mathbf{0}}$.

Then $\beta_{C,\mathbf{X}}$ is a numbering of some subfamily $\mathcal{A}_0 \subseteq \mathcal{A}$. Clearly, for every pair of $\mathbf{X}$–computable function f and g such that range$(f) =$ range$(g) = C$, we have

$$\beta \circ f \equiv_{\mathbf{X}} \beta \circ g.$$

Thus, in contexts in which we are interested in comparing numberings with respect to $\leqslant_{\mathbf{X}}$, we are justified in writing $\beta_{C,\mathbf{X}}$ without mentioning any particular $\mathbf{X}-$computable function f such that $\mathrm{range}(f) = C$.

The following lemma holds:

Lemma 2.1. *For every triple of sets X, Y, Z such that $X \oplus Y \leqslant_T Z$ and for every pair of numberings β, γ we have:*

(1) *if A is $\mathbf{X}-$c.e. then $\beta_{A,\mathbf{X}} \leqslant_{\mathbf{X}} \beta$;*

(2) *if A is $\mathbf{X}-$c.e., B is $\mathbf{Y}-$c.e. then the following are equivalent:*

   (a) *$\beta_{A,\mathbf{X}} \leqslant_{\mathbf{Z}} \beta_{B,\mathbf{Y}}$;*

   (b) *there exists a partial $\mathbf{Z}-$computable function φ satisfying the conditions: $\mathrm{dom}(\varphi) \supseteq A$, $\varphi(A) \subseteq B$ and for all $x \in A$, $\beta(x) = \beta(\varphi(x))$;*

   (c) *there exists a $\mathbf{Z}-$c.e. equivalence relation η such that*

$$\forall x, y((x, y) \in \eta \;\&\; x \neq y \Rightarrow x, y \in A \cup B),$$
$$\forall x, y((x, y) \in \eta \Rightarrow \beta(x) = \beta(y)),$$
$$\forall x \in A \exists y \in B((x, y) \in \eta);$$

(3) *if A is $\mathbf{X}-$c.e., B is $\mathbf{Y}-$c.e., and $A \subseteq B$ then $\beta_{A,\mathbf{X}} \leqslant_{\mathbf{Z}} \beta_{B,\mathbf{Y}}$;*

(4) *if A is $\mathbf{X}-$c.e., B is $\mathbf{Y}-$c.e. then $\beta_{A\cup B,\mathbf{Z}} \equiv_{\mathbf{Z}} \beta_{A,\mathbf{X}} \oplus \beta_{B,\mathbf{Y}}$;*

(5) *if A is $\mathbf{X}-$c.e., B is $\mathbf{Y}-$c.e., and if $\beta_{A,\mathbf{X}} \leqslant_{\mathbf{Z}} \beta_{B,\mathbf{Y}}$, then $\beta_{B,\mathbf{Y}} \equiv_{\mathbf{Z}} \beta_{A\cup B,\mathbf{Z}}$;*

(6) *if $\gamma \leqslant_{\mathbf{X}} \beta$ then there exists an $\mathbf{X}-$c.e. set A such that $\gamma \equiv_{\mathbf{X}} \beta_{A,\mathbf{X}}$;*

(7) *if $\gamma \leqslant_{\mathbf{X}} \beta$, and $\gamma \equiv_{\mathbf{X}} \beta_{A,\mathbf{X}}$, for some $\mathbf{X}-$c.e. A, then for every α such that $\gamma \leqslant_{\mathbf{X}} \alpha \leqslant_{\mathbf{X}} \beta$ there exists an $\mathbf{X}-$c.e. set B with $A \subseteq B$ and $\alpha \equiv_{\mathbf{X}} \beta_{B,\mathbf{X}}$;*

(8) *if $\gamma \leqslant_{\mathbf{X}} \beta$ via some $\mathbf{X}-$computable function f, then $\gamma \leqslant_{\mathbf{Z}} \beta_{A,\mathbf{Y}}$, for every $\mathbf{Y}-$c.e. set A such that $A \supseteq \mathrm{range}(f)$;*

(9) *if A and B are $\mathbf{X}-$c.e. and for some $a \in B$, $\beta[A] = \{\beta(a)\}$ then $\beta_{A\cup B,\mathbf{X}} \equiv_{\mathbf{X}} \beta_{B,\mathbf{X}}$;*

(10) *if A and B are $\mathbf{X}-$c.e., A is finite and $\beta[A] \subseteq \beta[B]$ then we have $\beta_{A\cup B,\mathbf{X}} \equiv_{\mathbf{X}} \beta_{B,\mathbf{X}}$.*

Proof. We now show the various items, one by one.

(1) Immediate.

(2) Let f and g be an $\mathbf{X}$– computable function and a $\mathbf{Y}$– computable function, respectively, such that $A = \mathrm{range}(f)$, $B = \mathrm{range}(g)$.

(a) $\Rightarrow$ (b). Assume that $\beta_{A,\mathbf{X}} \leqslant_{\mathbf{Z}} \beta_{B,\mathbf{Y}}$ via some $\mathbf{Z}$– computable function h. Then $\beta \circ f = \beta \circ g \circ h$. Define

$$\varphi(x) = \begin{cases} g(h(\mu y\ (x = f(y)))) & \text{if } x \in A, \\ \uparrow & \text{otherwise.} \end{cases}$$

Then φ is the desired partial $\mathbf{Z}$– computable function. (Notice that $\mathrm{dom}(\varphi) = A$.)

(b) $\Rightarrow$ (c). Let η be the $\mathbf{Z}$– c.e. equivalence relation generated by the set of pairs

$$\{(x, \varphi(x)) \mid x \in A\}.$$

It is easy to see that η satisfies (c).

(c) $\Rightarrow$ (a). Let η satisfy the conditions in (c); let $\eta = \bigcup_s \eta^s$ where $\{\eta^s\}_{s\in\mathbb{N}}$ is a $\mathbf{Z}$– computable sequence of finite sets approximating η. Define a function h as follows:

$$h(x) \leftrightharpoons \pi_0(\mu s((f(x), g(\pi_0(s))) \in \eta^{\pi_1(s)})).$$

It is now easy to see that h is a $\mathbf{Z}$– computable function reducing $\beta_{A,\mathbf{X}}$ to $\beta_{B,\mathbf{Y}}$.

(3) This item follows from (2b) taking φ to be the identity embedding of A into B, i.e.

$$\varphi(x) \leftrightharpoons \begin{cases} x & \text{if } x \in A, \\ \uparrow & \text{otherwise.} \end{cases}$$

Notice that φ is in this case partial $\mathbf{X}$– computable.

(4) Claim (3) implies that $\beta_{A,\mathbf{X}} \leqslant_{\mathbf{Z}} \beta_{A\cup B,\mathbf{Z}}$ and $\beta_{B,\mathbf{Y}} \leqslant_{\mathbf{Z}} \beta_{A\cup B,\mathbf{Z}}$. Therefore, $\beta_{A,\mathbf{X}} \oplus \beta_{B,\mathbf{Y}} \leqslant_{\mathbf{Z}} \beta_{A\cup B,\mathbf{Z}}$.

To prove that $\beta_{A\cup B,\mathbf{Z}} \leqslant_{\mathbf{Z}} \beta_{A,\mathbf{X}} \oplus \beta_{B,\mathbf{Y}}$ and let f_1 be an $\mathbf{X}$– computable function, f_2 a $\mathbf{Y}$– computable function and f a $\mathbf{Z}$– computable function whose ranges are A, B and $A \cup B$ respectively. By the Reduction Theorem there exist $\mathbf{Z}$– c.e. sets A_1 and B_1 such that $A_1 \subseteq A$, $B_1 \subseteq B$, $A_1 \cap \mathcal{B}_1 = \emptyset$ and $A_1 \cup B_1 = A \cup B$. Then $f^{-1}(A_1) \cup f^{-1}(B_1) = \mathbb{N}$ and $f^{-1}(A_1) \cap f^{-1}(B_1) = \emptyset$. Hence, $f^{-1}(A_1), f^{-1}(B_1)$ are $\mathbf{Z}$– computable sets. For $i = 1, 2$, let $g_i(x) \leftrightharpoons \mu y(f_i(y) = f(x))$ and define

$$g(x) \leftrightharpoons \begin{cases} 2g_1(x) & \text{if } x \in f^{-1}(A_1), \\ 2g_2(x) + 1 & \text{if } x \in f^{-1}(B_1) \end{cases}.$$

It is easy to see that $\beta_{A\cup B,\mathbf{Z}}$ is reducible to $\beta_{A,\mathbf{X}} \oplus \beta_{B,\mathbf{Y}}$ by the $\mathbf{Z}$–computable function g.

(5) Immediately from (4).

(6) A numbering $\gamma \leqslant_{\mathbf{X}} \beta$ can be considered of the form $\gamma = \beta_{A,\mathbf{X}}$, where $A = \mathrm{range}(f)$, and f is any $\mathbf{X}$–computable function that reduces γ to β.

(7) This follows from (5) and (6).

(8) Let $\gamma = \beta \circ f$, where f is some $\mathbf{X}$–computable function, and let $A \supseteq \mathrm{range}(f)$ be any $\mathbf{Y}$–c.e. set, with say $A = \mathrm{range}(g)$ and g is $\mathbf{Y}$–computable. Then

$$h(x) \leftrightharpoons \mu y\ (g(y) = f(x))$$

is the desired $\mathbf{Z}$–computable function that reduces γ to $\beta_{A,\mathbf{Y}}$.

(9) Let η be the least $\mathbf{X}$–c.e. equivalence relation containing the set $\{(x, a) \mid x \in A\}$. Then by (2), $\beta_{A,\mathbf{X}} \leqslant_{\mathbf{X}} \beta_{B,\mathbf{X}}$. Thus by (5), $\beta_{A\cup B,\mathbf{X}} \equiv_{\mathbf{X}} \beta_{B,\mathbf{X}}$, and statement (9) is proved.

(10) This is a consequence of (9).

□

Particularly important is the case in which $X = Y = \emptyset$. In this case the previous lemma gives:

Lemma 2.2. *For every pair A, B of c.e. sets and for every pair of numberings α, β, we have*

(1) The following are equivalent:

- *(a) $\beta_A \leqslant \beta_B$;*
- *(b) there is a partial computable function φ satisfying $\mathrm{dom}(\varphi) \supseteq A$, $\varphi(A) \subseteq B$ and for all $x \in A$, $\beta(x) = \beta(\varphi(x))$;*
- *(c) there exists a c.e. equivalence relation η such that*

$$\forall x, y((x, y) \in \eta \;\&\; x \neq y \Rightarrow x, y \in A \cup B),$$
$$\forall x, y((x, y) \in \eta \Rightarrow \beta(x) = \beta(y)),$$
$$\forall x \in A \exists y \in B((x, y) \in \eta);$$

(2) if $A \subseteq B$ then $\beta_A \leqslant \beta_B$;

(3) if $\beta_A \leqslant \beta_B$, then $\beta_B \equiv \beta_{A\cup B}$;

(4) *if* $\alpha \leqslant \beta$ *then* $\alpha = \beta_C$ *for some c.e. set* C*;*

(5) *if* $\alpha \leqslant \beta$*, and* $\alpha \equiv \beta_C$*, for some c.e. set* C*, then for every* γ *such that* $\alpha \leqslant \gamma \leqslant \beta$ *there exists a c.e. set* D *with* $C \subseteq D$ *and* $\gamma \equiv \beta_D$*;*

(6) $\beta_{A\cup B} \equiv \beta_A \oplus \beta_B$*;*

(7) *let* U *be any set; if* $\gamma \leqslant_{\mathbf{U}} \beta$ *via some* $\mathbf{U}$*–computable function* f*, then* $\gamma \leqslant_{\mathbf{U}} \beta_C$*, for every c.e. set* C *such that* $C \supseteq \mathrm{range}(f)$*.*

(8) *if for some* $a \in B$*,* $\beta[A] = \{\beta(a)\}$ *then* $\beta_{A\cup B} \equiv \beta_B$*;*

(9) *if* A *is finite set and* $\beta[A] \subseteq \beta[B]$ *then* $\beta_{A\cup B} \equiv \beta_B$*.*

Proof. The statements follow immediately from the previous lemma. □

We will consider mainly intervals and ideals of Rogers semilattices $\mathcal{R}^0_m(\mathcal{A})$ with $m \geq 2$. It should be mentioned that the principal ideals of $\mathcal{R}^0_1(\mathcal{A})$, in the case of finite families $\mathcal{A}$ of c.e. sets, were completely described by Yu.L. Ershov and I.A. Lavrov in [8]. V.V. V'jugin studied initial segments of $\mathcal{R}^0_1(\mathcal{A})$ for some infinite families $\mathcal{A} \subseteq \Sigma^0_1$ (see [24] and [25]). A first insight into the ideals of $\mathcal{R}^0_m(\mathcal{A})$, for $m \geq 2$, is provided by the following theorem.

Theorem 2.1 (S.S.Goncharov). *Let* $\mathfrak{B}_{k+1}$ *be the finite Boolean algebra with* $k+1$ *atoms. Then*

(1) *for every infinite* Σ^0_{n+2}*–computable family* $\mathcal{A}$*,* $\mathcal{R}^0_{n+2}(\mathcal{A})$ *contains an element* $\deg(\beta)$ *such that the subsemilattice* $\hat{\beta}$ *is isomorphic to* $\mathfrak{B}_{k+1} \setminus \{0\}$*;*

(2) *for every finite* Σ^0_{n+2}*–computable family* $\mathcal{A}$*,* $\mathcal{R}^0_{n+2}(\mathcal{A})$ *contains an element* $\deg(\beta)$ *such that the subsemilattice* $\hat{\beta}$ *is isomorphic to* $\mathfrak{B}_{k+1}$*.*

Proof. Let $R_0, R_1, \ldots, R_k$ be infinite computable sets such that

$$\bigcup_{i\leq k} R_i = \mathbb{N} \qquad \text{and} \qquad R_i \cap R_j = \emptyset \text{ if } i \neq j.$$

For every $i \leq k$ let f_i be a computable 1–1 function such that $\mathrm{range}(f_i) = R_i$. We will consider the Boolean algebra $\mathfrak{B}_{k+1}$ as the subalgebra of the Boolean algebra of all subsets of $\mathbb{N}$, generated by $R_0, R_1, \ldots, R_k$.

(1) Let $\mathcal{A}$ be an infinite Σ^0_{n+2}–computable family, and let $A_0, \ldots, A_k$ be distinct elements of $\mathcal{A}$. Choose $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$, and let M be a maximal set. For every $i \leq k$, let $M_i \leftrightharpoons f_i[M]$, and let

$$R_i \setminus M_i \leftrightharpoons \{m^i_0 < m^i_1 < m^i_2 < \ldots\}.$$

Define a numbering β of $\mathcal{A}$ as follows: For every $x \in \mathbb{N}$, find $i \leq k$ such that $x \in R_i$, and let

$$\beta(x) \leftrightharpoons \begin{cases} A_i & \text{if } x \in M_i, \\ \alpha(j) & \text{if } x = m^i_j. \end{cases}$$

Following the proof of Theorem 1.3, [4], one easily shows that for every $i \leq k$, β_{R_i} is a minimal $\Sigma^0_{n+2}-$computable numbering of $\mathcal{A}$, and $\beta_{R_i} \nleqslant \beta_{R_j}$ if $i \neq j$. Lemma 2.2(6) implies that

$$\beta \equiv \beta_{R_0 \cup R_1 \cup \ldots R_k} \equiv \beta_{R_0} \oplus \beta_{R_1} \oplus \cdots \oplus \beta_{R_k}.$$

We will now prove that $\hat{\beta}$ is isomorphic to $\mathfrak{B}_{k+1} \setminus \{0\}$. It is sufficient to show that for every $\gamma \in \mathrm{Com}^0_{n+2}(\mathcal{A})$, if $\gamma \leqslant \beta$ then

$$\gamma \equiv \beta_{R_{i_0}} \oplus \beta_{R_{i_1}} \oplus \cdots \oplus \beta_{R_{i_s}}$$

for some unique sequence $0 \leq i_0 < i_1 < \ldots < i_s \leq k$.

So, let $\gamma \leqslant \beta$ and choose by Lemma 2.2(4) some c.e. set Q such that $\gamma \equiv \beta_Q$. Denote by I_0 and I_1 the sets

$$I_0 \leftrightharpoons \{i \mid i \leq k \;\&\; Q \cap (R_i \setminus M_i) \text{ is infinite}\}$$

and

$$I_1 \leftrightharpoons \{0, 1, \ldots, k\} \setminus I_0,$$

respectively. Note that $I_0 \neq \emptyset$ since $\mathcal{A}$ is infinite. Define

$$Q_0 \leftrightharpoons \bigcup_{i \in I_0} (R_i \setminus M_i) \cap \overline{Q},$$

$$Q_1 \leftrightharpoons \bigcup_{i \in I_1} (R_i \setminus M_i) \cap Q,$$

$$Q_2 \leftrightharpoons \bigcup_{i \in I_1} (Q \cap M_i),$$

$$Q_3 \leftrightharpoons \bigcup_{i \in I_0} ((Q \cup Q_0) \cap R_i).$$

Since M is maximal, it follows that Q_0 is finite. Then, by Lemma 2.2(9),

$$\beta_{Q \cup Q_0} \equiv \beta_Q.$$

We have $Q \cup Q_0 = Q_1 \cup Q_2 \cup Q_3$. The set Q_1 is finite by choice of I_1, therefore, by Lemma 2.2(9), $\beta_Q \equiv \beta_{Q_2 \cup Q_3}$. Since $\beta[Q \cap M_i] \subseteq \{A_i\}$ for all $i \leq k$ it follows by Lemma 2.2(6) and (8) that

$$\beta_Q \equiv \beta_{Q_3} \equiv \beta_{R_{i_0}} \oplus \beta_{R_{i_1}} \oplus \cdots \oplus \beta_{R_{i_s}},$$

where $i_0, i_1, \ldots, i_s$ are all the elements of I_0. Thus, the ideal $\hat{\beta}$ of the semilattice $\mathcal{R}^0_{n+2}(\mathcal{A})$ is isomorphic to $\mathfrak{B}_{k+1} \setminus \{0\}$.

For uniqueness it is sufficient to show that, for$i \leq k$,

$$\beta_{R_i} \not\leqslant \beta_{R_0} \oplus \ldots \oplus \beta_{R_{i-1}} \oplus \beta_{R_{i+1}} \oplus \ldots \oplus \beta_{R_k}.$$

By Lemma 2.2(6) it is sufficient to show that $\beta_{R_i} \not\leqslant \beta_{R'_i}$ where $R'_i = \bigcup_{j \neq i} R_j$. Let $\beta \circ f_i = \beta \circ g \circ h$ where g and h are computable functions and range$(g) = R'_i$. Then $(g \circ h)[M] \subseteq \bigcup_{j \neq i}(R_j \setminus M_j)$ and $(g \circ h)[M]$ is finite since $\bigcup_{j \neq i}(R_j \setminus M_j)$ is immune. But this implies that M is computable, contradicting the choice of M. Thus, the ideal $\hat{\beta}$ of the semilattice $\mathcal{R}^0_{n+2}(\mathcal{A})$ is isomorphic to $\mathfrak{B}_{k+1} \setminus \{0\}$.

(2) If $\mathcal{A}$ is finite then, as in Theorem 1.3, we can easily show that $\mathcal{R}^0_{n+2}(\mathcal{A})$ contains any principal ideal of the c.e. $m-$degrees. It follows from a result of Lachlan, [17], that there exists a c.e. set such that the principal ideal of $m-$degrees generated by the $m-$degree of this set is isomorphic to $\mathfrak{B}_{k+1}$. □

We are now going to exhibit a great variety of intervals and ideals of $\mathcal{R}^0_{n+2}(\mathcal{A})$, for every $n \geq 0$. We still need some definitions and notations.

Definition 2.2. *Let ε denote the poset of all c.e. subsets of ω with respect to inclusion. As is well known, ε is a bounded (i.e. with least and greatest element) distributive lattice.*

Let ε^* denote the bounded distributive lattice obtained by dividing ε modulo the ideal of all finite sets.

If X is any c.e. set, then ε_X denotes the principal filter of ε generated by X, i.e. the collection of all c.e. supersets of X: Clearly ε_X is still a bounded distributive lattice. Likewise, ε^*_X denotes the bounded distributive lattice obtained by dividing ε_X modulo the finite sets.

The following three lemmas play a key role in understanding the constructions in the proofs of several theorems in the sequel of the paper.

Lemma 2.3. *Let α, β be $\Sigma^0_{n+2}-$computable numberings of a family $\mathcal{A}$. Then the mapping $\psi_1 : \varepsilon \longrightarrow \mathcal{R}^0_{n+2}(\mathcal{A})$ defined by $\psi_1(C) \leftrightharpoons \deg(\alpha \oplus \beta_C)$ for every $C \in \varepsilon$, induces an isomorphism from ε^* onto $[\alpha, \alpha \oplus \beta]$ if and only if*

(1) for every infinite $C \in \varepsilon$, $\beta_C \not\leqslant \alpha$;

(2) for every $C_1, C_2 \in \varepsilon$ if $C_1 \setminus C_2$ is infinite then $\beta_{C_1} \not\leqslant \beta_{C_2}$.

Proof. The conditions (1) and (2) are clearly necessary. As to sufficiency, let (1) and (2) hold. Clearly, by (2) and (9) of Lemma 2.2, if $C_1 \subseteq^* C_2$ then $\alpha \oplus \beta_{C_1} \leqslant \alpha \oplus \beta_{C_2}$.

Moreover, we can use (5) and (6) of Lemma 2.2 to show that the mapping induced by ψ_1 is onto. In fact, let γ be a numbering of $\mathcal{A}$ such that $\alpha \leqslant \gamma \leqslant \alpha \oplus \beta$ and let C_1 be the set of all even numbers. Then $\alpha \equiv (\alpha \oplus \beta)_{C_1}$. By Lemma 2.2 (5), there exists a c.e. superset C_2 of the set C_1 such that $\gamma \equiv (\alpha \oplus \beta)_{C_2}$. Then by Lemma 2.2 (6), $(\alpha \oplus \beta)_{C_2} \equiv (\alpha \oplus \beta)_{C_1} \oplus (\alpha \oplus \beta)_{C_2 \setminus C_1}$. Therefore $\gamma \equiv \alpha \oplus \beta_D$ where $D = \{x : 2x+1 \in C_2 \setminus C_1\}$.

We are only left to show that if $C_1 \setminus C_2$ is infinite then $\alpha \oplus \beta_{C_1} \nleqslant \alpha \oplus \beta_{C_2}$. For the sake of a contradiction, suppose otherwise, and let $C_1 \setminus C_2$ be infinite and $\alpha \oplus \beta_{C_1} \leqslant \alpha \oplus \beta_{C_2}$. Then $\beta_{C_1} \leqslant \alpha \oplus \beta_{C_2}$. Let f be a computable function such that $\mathrm{range}(f) = C_1$, and let g be a computable function that reduces β_{C_1} to $\alpha \oplus \beta_{C_2}$. Define

$$C_3 \leftrightharpoons \{f(x) \mid g(x) \text{ even}\}$$
$$C_4 \leftrightharpoons \{f(x) \mid g(x) \text{ odd}\}.$$

Then $C_1 = C_3 \cup C_4$ and $C_3 \cap C_4 = \emptyset$. Clearly $\beta_{C_3} \leqslant \alpha$. If C_3 were infinite, then by condition (1) we would get $\beta_{C_3} \nleqslant \alpha$. Thus C_3 is finite. But then $C_1 =^* C_4$ and thus, being $\beta_{C_4} \leqslant \beta_{C_2}$, we would have $\beta_{C_1} \equiv \beta_{C_4} \leqslant \beta_{C_2}$, contradicting (2). □

Lemma 2.4. *Let β be a Σ^0_{n+2}–computable numbering of an infinite family $\mathcal{A}$ such that every set of $\mathcal{A}$ has infinitely many β-indices. Let $\psi_2 : \varepsilon \longrightarrow \mathcal{R}^0_{n+2}(\mathcal{A})$ be a mapping defined as follows: $\psi_2(C) \leftrightharpoons \deg(\beta_C)$, $C \in \varepsilon$. Then*

(1) ψ_2 induces an epimorphism of upper semilattices from the semilattice $\langle \varepsilon^* \setminus \{\bot\}, \cup^*, \subseteq^* \rangle$ onto the ideal $\hat{\beta}$ of the semilattice $\mathcal{R}^0_{n+2}(\mathcal{A})$ if and only if for every infinite $C \in \varepsilon$, β_C is a numbering of the whole family;

(2) ψ_2 induces an isomorphism between $\langle \varepsilon^* \setminus \{\bot\}, \cup^*, \subseteq^* \rangle$ and $\mathcal{R}^0_{n+2}(\mathcal{A})$ if and only if for every pair of infinite sets $C_1, C_2 \in \varepsilon$, if $C_1 \setminus C_2$ is infinite then $\beta_{C_1} \nleqslant \beta_{C_2}$.

Proof. Let $\mathcal{A}$ and β be chosen as in statement of the lemma.

(1) Necessity is obvious (if the condition does not hold then ψ_2 is not well defined). For sufficiency we only need to show that ψ_2 preserves the binary operation and induces a mapping onto $\hat{\beta}$. This follows immediately from (4) and (6) of Lemma 2.2 (notice that ψ_2 is well defined by (9) of lemma 2.2).

(2) Sufficiency is evident. Let us prove necessity. Let ψ_2 induce an isomorphism of $\langle \varepsilon^* \setminus \{\bot\}, \cup^*, \subseteq^* \rangle$ onto $\mathcal{R}^0_{n+2}(\mathcal{A})$ and assume that

$\beta_{C_1} \leqslant \beta_{C_2}$ for some pair of infinite c.e. sets C_1, C_2 such that $C_1 \setminus C_2$ is infinite. Then by (3) of Lemma 2.2, $\beta_{C_2} \equiv \beta_{C_1 \cup C_2}$. Therefore, $\psi_2[C_2] = \psi_2[C_1 \cup C_2]$ but $C_2 \neq^* C_1 \cup C_2$. This is in contradiction with the injectivity of the mapping induced by ψ_2 on $\varepsilon^* \setminus \{\bot\}$.

□

Lemma 2.5. *Let $\mathcal{A}$ be a finite family of Σ^0_{n+2}–sets, and suppose that for some $\beta \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ and some finite set F we have that $\beta[F] = \mathcal{A}$. Let $\psi_2 : \varepsilon \longrightarrow \mathcal{R}^0_{n+2}(\mathcal{A})$ be defined as in the previous lemma. Then*

(1) ψ_2 induces an epimorphism of upper semilattices from $\langle \varepsilon^*_F, \cup^*, \subseteq^* \rangle$ onto the ideal $\hat{\beta}$ of the semilattice $\mathcal{R}^0_{n+2}(\mathcal{A})$;

(2) ψ_2 induces an isomorphism between $\langle \varepsilon^*_F, \cup^*, \subseteq^* \rangle$ and $\mathcal{R}^0_{n+2}(\mathcal{A})$ if and only if for every pair of sets $C_1, C_2 \in \varepsilon_F$, if $C_1 \setminus C_2$ is infinite then $\beta_{C_1} \nleqslant \beta_{C_2}$.

Proof. Let $\mathcal{A}$, β, and F be chosen as in statement of lemma.

(1) The mapping ψ_2 is onto by (4) and (9) of Lemma 2.2 and preserves the binary operation by (6) of the same lemma. (Notice that $\varepsilon^*_F = \varepsilon^*$).

(2) Similar to the proof of (2) in Lemma 2.4.

□

Theorem 2.2. *Let $\mathcal{A}$ be a Σ^0_{n+2}–computable family. If $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ is not $\mathbf{0}'$–universal in $\mathrm{Com}^0_{n+2}(\mathcal{A})$ then there exists a numbering $\beta \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ such that the following conditions hold:*

- *The interval $[\alpha, \alpha \oplus \beta]$ is isomorphic to ε^*;*
- *the ideal $\hat{\beta}$ is isomorphic to $\varepsilon^* \setminus \{\bot\}$ if $\mathcal{A}$ is infinite, and $\hat{\beta}$ is isomorphic to ε^* if $\mathcal{A}$ is finite family.*

Proof. Let $\mathcal{A}$ and α be given as in the statement of the theorem. We will construct the numbering β by stages. Let $\gamma \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ be such that $\gamma \nleqslant_{\mathbf{0}'} \alpha$, and let a_1, a_2 be two numbers such that $\gamma(a_1) \neq \gamma(a_2)$. Let $\{\epsilon_t\}_{t \in \mathbb{N}}$ be a standard listing of all c.e. equivalence relations. Firstly we will consider the case when $\mathcal{A}$ is infinite family.

The requirements. In view of Lemma 2.3 and Lemma 2.4, we will build a numbering β of $\mathcal{A}$ so as to satisfy the following requirements, for every p, q, t:

$$P : \beta \leqslant_{\mathbf{0}'} \gamma$$
$$Q_p : W_p \text{ infinite } \Rightarrow \gamma \leqslant_{\mathbf{0}'} \beta_{W_p}$$
$$R_{t,p,q} : W_p \setminus W_q \text{ infinite } \Rightarrow \beta_{W_p} \nleqslant \beta_{W_q} \text{ via } \epsilon_t$$

where we say that $\beta_{W_p} \nleqslant \beta_{W_q}$ *via* ϵ_t if ϵ_t does not witness the reduction of β_{W_p} to β_{W_q} in the sense of (1c) of Lemma 2.2.

We will satisfy P by building a $\mathbf{0}'$–computable function h such that $\beta = \gamma \circ h$. This feature of β (i.e. $\beta \leqslant_{\mathbf{0}'} \gamma$) will be used in the verification.

For every p we will satisfy Q_p by building, whenever W_p is infinite, a $\mathbf{0}'$–computable function f_p such that range$(f_p) \subseteq W_p$ and $\gamma = \beta \circ f_p$. Thus, by (7) of Lemma 2.2, $\gamma \leqslant_{\mathbf{0}'} \beta_{W_p}$, which implies that β_{W_p} is a numbering of the whole family, and $\beta_{W_p} \nleqslant \alpha$, otherwise $\gamma \leqslant_{\mathbf{0}'} \alpha$.

Finally, if $W_p \setminus W_q$ is infinite then we will satisfy $R_{t,p,q}$ by choosing in the interval $(k(t), k(t+1))$ – where k is a suitable increasing $\mathbf{0}'$–computable function – numbers x, y with $(x, y) \in \epsilon_t$, and letting $\beta(x) \neq \beta(y)$.

Together with the numbering β, and the functions h, f_p, during the construction we will also define the values of a counter function $\lambda t\, s(p,t)$, for every number p. In fact, the definition of β will follow automatically from h by letting $\beta^0 \leftrightharpoons \{\langle 0, \gamma(h(0))\rangle\}$ and

$$\beta^{t+1} \leftrightharpoons \beta^t \cup \{\langle x, \gamma(h(x))\rangle \mid k(t) < x \leq k(t+1)\}$$

for all t.

The construction. By stages:
Stage 0) Define

$$k(0) = h(0) \leftrightharpoons 0, f_p^0 \leftrightharpoons \emptyset, s(p,0) \leftrightharpoons p$$

for every $p \in \mathbb{N}$ and go to next stage.

Stage $t+1$). Go through Steps 1, 2), 3) in order.
Step 1) Check whether

$$\exists x_0 \exists x_1 \exists y_0 \exists y_1 [(\forall i \leq 1)\,(k(t) < x_i, y_i \\ \& \ |\{x_0, x_1, y_0, y_1\}| = 4 \ \& \ (x_i, y_i) \in \epsilon_t)] \quad \text{(A)}$$

If (A) does not hold then define $k(t+1) \leftrightharpoons k(t)+1$. If (A) holds then fix such numbers x_0, x_1, y_0, y_1 so that, say, $\langle x_0, x_1, y_0, y_1\rangle$ is the least such quadruple, and define

$$k(t+1) \leftrightharpoons \max\{x_0, x_1, y_0, y_1\} + 1.$$

Move to Step 2).

Step 2) Check if the following condition $B(p)$ holds, for any $p \leq t$:

$$B(p): \qquad \exists z(k(t) < z \leq k(t+1) \ \&\ z \in W_p).$$

If $B(p)$ holds of some p, then fix p_0 such that $B(p_0)$ holds and

$$\forall p \leq t(B(p) \ \&\ p \neq p_0 \Rightarrow (s(p_0,t) < s(p,t) \\ \vee\ (s(p_0,t) = s(p,t) \ \&\ p_0 < p)).$$

Let z_0 be the least z for which $B(p_0)$ holds. Choose the least number $u_0 \notin \mathrm{dom}(f^t_{p_0})$. Fix the least $i_0 \in \{0,1\}$ such that $z_0 \notin \{x_{i_0}, y_{i_0}\}$. Go to Step 3).

Step 3) We distinguish the following cases:

(a) If (A) is not true and $B(p)$ fails for every $p \leq t$ then choose the least number $y \notin \mathrm{range}(h^t)$ and let

$$h^{t+1} \leftrightharpoons h^t \cup \{(k(t+1), y)\}, f^{t+1}_p \leftrightharpoons f^t_p, s(p,t+1) \leftrightharpoons s(p,t) \quad \text{for all} \quad p.$$

(b) If (A) holds and, for some p, $B(p)$ holds then define

$$h^{t+1} \leftrightharpoons h^t \cup \{(x_{i_0}, a_1), (y_{i_0}, a_2), (z_0, u_0)\} \\ \cup \{(x, a_1) \mid k(t) < x \leq k(t+1) \ \&\ x \notin \{x_{i_0}, y_{i_0}, z_0\}\}$$

and

$$f^{t+1}_{p_0} \leftrightharpoons f^t_{p_0} \cup \{(u_0, z_0)\}, s(p_0, t+1) \leftrightharpoons s(p_0,t) + 1, f^{t+1}_p \leftrightharpoons f^t_p \\ s(p,t+1) \leftrightharpoons s(p,t) \quad \text{for all} \quad p \neq p_0$$

(c) If (A) holds but $B(p)$ fails for all $p \leq t$ then define

$$h^{t+1} \leftrightharpoons h^t \cup \{(x_0, a_1), (y_0, a_2)\} \cup \{(x, a_1) \mid \\ k(t) < x \leq k(t+1) \ \&\ x \notin \{x_0, y_0\}\}$$

and

$$f^{t+1}_p \leftrightharpoons f^t_p, s(p,t+1) \leftrightharpoons s(p,t) \quad \text{for all} \quad p.$$

(d) If (A) fails and $B(p)$ holds for some p then define

$$h^{t+1} \leftrightharpoons h^t \cup \{(z_0, u_0)\}$$

and

$$f_{p_0}^{t+1} \leftrightharpoons f_{p_0}^t \cup \{(u_0, z_0)\}, s(p_0, t+1) \leftrightharpoons s(p_0, t) + 1$$
$$f_p^{t+1} \leftrightharpoons f_p^t, s(p, t+1) \leftrightharpoons s(p, t) \quad \text{for all} \quad p \neq p_0$$

Notice that in this case $z_0 = k(t+1)$. Go to stage $t+2$.

The verification. We show that the following conditions hold:
(1) For every t, $k(t) < k(t+1)$.
(2) The functions h, k, f_p, with $p \in \mathbb{N}$, are $\mathbf{0}'$–computable.
(3) For every x, we have that $\beta(x) = \gamma(h(x))$. Thus β is a Σ^0_{n+2}–computable numbering of some subfamily $\mathcal{A}_0 \subseteq \mathcal{A}$.
(4) For every p, $s(p) \leq s(p+1) \leq s(p) + 1$.
(1)–(4) are obvious.
(5) For every p,

$$W_p \text{ infinite } \Rightarrow \lim_t s(p,t) = \infty.$$

To see this, assume that W_p is infinite and at a stage t_0 we have $s(p, t_0) = m$ for some m. We will show that for some $t_1 \geq t_0$, $s(p, t_1{+}1) > m$.

For all t and $p' > m$ we have $s(p', t) > m$ (recall that $s(p', 0) = p'$). Let $t_1 \geq t_0$ be a stage such that $t_1 \geq p$ and for every $p' \leq m$,

- if $\lim_t s(p', t) < \infty$ than $(\forall t \geq t_1)[s(p', t) = s(p', t_1)]$;
- if $\lim_t s(p', t) = \infty$ than $s(p', t_1) > m$;
- $(k(t_1), k(t_1 + 1)] \cap W_p \neq \emptyset$.

So, either $s(p, t_1) > m$, or $s(p, t_1) = m$. In the latter case, we choose $p = p_0$ at t_1 and increase the counter $s(p, t)$, i.e. $s(p, t_1 + 1) = m + 1$.
(6) For every p, if W_p is infinite then f_p is total, range$(f_p) \subseteq W_p$, and $\gamma = \beta \circ f_p$.

This follows from the way we define f_p and β and the fact that $\lim_t s(p,t) = +\infty$ by (5) and thus every time we increase $s(p,t)$ we also define f_p on the least element u_0 which is not in the domain of f_p^t:

$$\beta(f_p(u_0)) = \beta(z_0) = \gamma(h(z_0)) = \gamma(u_0).$$

(7) For every p, if W_p is infinite then β is a numbering of the whole family and $\beta_{W_p} \not\leq \alpha$.

This follows immediately from the previous point, and the choice of $\gamma \not\leq_{\mathbf{0}'} \alpha$.
(8) For every p, q if $W_p \setminus W_q$ is infinite then $\beta_{W_p} \not\leq \beta_{W_p}$.

To see this, assume that $W_p \setminus W_q$ is infinite, and $\beta_{W_p} \leqslant \beta_{W_p}$. By (1c) of Lemma 2.2, let η be a c.e. equivalence relation such that

$$\begin{aligned}
&\forall x, y((x,y) \in \eta \,\&\, x \neq y \Rightarrow x, y \in W_p \cup W_q),\\
&\forall x, y((x,y) \in \eta \Rightarrow \beta(x) = \beta(y)),\\
&\forall x \in W_p \exists y \in W_q((x,y) \in \eta).
\end{aligned}$$

Let $\eta = \epsilon_t$ for some $t \in \mathbb{N}$. Since $W_p \setminus W_q$ is infinite it follows that either η has infinitely many nontrivial equivalence classes, or it has an infinite equivalence class. Therefore (A) holds at stage t. By construction, at stage t we proceed through (b) or (c) in Step 3. We then define $h(x) = a_1$, $h(y) = a_2$ for some $x, y > k(t)$ such that $(x,y) \in \eta$. But $(x,y) \in \eta$ implies $\beta(x) = \beta(y)$, and therefore $\gamma(a_1) = \gamma(a_2)$, since $\beta(x) = \gamma(h(x))$ and $\beta(y) = \gamma(h(y))$. This contradicts our choice of $\gamma(a_1) \neq \gamma(a_2)$.

To finish off the proof in the case of an infinite family $\mathcal{A}$, we observe that β satisfies both Lemma 2.3 and Lemma 2.4. (Note that every set of $\mathcal{A}$ has infinitely many β-indices. This is consequence of (6), since there are infinitely many pairwise disjoint infinite c.e. sets.) Hence the interval $[\alpha, \alpha \oplus \beta]$ is isomorphic to ε^* and the ideal $\hat{\beta}$ is isomorphic to $\varepsilon^* \setminus \{\bot\}$.

Finally, if $\mathcal{A}$ is finite, then we may slightly modify stage 0 of our construction as follows. Let $\mathcal{A}$ consist of $m+1$ sets and let $b_0, b_1, \ldots, b_m$ be γ-indices of these sets. Then at stage 0 we define $k(0) \leftrightharpoons m$, $h(i) \leftrightharpoons b_i$ for all $i \leq m$, and let $f_p^0 \leftrightharpoons \emptyset, s(p,0) \leftrightharpoons p$ for every $p \in \mathbb{N}$.

It is easy to check that the modified numbering β satisfies both Lemma 2.3 and Lemma 2.5 and thus in this case the ideal $\hat{\beta}$ is isomorphic to ε_F^* for $F = \{0, 1, \ldots, m\}$. The result then follows by observing the trivial fact that the lattices ε_F^* and ε^* are isomorphic. □

Corollary 2.2.1 (S.Podzorov, [19]). *Let $\mathcal{A}$ be any Σ^0_{n+2}– computable family. For every numbering $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ there exists a numbering $\beta \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ which is $\mathbf{0}'$– equivalent to α and such that*

(1) $\hat{\beta}$ is isomorphic to $\varepsilon^* \setminus \{\bot\}$ if the family $\mathcal{A}$ is infinite;

(2) $\hat{\beta}$ is isomorphic to ε if the family $\mathcal{A}$ is finite.

Proof. Note that numbering β constructed in Theorem 2.2 is $\mathbf{0}'$–equivalent to γ. The proof is now immediate, since if we do not require that the interval $[\alpha, \alpha \oplus \beta]$ be isomorphic to ε^*, then we do not need the feature that $\gamma \nleq_{\mathbf{0}'} \alpha$. In other words, we can carry on the construction of Theorem 2.2 starting from the numbering α instead of γ, and thus obtaining the desired conclusions (1) and (2). □

Corollary 2.2.1 and the fact that the elementary theory of ε^* is hereditarily undecidable, [13], immediately yield:

Corollary 2.2.2. *The elementary theory of every non-trivial Rogers semilattice $\mathcal{R}^0_{n+2}(\mathcal{A})$ is hereditarily undecidable.*

Corollary 2.2.1 and Corollary 2.2.2 give us a deep insight into the complexity of Rogers semilattices of Σ^0_{n+2}–computable families. The case of Σ^0_1–computable families is still open:

Question 3. *Is the elementary theory of any non-trivial Rogers semilattice of a Σ^0_1–computable family hereditarily undecidable, or at least undecidable?*

We know that the range of isomorphism types of ε^*_A, for different c.e. sets A, is very wide. For instance, we know that every countable Boolean algebra with a sufficiently effective presentation is isomorphic to some ε^*_A (namely every Σ^0_3– presentable Boolean algebra, see [15]; see also [23]). Under reasonable conditions, we can realize these isomorphism types as intervals of Rogers semilattices of arithmetical numberings.

Theorem 2.3. *Let $\mathcal{A}$ be a Σ^0_{n+3}– computable family, and let A be both an infinite and coinfinite c.e. set. Then for every $\alpha \in \mathrm{Com}^0_{n+3}(\mathcal{A})$ such that α is not $\mathbf{0}^{(2)}$– universal in $\mathrm{Com}^0_{n+3}(\mathcal{A})$, there exists some numbering $\beta \in \mathrm{Com}^0_{n+3}(\mathcal{A})$ such that the subsemilattice $[\alpha, \beta]$ is isomorphic to ε^*_A.*

Proof. Let $\mathcal{A}$, A, and α be given, and let $f : \mathbb{N} \longrightarrow A$ be a computable bijection. We will construct a numbering β, as required. First of all, we define

$$\beta(f(n)) \leftrightharpoons \alpha(n)$$

so that $\alpha \leqslant \beta$ and $\alpha \equiv \beta_A$. It remains to define $\beta(n)$ on each $n \in \overline{A}$: This will be done, stage by stage, by the construction below. Fix some $\gamma \in \mathrm{Com}^0_{n+3}(\mathcal{A})$ such that $\gamma \nleqslant_{\mathbf{0}^{(2)}} \alpha$; define a listing of all c.e. supersets of A by letting, for every i,

$$V_i \leftrightharpoons W_i \cup A;$$

and fix some distinct $C, D \in \mathcal{A}$.

We sketch the strategy for the construction. We define β so that the mapping

$$\psi(V_i) \leftrightharpoons \deg(\beta_{V_i})$$

induces an isomorphism from ε^*_A onto $[\alpha, \beta]$. For this, we first prove:

Lemma 2.6. *ψ (as defined above) induces an isomorphism from ε_A^* onto $[\alpha, \beta]$ if and only if*

$$\forall V_i, V_j \in \varepsilon_A \left(V_i \setminus V_j \textit{ infinite } \Rightarrow \beta_{V_i} \nleqslant \beta_{V_j}\right). \tag{†}$$

Proof. The condition (†) is clearly necessary. Suppose now that ψ satisfies condition (†). It follows from (2) and (9) of Lemma 2.2 that

$$V_i \subseteq^* V_j \Rightarrow \beta_{V_i} \leqslant \beta_{V_j}.$$

By (5) of Lemma 2.2 every γ such that $\alpha \leqslant \gamma \leqslant \beta$ satisfies $\gamma \equiv \beta_B$, for some c.e. $B \supseteq A$. So ψ is onto. On the other hand, if $V_i \nsubseteq^* V_j$, then $V_i \setminus V_j$ is infinite, and thus $\beta_{V_i} \nleqslant \beta_{V_j}$. So ψ is an isomorphism. □

The requirements. In view of Lemma 2.6 the construction aims to satisfy the following requirements:

$$R_{p,i,j}: V_i \setminus V_j \text{ infinite and } \varphi_p[V_i] \nsubseteq^* A \Rightarrow \beta_{V_i} \nleqslant \beta_{V_j} \text{ via } \varphi_p$$
$$Q_i: V_i \setminus A \text{ infinite } \Rightarrow \gamma \leqslant_{\mathbf{0}^{(2)}} \beta_{V_i}$$

where we say that $\beta_{V_i} \nleqslant \beta_{V_j}$ *via* φ_p if the following does not happen (see (1b) of Lemma 2.2): $V_i \subseteq \mathrm{dom}(\varphi_p)$, $\varphi_p[V_i] \subseteq V_j$, and, for every $x \in V_i$, $\beta(x) = \beta(\varphi_p(x))$.

The requirement $R_{p,i,j}$ will be satisfied at stage $2t+1$ with $t = \langle p, i, j\rangle$, by looking for some $k \in V_i \setminus A$ such that $\varphi_p(k) \in V_j \setminus A$, and by defining $\beta(k) \neq \beta(\varphi_p(k))$.

The requirement Q_i will be attacked at infinitely many stages $2t+2$, with $\pi_0(t) = i$, each time extending (if $V_i \setminus A$ is infinite), the values of a $\mathbf{0}^{(2)}$–computable function f_i such that $\mathrm{range}(f_i) \subseteq \overline{A}$ and $\gamma = \beta \circ f_i$. This is used to show that $\gamma \leqslant_{\mathbf{0}^{(2)}} \alpha$ (as shown in details later), contrary to the assumptions.

Step by step, we will construct an increasing sequence $\{A^t\}_{t\in\mathbb{N}}$ of finite sets such that $\overline{A} = \bigcup_t A^t$, and the (in general, partial) $\mathbf{0}^{(2)}$–computable functions $\{f_i\}_{i\in\mathbb{N}}$.

The construction. By stages:

Stage 0) Define $\beta(f(n)) \leftrightharpoons \alpha(n)$ for every n. Let $f_i^0 \leftrightharpoons \emptyset$ for all i, and let $A^0 \leftrightharpoons \emptyset$.

Stage $2t+1$). Assume that $t = \langle p, i, j\rangle$. If $i \neq j$ and the following conditions hold:

(1) $V_i \subseteq \mathrm{dom}(\varphi_p)$ & $\forall y(y \in V_i \Rightarrow \varphi_p(y) \in V_j)$;

(2) $\exists k(k \in V_i \setminus A$ & $\varphi_p(k) \in V_j \setminus A)$ & $k \notin A^{2t}$ & $\varphi_p(k) \notin A^{2t} \cup \{k\})$

then choose the least such number k, let

$$\begin{aligned} A^{2t+1} &\leftrightharpoons A^{2t} \cup \{k, \varphi_p(k)\} \\ \beta(k) &\leftrightharpoons C \\ \beta(\varphi_p(k)) &\leftrightharpoons D \end{aligned}$$

and move to next stage.

If $i = j$, or either of (1) or (2) does not hold, then do nothing and move to next stage.

Stage $2t + 2$). Suppose that $\pi_0(t) = i$, and there exists $x \in V_i \setminus A$ such that $\beta(x)$ has not yet been defined. Fix the least such x, and define

$$\begin{aligned} A^{2t+2} &\leftrightharpoons A^{2t+1} \cup \{x\} \\ f_i^{2t+2} &\leftrightharpoons f_i^{2t+1} \cup \{(s, x)\} \end{aligned}$$

where s is the least number such that $s \notin \mathrm{dom}(f_i^{2t+1})$; finally define

$$\beta(x) \leftrightharpoons \gamma(s).$$

Move to next stage.

Finally, define $f_i = \bigcup_t f_i^t$, for every i.

The verification. The following properties hold:

(1) For every x there exists a stage t such that $\beta(x)$ is defined at stage t.

To see this, notice that if $x \in A$ then $\beta(x)$ is defined at stage 0 of the construction. On the other hand, let i_0 be such that $\mathbb{N} = V_{i_0}$. At every stage $2t + 2$, with $\pi_0(t) = i_0$, we define $\beta(x)$ for the least x such that $x \notin A$ and $\beta(x)$ is undefined by stage $2t + 1$. So if $x \notin A$, then there exists some t such that we define $\beta(x)$ at stage $2t + 2$.

(2) $\beta \in \mathrm{Com}^0_{n+3}(\mathcal{A})$.

Indeed, range$(\beta) = \mathcal{A}$. This follows from the way β is defined at stage 0. Moreover, the relation "$y \in \beta(x)$" is $\mathbf{0}^{(3)}$–computably enumerable, since $\alpha, \gamma \in \mathrm{Com}^0_{n+3}(\mathcal{A})$ and the conditions (1) and (2) of odd stages, as well as the test relative to each even stage can be effectively tested with oracle $\mathbf{0}^{(2)}$.

(3) For every i, if $V_i \setminus A$ is infinite, then $\beta_{V_i} \nleqslant \alpha$.

Indeed, if $V_i \setminus A$ is infinite then at every stage $2t+2$ such that $\pi_0(t) = i$, we have that $f_i^{2t+2} = f^{2t+1} \cup \{(s, x)\}$, for some s and $x \in V_i$. It is thus easy to see that f_i is $\mathbf{0}^{(2)}$–computable, and $\gamma = \beta \circ f_i$. Therefore $\gamma \leqslant_{\mathbf{0}^{(2)}} \beta_{V_i}$ by (7) of Lemma 2.2. This implies that $\beta_{V_i} \nleqslant \alpha$, since otherwise it would follow that $\gamma \leqslant_{\mathbf{0}^{(2)}} \alpha$.

(4) If $V_i \setminus V_j$ is infinite then $\beta_{V_i} \nleqslant \beta_{V_j}$.

We prove this by contradiction. Assume that $\beta_{V_i} \leqslant \beta_{V_j}$. Then by (1b) of Lemma 2.2 there exists a partial computable function φ_p such that $V_i \subseteq \mathrm{dom}(\varphi_p)$, $\varphi_p[V_i] \subseteq V_j$, and $\beta(x) = \beta(\varphi_p(x))$, for every $x \in V_i$.

Let $V_m = \{x \in V_i \mid \varphi_p(x) \neq x\} \cup A$ and $t = \langle p, m, j \rangle$. It is easy to see that $V_m \setminus A$ is infinite since it contains $V_i \setminus V_j$. We have $\varphi_p[V_m] \nsubseteq^* A$, otherwise $\beta_{V_m} \leqslant \beta_A$ via φ_p, but this contradicts (3). Moreover, $\varphi_p[V_m \setminus (A \cup A^{2t})] \nsubseteq A \cup A^{2t}$. To show this fact suppose that this inclusion holds and consider the partial computable function

$$\varphi_q(x) \leftrightharpoons \begin{cases} \varphi_p(x) & \text{if } \exists y(g(y) = \varphi_p(x) \& \forall z < y(g(z) \notin \{x, \varphi_p(x)\})), \\ x & \text{if } \exists y(g(y) = x \& \forall z < y(g(z) \notin \{x, \varphi_p(x)\})), \\ \uparrow & \text{otherwise,} \end{cases}$$

where g is a computable function from $\mathbb{N}$ onto $A \cup A^{2t}$. It's easy to see that $\mathrm{dom}(\varphi_q) \supseteq V_m$ and $\beta_{V_m} \leqslant \beta_A$ via φ_q which contradicts (3) again.

Now consider the stage $2t+1$. We see that all conditions of this stage hold and we have $\beta(k) = C \neq D = \beta(\varphi_p(k))$ for some $k \in V_m$. But $k \in V_i$ since $V_m \subseteq V_i$ by choice of V_m. □

By requesting more of α, we can show that a result similar to Theorem 2.3 holds also for Σ^0_2–computable families. (We observe that next theorem is a trivial consequence of the previous theorem if $n > 0$; for $n = 0$ it provides a stronger result. We give a proof here for any arbitrary n.)

Theorem 2.4. *Let $\mathcal{A}$ be a Σ^0_{n+2}–computable family and let A be both an infinite and coinfinite c.e. set. If $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ and α is not $\mathbf{0}^{(n+1)}$–universal in $Com^0_{n+2}(\mathcal{A})$ then there exists a numbering $\beta \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ such that the interval $[\alpha, \beta]$ is isomorphic to ε^*_A.*

Proof. Let $\mathcal{A}$, A, and α be given as in the statement of the theorem. In view of Lemma 2.6, we construct β so as to satisfy the following requirements, for every p, i, j:

$$R_{p,i,j} : V_i \setminus V_j \text{ infinite } \Rightarrow \beta_{V_i} \nleqslant \beta_{V_j} \text{ via } \varphi_p$$

where $V_e \leftrightharpoons W_e \cup A$, as in the proof of Theorem 2.3, and "$\beta_{V_i} \nleqslant \beta_{V_j}$ via φ_p" has the same meaning as in the proof of Theorem 2.3. Thus if $V_i \setminus V_j$ is infinite, satisfaction of $R_{p,i,j}$, for any p, will guarantee that φ_p does not reduce β_{V_i} to β_{V_j} in the sense of (1b) of Lemma 2.2.

For this proof we employ a finite priority argument to computably enumerate, relatively to the oracle $\mathbf{0}^{(n+1)}$, a numbering β so that $\beta \in \mathrm{Com}^0_{n+2}(\mathcal{A})$.

We briefly sketch the strategy to meet $R_{p,i,j}$.

Let $\gamma \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ be such that $\gamma \nleqslant_{\mathbf{0}^{(n+1)}} \alpha$. Using that $\alpha, \gamma \in \mathrm{Com}^0_{n+2}(\mathcal{A})$, fix $\mathbf{0}^{(n+1)}$–computable sequences of finite sets $\{\alpha^t(x)\}_{t,x\in\mathbb{N}}$ and $\{\gamma^t(x)\}_{t,x\in\mathbb{N}}$ such that, for every t, x,

$$\alpha^t(x) \subseteq \alpha^{t+1}(x), \qquad \alpha(x) = \bigcup_t \alpha^t(x)$$

and

$$\gamma^t(x) \subseteq \gamma^{t+1}(x), \qquad \gamma(x) = \bigcup_t \gamma^t(x).$$

First of all, we make β "look like" α on A, to get $\alpha \leqslant \beta$. Then we monitor (via the notion of t–reducibility, and a suitable counter function) the step by step progress towards obtaining that $V_i \setminus V_j$ is infinite and φ_p reduces β_{V_i} to β_{V_i} (via monitoring, in turn, the progress towards obtaining $V_i \subseteq \mathrm{dom}(\varphi_p)$, $\varphi_p[V_i] \subseteq V_j$ and $\beta(x) = \beta(\varphi_p(x))$, for every $x \in V_i$). If this progress threatens to tend to infinity, then we select (by suitably labelling its elements) a set $G \subseteq V_i \setminus V_j$ (hence $G \subseteq \overline{A}$) on which we define a partial $\mathbf{0}^{(n+1)}$–computable function F_γ such that, for every $x \in G$, $\beta(x) = \gamma(F_\gamma(x))$. On the other hand, if $F_\gamma(x)$ is defined and $\varphi_p(x) \in V_j$, then we define on $\varphi_p(x)$ the value $F_\alpha(\varphi_p(x))$ of a $\mathbf{0}^{(n+1)}$–computable function F_α, so that $\beta(\varphi_p(x)) = \alpha(F_\alpha(\varphi_p(x)))$. As shown in details later, this implies that $\gamma \leqslant_{\mathbf{0}^{(n+1)}} \alpha$, contradiction. Thus φ_p does not reduce β_{V_i} to β_{V_j} (in the sense of (1b) of Lemma 2.2), and eventually the requirement is satisfied, and thus injures only finitely many times the lower priority requirements.

Some of the β-indices will be marked by labels of the form $\boxed{m}$, with $m \in \mathbb{N}$ (where m must be viewed as coding a triple, $m = \langle p, i, j\rangle$). Together with the values of the finite approximation β^t to β at stage t, the construction will define a counter function $s(m,t)$, for every m.

We will say that φ_p *t-reduces* β_{V_i} *to* β_{V_j} if (where $m = \langle p, i, j\rangle$)

(1) $s(m,t) < \max(V_i \setminus V_j \cap [0,t])$;

(2) for every $x \in V_i \cap [0, s(m,t)]$ we have that $x \in \mathrm{dom}(\varphi_p)$ and $\varphi_p(x) \in V_j$;

(3) for every $x \in V_i \cap [0, s(m,t)]$ we have

$$\beta^t(x) \cap [0, s(m,t)] = \beta^t(\varphi_p(x)) \cap [0, s(m,t)].$$

Fix also a partial computable 1–1 and onto function $\varphi : A \longrightarrow \mathbb{N}$, and a $\mathbf{0}'$–computable function k enumerating $\overline{A}$ in order of magnitude, i.e.

$$\overline{A} = \{k(0) < k(1) < \cdots\}.$$

Finally, at stage t we will define also the approximations F_α^t, F_γ^t of partial $\mathbf{0}^{(n+1)}$–computable functions F_α, F_γ. The arguments of these functions are β–indices, the values of F_α are α–indices, and the values of F_γ are γ–indices. Moreover, we shall have, for every t,

$$\mathrm{dom}(F_\alpha^t) \cap \mathrm{dom}(F_\gamma^t) = \emptyset$$

and

$$\mathrm{dom}(F_\alpha^t) \cup \mathrm{dom}(F_\gamma^t) = \{k(i) \mid i < t\}.$$

The construction. By stages:

Stage 0) Define $\beta^0(x) \leftrightharpoons \alpha^0(\varphi(x))$ for every $x \in A$. Let $\beta^0(x) \leftrightharpoons \emptyset$, for every $x \in \overline{A}$. Let $F_\alpha^0 = \emptyset$ and $F_\gamma^0 = \emptyset$. Define $s(m,0) = 0$ for all m.

Move to next stage.

Stage $t+1$) For every $m \le t$ (with, say, $m = \langle p, i, j\rangle$) check whether the following conditions hold.

- Any number, marked by $\boxed{m'}$ with $m' \le m$, is less then $k(t)$;
- φ_p t–reduces β_{V_i} to β_{V_j};
- $k(t) \in V_i \setminus V_j$;
- $k(t) \in \mathrm{dom}(\varphi_p)$ and $\varphi_p(k(t)) \in V_j$.

If there is no $m \le t$ with those four properties then go directly to Procedure E below. Otherwise let $m_0 = \langle p_0, i_0, j_0\rangle$ be the least number satisfying these conditions. Act as follows.

- Remove all labels $\boxed{m}$ with $m > m_0$ from all numbers currently marked by these labels;
- Increase $s(m_0, t)$ (i.e. define $s(m_0, t+1) = s(m_0, t) + 1$);
- Put the label $\boxed{m_0}$ on $k(t)$ (notice that $k(t)$ is not currently marked by any label since we have already removed all markers $\boxed{m}$, with $m > m_0$);
- If $\varphi_{p_0}(k(t)) \notin A$ and $\varphi_{p_0}(k(t))$ is not marked by any markers then put the marker $\boxed{m_0}$ on $\varphi_{p_0}(k(t))$;
- If $\varphi_{p_0}(k(t))$ is marked by $\boxed{m_0}$ (it should be kept in mind that we have already executed the previous item) and $\varphi_{p_0}(k(t)) \in \mathrm{dom}(F_\gamma^t)$ then remove the pair $(\varphi_{p_0}(k(t)), F_\gamma^t(\varphi_{p_0}(k(t))))$from F_γ, find the least pseudopair $\langle y, s\rangle$ such that $\beta^t(\varphi_{p_0}(k(t))) \subseteq \alpha^s(y)$ and put the pair $(\varphi_{p_0}(k(t)), y)$ into F_α.

Go to Procedure E, below.

Procedure E. If $k(t)$ is not marked by any markers then put the pair $(k(t), 0)$ into F_α. If $k(t)$ is marked by $\boxed{m}$ with $m = \langle p, i, j\rangle$, consider the following possibilities:

- $k(t) \in V_i \setminus V_j$. Find the least y such that $y \notin \text{range}(F_\gamma^t)$ or $y \in \text{range}(F_\gamma^t)$ but for every x such that $F_\gamma^t(x) = y$ we have that x is not marked by $\boxed{m}$ and put the pair $(k(t), y)$ into F_γ;
- $k(t) \in V_j$. Put the pair $(k(t), 0)$ into F_α.

Let F_α^{t+1} and F_γ^{t+1} be the sets of all pairs which are currently in F_α and F_γ, respectively, as a consequence of the actions undertaken in the previous stages, and of the action so far undertaken at the current stage. Let $z > t$ be the least number such that for all x, y,

$$(x, y) \in F_\alpha^{t+1} \Rightarrow \beta^t(x) \subseteq \alpha^z(y)$$
$$(x, y) \in F_\gamma^{t+1} \Rightarrow \beta^t(x) \subseteq \gamma^z(y).$$

For every x, let

$$\beta^{t+1}(x) \rightleftharpoons \begin{cases} \alpha^{t+1}(\varphi(x)) & \text{if } x \in A, \\ \alpha^z(y) & \text{if } (x, y) \in F_\alpha^{t+1} \text{ for some } y \\ \gamma^z(u) & \text{if } (x, u) \in F_\gamma^{t+1} \text{ for some } u \\ \emptyset & \text{otherwise.} \end{cases}$$

For any m, if $s(m, t+1)$ has not yet been defined then let $s(m, t+1) = s(m, t)$. Go to the next stage.

The Verification. Let $\beta(x) = \bigcup_{t \in \mathbb{N}} \beta^t(x)$. The construction satisfies the following properties:

(1) All parameters in the construction are $\mathbf{0}^{(n+1)}$– computable and β is Σ^0_{n+2}– computable numbering of the family $\beta[\mathbb{N}]$.

(2) For every t any number can be labelled by at most one marker after stage t. Any marked number must be in $\overline{A}$.

(3) We can remove marker only if we put some smaller marker. If we put a marker $\boxed{m}$ then we increase $s(m, t)$. At every stage we put at most one marker and we mark by it at most two numbers.

(4) For every m, t, $s(m, t) \leq s(m, t+1) \leq s(m, t) + 1$.

(5) For every t, $\text{dom}(F_\alpha^{t+1}) \cap \text{dom}(F_\gamma^{t+1}) = \emptyset$, and

$$\text{dom}(F_\alpha^{t+1}) \cup \text{dom}(F_\gamma^{t+1}) = \{k(0), \ldots, k(t)\}.$$

(6) For every $x \notin A$ if, for some t_0, $x \in \mathrm{dom}(F_\alpha^{t_0})$ then for all $t \geq t_0$ $x \in \mathrm{dom}(F_\alpha^t)$ and $F_\alpha^t(x) = F_\alpha^{t_0}(x)$.

(7) For every $x \notin A$ if, for some t_0, $x \in \mathrm{dom}(F_\gamma^{t_0})$ then for all $t \geq t_0$ such that $x \in \mathrm{dom}(F_\gamma^t)$ we have $F_\gamma^t(x) = F_\gamma^{t_0}(x)$.

In view of properties (5)–(7) above we can introduce two functions: $F_\alpha = \lim_t F_\alpha^t$ and $F_\gamma = \lim_t F_\gamma^t$.

(8) $\mathrm{dom}(F_\alpha) \cap \mathrm{dom}(F_\gamma) = \emptyset$ and $\mathrm{dom}(F_\alpha) \cup \mathrm{dom}(F_\gamma) = \overline{A}$. F_α is a $\mathbf{0}^{(n+1)}$– computable partial function and for every $\mathbf{0}^{(n+1)}$– computably enumerable G such that $G \subseteq \mathrm{dom}(F_\gamma)$, the partial function $F_\gamma \upharpoonright G$ is $\mathbf{0}^{(n+1)}$– computable.

(9) For every $x \in \mathrm{dom}(F_\alpha)$, $\beta(x) = \alpha(F_\alpha(x))$, and for every $x \in \mathrm{dom}(F_\gamma)$, $\beta(x) = \gamma(F_\gamma(x))$.

(10) β is a Σ^0_{n+2}– computable numbering of $\mathcal{A}$.

We have already noticed in (1) that β is Σ^0_{n+2}– computable. By the definition of β we have that $\beta[A] = \mathcal{A}$. From (8) and (9) we have that $\beta[\overline{A}] \subseteq \mathcal{A}$.

(11) For every m (with, say, $m = \langle p, i, j \rangle$) if $\lim_t s(m,t) = \infty$ then $V_i \setminus V_j$ is infinite and $\beta_{V_i} \leqslant \beta_{V_j}$ via φ_p (in the sense of Lemma 2.2 (1b)).

Suppose that $\lim_t s(m,t) = \infty$, and $V_i \setminus V_j$ is finite or $\beta_{V_i} \not\leqslant \beta_{V_j}$ via φ_p. If $\lim_t s(m,t) = \infty$ then let $t_0 \geq m$ be a step such that

- if $V_i \setminus V_j$ is finite than $s(m,t_0) \geq \max(V_i \setminus V_j)$;
- if $V_i \not\subseteq \mathrm{dom}(\varphi_p)$ then there exists $x \in V_i$ such that $x \notin \mathrm{dom}(\varphi_p)$ and $x \leq s(m,t_0)$;
- if $\varphi_p(V_i) \not\subseteq V_j$ then there exists $x \in V_i$ such that $\varphi_p(x) \notin V_j$ and $x \leq s(m,t_0)$;
- if $\beta(x) \neq \beta(\varphi_p(x))$ for some $x \in V_i$ then $x \leq s(m,t_0)$ and for some $y \in (\beta(x) \cup \beta(\varphi_p(x))) \setminus (\beta(x) \cap \beta(\varphi_p(x)))$ we have $y \leq s(m,t_0)$ and $y \in \beta^{t_0}(x) \cup \beta^{t_0}(\varphi_p(x))$.

According to the definition of t-reducibility and in view of property (4) above, we have that for all $t \geq t_0$ φ_p does not t-reduces β_{V_i} to β_{V_j}. So for any such t, m does not satisfy the second condition of stage $t+1$ and we have $s(m,t+1) = s(m,t)$. A contradiction.

(12) For any $m = \langle p, i, j \rangle$ if m is the least number such that $V_i \setminus V_j$ is infinite and $\beta_{V_i} \leqslant \beta_{V_j}$ via φ_p then $\lim_t s(m,t) = \infty$.

From (11) we have that $\lim_t s(m',t) < \infty$ for all $m' < m$. Suppose that t_0 is so big that $m \leq t_0$ and $s(m',t_0) = \lim_t s(m',t_0)$ for all $m' < m$.

In order to obtain a contradiction suppose that $\lim_t s(m,t) = s(m,t_1)$ for some $t_1 \geq t_0$. In view of (3) after stage t_1 we do not put markers

$\boxed{m'}$ for $m' \leq m$. Let $t \geq t_1$ be a step such that $k(t) \in V_i \setminus V_j$, $k(t)$ is greater that any number marked by $\boxed{m'}$ with $m' \leq m$ at stage t_1, $\max(V_i \setminus V_j \cap [0,t]) > s(m,t_1)$ and for all $x \in V_i \cap [0, s(m,t_1)]$ we have $\beta^t(x) \cap [0, s(m,t_1)] = \beta(x) \cap [0, s(m,t_1)]$ and $\beta^t(\varphi_p(x)) \cap [0, s(m,t_1)] = \beta(\varphi_p(x)) \cap [0, s(m,t_1)]$. Than all four conditions of step $t+1$ are satisfied for m. They are not satisfied for any $m' < m$ since $s(m', t+1) = s(m', t)$, by choice of t_0. Then m is the least number such that $m \leq t$ and all conditions of stage $t+1$ hold for m; therefore $s(m, t+1) = s(m,t)+1$ and we have a contradiction with the choice of t_1.

(13) For every m, $\lim_t s(m,t) < \infty$.

Let $m = \langle p, i, j \rangle$ be the least number with $\lim_t s(m,t) = \infty$. By (11) we have that $V_i \setminus V_j$ is infinite and $\beta_{V_i} \leqslant \beta_{V_j}$ via φ_p.

Let t_0 be such that $s(m', t_0) = \lim_t s(m', t)$ for all $m' < m$. In view of (3) we may assume that $k(t_0)$ is greater than any number ever marked by $\boxed{m'}$ with $m' < m$. Let M be the set of all numbers marked by $\boxed{m'}$ with $m' < m$ at the end of stage t_0.

Let $G = \{k(t) : t \geq t_0 \text{ and } s(m, t+1) = s(m,t)+1\}$. Then G is $\mathbf{0}^{(n+1)}$– computably enumerable. For every $t \geq t_0$ if $k(t) \in G$ then at stage $t+1$ we put the label $\boxed{m}$ on $k(t)$ and put $k(t)$ into $\mathrm{dom}(F_\gamma^{t+1})$. After this, the marker $\boxed{m}$ can not be removed from $k(t)$ and we have $k(t) \in \mathrm{dom}(F_\gamma^s)$ for all $s > t$. So, $G \subseteq \mathrm{dom}(F_\gamma)$ and $G \subseteq V_i \setminus V_j$.

Notice that G contains almost all numbers in $\mathrm{dom}(F_\gamma)$ that are ever marked by $\boxed{m}$. Indeed if we put $\boxed{m}$ on some $x \notin G$ at some stage $t+1 > t_0$ then $x \in V_j$ and either $x < k(t)$ and $x \in \mathrm{dom}(F_\alpha^{t+1})$ or $x = k(t')$ for some $t' > t$ and we put x into $\mathrm{dom}(F_\alpha)$ at stage $t'+1$. Then from Procedure E of the construction we have that $F_\gamma[G]$ is a cofinite subset of $\mathbb{N}$.

On the other hand, for any $x \in G$ $\varphi_p(x) \in V_j$, and either $\varphi_p(x) \in A$ or $\varphi_p(x) \in \mathrm{dom}(F_\alpha)$ or $\varphi_p(x) \in M$. In all three cases we can easily compute an α–index of $\varphi_p(x)$ for any $x \in G$ with oracle $\mathbf{0}^{(n+1)}$ since M is finite, A is $\mathbf{0}^{(n+1)}$– computable and $\beta(x) = \alpha(\varphi^{-1}(x))$ for all $x \in A$.

Putting things together we have that there is a $\mathbf{0}^{(n+1)}$– computable function g, defined on $F_\gamma[G]$, such that $\gamma(x) = \alpha(g(x))$ for all $x \in F_\gamma[G]$. In order to compute $g(x)$ with oracle $\mathbf{0}^{(n+1)}$, in view of (8) and (9) we may enumerate G until we find $y \in G$ such that $F_\gamma(y) = x$ and then define $g(x)$ to be an α–index of $\varphi_p(y)$. Now defining $g(x)$ on $\mathbb{N} \setminus F_\gamma[G]$ in an obvious way we obtain that $\gamma \leqslant_{\mathbf{0}^{(n+1)}} \alpha$ which contradicts the choice of γ.

(14) All requirements $R_{p,i,j}$ are satisfied.

For a contradiction let $m = \langle p, i, j\rangle$ be the least number such that the requirement $R_{p,i,j}$ is not satisfied. Then $\lim_t s(m,t) = \infty$ by (12) and $\lim_t s(m,t) < \infty$ by (13).

Properties (10) and (14) imply that the theorem is true. □

The previous three theorems have been proved under the assumption that the numbering α has special properties of relativized non-universality. We do not know whether in any of these theorems this assumption may be replaced by the weaker assumption of simple non-universality. The following question therefore arises naturally:

Question 4. *Do the claims of the previous three theorems remain true if we just assume that the numbering α is not universal?*

3. Empty intervals. Minimal elements and minimal covers of Rogers semilattices

We now turn our attention to empty intervals of Rogers semilattices, i.e. to intervals of the form $(\alpha, \beta) = \emptyset$, with $\alpha < \beta$ (such a pair of numberings is usually described by saying that β *is a minimal cover of* α), and of the form $\hat{\hat{\alpha}} = \emptyset$, i.e. α is minimal. We say that β is a *strong minimal cover of* α if $\alpha < \beta$ and $\hat{\hat{\beta}} = \hat{\alpha}$.

3.1 Minimal elements of Rogers semilattices

As to the number of minimal elements of Rogers semilattices of Σ^0_{n+1}–computable families, the question has been completely settled for infinite families in the case $n \geq 1$ by Badaev and Goncharov, see Theorem 1.3, [4].

The case $n = 0$ is still open:

Question 5. *Let $\mathcal{A}$ be a Σ^0_1–computable family such that the Rogers semilattice $\mathcal{R}^0_1(\mathcal{A})$ contains at least two minimal elements. Does $\mathcal{R}^0_1(\mathcal{A})$ have infinitely many minimal elements?*

There has been however a considerable progress towards answering this question since it was first posed by Ershov in the late sixties.

We recall the following complete answer to this problem, due to Goncharov, [10], [11], if one just considers Friedberg or positive numberings:

Theorem 3.1 (S.Goncharov). *For every n there exists a Σ^0_1–computable family $\mathcal{A}$ such that $\mathcal{R}^0_1(\mathcal{A})$ has exactly n degrees of Friedberg (positive) numberings.*

Proof. See [10], [11] □

3.2 Covers and strong minimal covers in Rogers semilattices

The following lemma was the starting point of an approach pursued by S. Badaev and S. Goncharov's, [3], towards searching for reasonable conditions under which a Σ^0_{n+2}–computable numbering has minimal covers.

Lemma 3.1. *Let β be any Σ^0_{n+2}–computable numbering of a family $\mathcal{A}$. If for some maximal set M, β_M is a numbering of the whole family $\mathcal{A}$ and $\beta \nleq \beta_M$ then β is a minimal cover of β_M.*

Proof. Immediate by Lemma 2.2(5), and by maximality of M. □

An evident way to construct a minimal cover β of a given numbering $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ is the following. Choose any maximal set M and let f be a computable bijection from $\mathbb{N}$ onto M. For every $x \in M$, define $\beta(x) = \alpha f^{-1}(x)$. It remains to define β on $\overline{M}$ in such a way that $\beta \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ and $\beta \nleq \alpha$.

The following theorem singles out some conditions under which this can be suitably done.

Theorem 3.2. *Let $\mathcal{A} \subseteq \Sigma^0_{n+2}$ be any Σ^0_{n+2}–computable family, and let α be a Σ^0_{n+2}–computable numbering of $\mathcal{A}$. If either*

(1) there exists a proper subfamily $\mathcal{A}_0$ of $\mathcal{A}$ whose index set $\alpha^{-1}[\mathcal{A}_0]$ is $\mathbf{0}^{(n+1)}$–computable, or

(2) the numbering α is not $\mathbf{0}'$–universal in $\mathrm{Com}^0_{n+2}(\mathcal{A})$, or

(3) there exists a $\mathbf{0}^{(n+1)}$–computable function f such that $\alpha(x) \neq \alpha(f(x))$ for every x,

then α has a minimal cover.

Proof. Let M be a maximal set, with $\overline{M} = \{m_0 < m_1 < m_2 \ldots\}$, and let f be a one-to-one computable function with range$(f) = M$. If for every $x \in M$ we define $\beta(x) \leftrightharpoons \alpha(f^{-1}(x))$ then clearly $\alpha \equiv \beta_M$. Let us now define the value $\beta(x)$ for an arbitrary $x \in \overline{M}$ according to which of the assumptions (1), (2) or (3), in the statement of the theorem, holds. Let $x = m_s$ for some $s \in \mathbb{N}$.

(1) Fix two sets A, B such that $A \in \mathcal{A}_0$ and $B \in \mathcal{A} \setminus \mathcal{A}_0$. If $m_s \in \mathrm{dom}(\varphi_s)$ and $\varphi_s(m_s) \in \alpha^{-1}(\mathcal{A}_0)$ then define $\beta(m_s) = B$, otherwise let $\beta(m_s) = A$. Since the sets $\overline{M}$, $\alpha^{-1}(\mathcal{A}_0)$, and $\{s \mid m_s \in \mathrm{dom}(\varphi_s)\}$ are $\mathbf{0}^{(n+1)}$–computable it follows that β is a Σ^0_{n+2}–computable numbering of $\mathcal{A}$.

To apply Lemma 3.1, it is sufficient to show that $\beta \not\leqslant \alpha$. Suppose $\beta(x) = \alpha(\varphi_s(x))$, $x \in \mathbb{N}$, for some computable function φ_s. Then $m_s \in \mathrm{dom}(\varphi_s)$. If $\varphi_s(m_s) \in \alpha^{-1}(\mathcal{A}_0)$ then $\beta(m_s) = B$. On the other hand, $\alpha(\varphi_s(m_s)) \in \mathcal{A}_0$, a contradiction. Similarly, if $\varphi_s(m_s) \notin \alpha^{-1}[\mathcal{A}_0]$ then $\beta(m_s) = A$, and this is in contradiction with $\alpha(\varphi_s(m_s)) \notin \mathcal{A}_0$.

(2) Since the numbering α is not $\mathbf{0}'$–universal in $\mathrm{Com}^0_{n+2}(\mathcal{A})$, there exists $\gamma \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ such that $\gamma \not\leqslant_{\mathbf{0}'} \alpha$.

Define $\beta(m_s) = \gamma(s)$. By Theorem 1.2, [4], $\beta \in \mathrm{Com}^0_{n+2}(\mathcal{A})$. Since the mapping $\lambda s\, m_s$ is $\mathbf{0}'$–computable, it follows that $\gamma \leqslant_{\mathbf{0}'} \beta$. Therefore, $\beta \not\leqslant \alpha$, otherwise we would obtain $\gamma \leqslant_{\mathbf{0}'} \alpha$.

(3) Let f be a $\mathbf{0}^{(n+1)}$–computable function, such that $\alpha(x) \neq \alpha(f(x))$ for every x. Define

$$\beta(m_s) = \begin{cases} \alpha(0) & \text{if } m_s \notin \mathrm{dom}(\varphi_s), \\ \alpha(f(\varphi_s(m_s))) & \text{if } m_s \in \mathrm{dom}(\varphi_s). \end{cases}$$

Again by Theorem 1.2, [4], $\beta \in \mathrm{Com}^0_{n+2}(\mathcal{A})$. Let us prove that $\beta \not\leqslant \alpha$. By contradiction, assume that β is reducible to α via a computable function φ_s. Then $\beta(x) = \alpha(\varphi_s(x))$ for all x. By definition of β, $\beta(m_s) = \alpha(f(\varphi_s(m_s)))$. Therefore, $\alpha(\varphi_s(m_s)) = \alpha(f(\varphi_s(m_s)))$, i.e. $\varphi_s(m_s)$ is a fixed point of the function f modulo the numbering α, contradicting assumption (3). □

Remark 3.1. The particular case of condition (1) in Theorem 3.2, when $\mathcal{A}_0$ consists of a single set A, was considered by S.Badaev, S.Goncharov in [3]. Conditions (2) and (3) were suggested by S.Badaev and S.Podzorov, [5].

Question 6. *Let $\mathcal{A}$ be any Σ^0_{n+2}–computable family, and assume that $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ is not universal in $\mathrm{Com}^0_{n+2}(\mathcal{A})$. How many (up to equivalence) minimal covers can α have?*

The next theorem shows another approach to constructing minimal covers based on the special minimal numberings considered in Theorem 1.3, [4].

Theorem 3.3. *If $\mathcal{A}$ is an infinite Σ^0_{n+2}–computable family and $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ is not $\mathbf{0}'$- universal in $\mathrm{Com}^0_{n+2}(\mathcal{A})$, then there exist infinitely many minimal covers (up to equivalence) of α.*

Proof. Let $\mathcal{A}$ be an infinite Σ^0_{n+2}–computable family, and let $\alpha, \beta \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ be numberings such that $\beta \not\leqslant_{\mathbf{0}'} \alpha$.

Let M be a maximal set and let $M' \leftrightharpoons \{2x+1 \mid x \in M\}$. For every $A \in \mathcal{A}$, let us consider the minimal numbering $\beta_{M,A}$ constructed in the proof of Theorem 1.3, [4]. We will now check that the numberings $\alpha\oplus\beta_{M,A}$, $A\in\mathcal{A}$, are minimal covers of β and that $\alpha\oplus\beta_{M,A}\nleqslant\alpha\oplus\beta_{M,B}$ if $A\neq B$.

Since $\beta_{M,A}\equiv_{\mathbf{0}'}\beta$, it follows that $\alpha<\alpha\oplus\beta_{M,A}$, otherwise we would obtain a contradiction with $\beta\nleqslant_{\mathbf{0}'}\alpha$. Let $\gamma\in\mathrm{Com}^0_{n+2}(\mathcal{A})$ be any numbering such that $\alpha\leqslant\gamma\leqslant\alpha\oplus\beta_{M,A}$. If C_1 stands for the set all even numbers then $\alpha\equiv(\alpha\oplus\beta_{M,A})_{C_1}$. By Lemma 2.2(5), $\gamma\equiv(\alpha\oplus\beta_{M,A})_{C_2}$ for some c.e. set $C_2\supseteq C_1$.

By construction of the numbering $\beta_{M,A}$, we have $(\alpha\oplus\beta_{M,A})(x)=A$ for every $x\in M'$. Therefore, by applying Lemma 2.2(8) if necessary, we may suppose that $C_2\supseteq M'$. By Lemma 2.2(6),

$$\gamma\equiv(\alpha\oplus\beta_{M,A})_{C_1}\oplus(\alpha\oplus\beta_{M,A})_{C_2\setminus C_1}.$$

If $(\alpha\oplus\beta_{M,A})[C_2\setminus C_1]$ is a finite family then $(C_2\setminus C_1)\setminus M'$ is a finite set since M is a maximal set. Applying Lemma 2.2(8,9), we have $(\alpha\oplus\beta_{M,A})_{C_1}\equiv(\alpha\oplus\beta_{M,A})_{C_2}$, and, therefore, $\gamma\equiv\alpha$. If $(\alpha\oplus\beta_{M,A})[C_2\setminus C_1]$ is an infinite family then, by maximality of M, C_2 is a co-finite set and, hence, by Lemma 2.2(9), $\gamma\equiv\alpha\oplus\beta_{M,A}$. Thus, for every $A\in\mathcal{A}$, the numbering $\alpha\oplus\beta_{M,A}$ is a minimal cover of α.

It remains to show that $\alpha\oplus\beta_{M,A}\nleqslant\alpha\oplus\beta_{M,B}$ if $A\neq B$. We prove this by contradiction. Assume that $A\neq B$ and $\alpha\oplus\beta_{M,A}\leqslant\alpha\oplus\beta_{M,B}$. Then $\beta_{M,A}\leqslant\alpha\oplus\beta_{M,B}$. By Lemma 2.2(4), $\beta_{M,A}\equiv(\alpha\oplus\beta_{M,B})_D$ for some c.e. set D. Let D_1 and D_2 consist of the even numbers of D and the odd numbers of D, respectively. By Lemma 2.2(6),

$$\beta_{M,A}\equiv(\alpha\oplus\beta_{M,B})_{D_1}\oplus(\alpha\oplus\beta_{M,B})_{D_2}.$$

Just the same arguments which we have used for the set C_2 above, can be applied to the set D_2 as well. So, we may suppose that $M'\subseteq D_2$. If the family $(\alpha\oplus\beta_{M,B})[D_2\setminus M']$ is finite then the set $D_2\setminus M'$ is finite. Then by Lemma 2.2(8,9), $(\alpha\oplus\beta_{M,B})_D\equiv(\alpha\oplus\beta_{M,B})_{D_1}$. Therefore, $\beta_{M,A}\equiv\alpha_{D'}$ with $D'=\{x\mid 2x\in D_1\}$. Hence, $\beta_{M,A}\leqslant\alpha$. Since $\beta_{M,A}\equiv_{\mathbf{0}'}\beta$, we obtain a contradiction with $\beta\nleqslant_{\mathbf{0}'}\alpha$.

If the family $(\alpha\oplus\beta_{M,B})[D_2\setminus M']$ is infinite then by maximality of M, the set D_2 is co-finite with respect to the set of all odd numbers. Then Lemma 2.2(6,9) implies that

$$\beta_{M,A}\equiv(\alpha\oplus\beta_{M,B})_D\equiv\ \alpha_{D'}\oplus\beta_{M,B}.$$

This is in contradiction with $\beta_{M,B}\nleqslant\beta_{M,A}$.

Thus, $\alpha\oplus\beta_{M,A}\nleqslant\alpha\oplus\beta_{M,B}$ if $A\neq B$. □

A useful condition guaranteeing the existence of strong minimal covers is provided by the following theorem.

Theorem 3.4 (S.Badaev, S.Podzorov). *Let $\mathcal{A}$ be any Σ^0_{n+2}– computable family, and let $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$. If there exist a subfamily $\mathcal{A}_0 \subseteq \mathcal{A}$, a numbering $\gamma \in \mathrm{Com}^0_{n+2}(\mathcal{A}_0)$ with $\gamma \nleq \alpha$, and a computable function f such that $\alpha(f(x)) \subseteq \gamma(x)$, for all x, then α has a strong minimal cover.*

Proof. See [5]. Notice that the proof is based on Lachlan's construction using Chinese boxes, [16]. □

Corollary 3.4.1. *If $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ and there exists a subfamily $\mathcal{A}_0 \subseteq \mathcal{A}$ such that $\mathcal{A}_0$ has least element under inclusion and there is a numbering $\gamma \in \mathrm{Com}^0_{n+2}(\mathcal{A}_0)$ with $\gamma \nleq \alpha$, then α has a strong minimal cover.*

Proof. Let $\mathcal{A}, \mathcal{A}_0, \alpha, \gamma$ be as in the statement of the corollary. Let $\bot \in \mathcal{A}_0$ be the least element of $\mathcal{A}_0$ under inclusion, and let a be such that $\alpha(a) = \bot$. Define $f(x) = a$, for every x. Then $\alpha(f(x)) \subseteq \gamma(x)$, for every x. Thus the hypotheses of Theorem 3.4 hold, and α has a strong minimal cover. □

Corollary 3.4.2. *If a Σ^0_{n+2}–family $\mathcal{A}$ has least element with respect to inclusion, then every non–universal numbering $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ has a strong minimal cover.*

Proof. Immediate by the previous corollary, taking $\mathcal{A}_0 = \mathcal{A}$. If α is not universal in $\mathrm{Com}^0_{n+2}(\mathcal{A})$, then there exists a numbering $\gamma \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ such that $\gamma \nleq \alpha$. □

Corollary 3.4.3. *For every finite family $\mathcal{A} \subseteq \Sigma^0_{n+2}$, for every numbering $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$, if α is not universal in $\mathrm{Com}^0_{n+2}(\mathcal{A})$ then α has a minimal cover.*

Proof. If $\mathcal{A}$ contains $\bot$, the least set under inclusion, then α has a minimal cover by the previous corollary.

Suppose that $\mathcal{A}$ has no least element under inclusion. Let $\mathcal{A} = \{A_0, A_1, \dots, A_k\}$. It can be easily shown (see [9] for details) that there exists a family $\mathcal{F} = \{F_0, F_1, \dots, F_k\}$ of finite sets such that for all $i, j \leq k$

$$F_i \subseteq F_j \Leftrightarrow F_i \subseteq A_j \Leftrightarrow A_i \subseteq A_j.$$

Without loss of generality we may assume that $A_0, A_1, \dots, A_m$, for some $m \leq k$, are all the minimal elements of $\mathcal{A}$ with respect to inclusion and $\alpha(x) = A_x$ for $x \leq m$. Let σ be a permutation of the set $\{0, 1, \dots, m\}$ with no fixed points. Let $\{\alpha^t(x)\}_{t \in \mathbb{N}}$ be a $\mathbf{0}^{(n+1)}$–computable sequence of finite sets such that for all x, t

$$\alpha^t(x) \subseteq \alpha^{t+1}(x); \qquad \alpha(x) = \bigcup_{t \in \mathbb{N}} \alpha^t(x).$$

It is easy to see that the functions

$$t_0(x) = \mu t(\exists i \leq m(F_i \subseteq \alpha^t(x))),$$
$$f(x) = \mu i \leq m(F_i \subseteq \alpha^{t_0(x)}(x))$$

are $\mathbf{0}^{(n+1)}-$computable and $\alpha(\sigma(f(x))) \neq \alpha(x)$ for all x. Finally we can apply Theorem 3.2(3) for the function $\sigma \circ f$. □

Question 7. *Let* $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ *be a numbering which is not universal in* $\mathrm{Com}^0_{n+2}(\mathcal{A})$*. How many strong minimal covers, up to equivalence, can the numbering* α *have?*

Question 8. *Let* $\mathcal{A}$ *be any finite family of* $\Sigma^0_{n+2}-$*sets, and let* $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ *be not universal in* $\mathrm{Com}^0_{n+2}(\mathcal{A})$*. Does* α *have a strong minimal cover?*

References

[1] S.A. Badaev, *On incomparable enumerations.* Sibirsk. Mat. Zh., 1974, vol. 15, no. 4, pp. 730–738 (Russian); Siberian Math. J. 1974, vol. 15, no. 4, pp. 519–524 (English translation).

[2] S.A. Badaev, S.S. Goncharov, *Theory of numberings: open problems.* In *Computability Theory and its Applications.* P. Cholak, S. Lempp, M. Lerman and R. Shore eds.—Contemporary Mathematics, American Mathematical Society, 2000, vol. 257, Providence, pp. 23-38.

[3] S.A. Badaev, S.S. Goncharov, *On Rogers semilattices of families of arithmetical sets.* Algebra i Logika, 2001, vol. 40, no. 5, pp. 507–522 (Russian); Algebra and Logic, 2001, vol. 40, no. 5, pp. 283–291 (English translation).

[4] S.A. Badaev, S.S. Goncharov, A. Sorbi, *Arithmetical numberings: completeness and universality.* This volume.

[5] S.A. Badaev, S.Yu. Podzorov, *Minimal covers in Rogers semilattices of* Σ^0_n- *computable numberings.* Siberian Math. J., to appear.

[6] Yu.L. Ershov, *Numerations of families of general recursive functions*, Sibirsk. Mat. Zh., 1967, vol. 8, no. 5., pp. 1015-1025 (Russian); Sib. Math. J., 1967, vol. 8, no. 5., pp. 771-778 (English translation).

[7] Yu.L. Ershov, *Hyperhypersimple* $m-$ *degrees.* Algebra i Logika, 1969, vol. 8, pp. 523–552 (Russian); Algebra and Logic, 1969, vol. 8, pp. 298–315 (English translation).

[8] Yu.L. Ershov, I.A. Lavrov, *The upper semilattice* $L(\gamma)$. Algebra i Logika, 1973, vol. 12, no. 2, pp. 167–189 (Russian); Algebra and Logic, 1973, vol. 12, no. 2, pp. 93–106 (English translation).

[9] Yu.L. Ershov, *Theory of Numberings.* Nauka, Moscow, 1977 (Russian).

[10] S.S. Goncharov, *Computable single-valued numerations.* Algebra i Logika, 1980, vol. 19, no. 5, pp. 507–551 (Russian); Algebra and Logic, 1980, vol. 19, no. 5, pp. 325–356 (English translation).

[11] S.S. Goncharov, *Positive computable enumerations.* Dokl. Akad. Nauk, 1993, vol. 332, no. 2, pp. 142–143 (Russian); Russian Acad. Sci. Dokl. Math., 1994, vol. 48, no. 2, pp. 268–270 (English translation).

[12] S.S. Goncharov, A. Sorbi, *Generalized computable numerations and non-trivial Rogers semilattices.* Algebra i Logika, 1997, vol. 36, no. 6, pp. 621–641 (Russian); Algebra and Logic, 1997, vol. 36, no. 6, pp. 359–369 (English translation).

[13] L. Harrington, A. Nies, *Coding in the lattice of enumerable sets.* Adv. Math., to appear

[14] A.B. Khutoretsky, *On the cardinality of the upper semilattice of computable enumerations*, Algebra i Logika, 1971, vol. 10, no. 5, pp. 561–569 (Russian); Algebra and Logic, 1971, vol. 10, no. 5, pp. 348–352 (English translation).

[15] A.H. Lachlan, *On the lattice of recursively enumerable sets.* Trans. Amer. Math. Soc., 1968, vol. 130, pp. 1–37.

[16] A.H. Lachlan, *Two theorems on many–one degrees of recursively enumerable sets.* Algebra i Logika, 1972, vol. 11, no. 2, pp. 216–229 (Russian); Algebra and Logic, 1972, vol. 11, no. 2, pp. 127–132 (English translation).

[17] A.H. Lachlan, *Recursively enumerable many–one degrees.* Algebra i Logika, 1972, vol. 11, no. 3, pp. 326–358 (Russian); Algebra and Logic, 1972, vol. 11, no. 3, pp. 186–202 (English translation).

[18] A.I. Mal'tsev, *Algorithms and Recursive Functions.* Nauka, Moscow, 1965 (Russian) ; Wolters-Noordoff Publishing, Groningen, 1970 (English translation).

[19] S.Yu. Podzorov, *Initial segments in Rogers semilattices of* Σ^0_n *–computable numberings,* Algebra and Logic, to appear.

[20] H. Rogers, *Gödel numberings of partial computable functions.* J. Symb. Logic, 1958, vol. 23, no. 3, pp.49–57.

[21] H. Rogers, Jr. *Theory of Recursive Functions and Effective Computability.* McGraw-Hill, New York, 1967.

[22] V.L. Selivanov, *Two theorems on computable enumerations*, Algebra i Logika, 1976, vol. 15, no. 4, pp. 470–484 (Russian); Algebra and Logic, 1976, vol. 15, no. 4, pp. 297–306 (English translation).

[23] R.I. Soare *Recursively Enumerable Sets and Degrees.* Springer-Verlag, Berlin Heidelberg, 1987.

[24] V.V. V'jugin, *On some examples of upper semilattices of computable enumerations.* Algebra i Logika, 1973, vol. 12, no. 5, pp. 512–529 (Russian); Algebra and Logic, 1973, vol. 12, no. 5, pp. 277–286 (English translation).

[25] V.V. V'jugin, *On upper semilattices of numberings.* Dokl. Akad. Nauk SSSR, 1974, vol. 217, no. 4, pp. 749–751 (Russian); Soviet Math. Dokl., 1974, vol. 15, pp. 1110–1113 (English translation).