COMPLETENESS AND UNIVERSALITY OF ARITHMETICAL NUMBERINGS

Serikzhan Badaev*
Kazakh National University, Almaty, Kazakhstan
badaev@math.kz

Sergey Goncharov†
Institute of Discrete Mathematics and Informatics, Novosibirsk State University, Novosibirsk, Russia
gonchar@math.nsu.ru

Andrea Sorbi‡
University of Siena, Siena, Italy
sorbi@unisi.it

Abstract We investigate completeness and universality notions, relative to different oracles, and the interconnection between these notions, with applications to arithmetical numberings. We prove that principal numberings are complete; completeness is independent of the oracle; the degree of any incomplete numbering is meet-reducible, uniformly complete numberings exist. We completely characterize which finite arithmetical families have a universal numbering.

Keywords: Numberings; arithmetical hierarchy; complete numberings; computable numberings; computable function; oracle; universal numberings.

*Partial funding provided by grant INTAS-RFBR *Computability and Models* no. 97-139 and by grant PICS-541-Kazakhstan.
†Partial funding provided by grant INTAS-RFBR *Computability and Models* no. 97-01-71001; and by grant of "University of Russia".
‡Partial funding provided by grant INTAS-RFBR *Computability and Models* no. 97-139; work done under the auspices of GNSAGA.

1. Basic notions and preliminaries

As is known, the s_n^m -theorem and the fixed point property hold for the universal numbering of the family of all partial computable functions, and these properties play an important role in the development of computability theory. The existence of universal numberings for different families of constructive objects as well as the study of the properties of such numberings have attracted the attention of many researchers (see, for instance, the monograph of Yu.L. Ershov [6] and the papers of A. Lachlan [8], A.I. Malt'sev [11] and H. Rogers [14]).

The notion of a complete numbering introduced by A.I. Malt'sev [9] became a very productive notion closely connected with both the fixed point property and the universality of numberings. Our paper is devoted to the study of these fundamental notions of completeness and universality for the class of arithmetical numberings.

For unexplained terminology and notations relative to computability theory, our main references are the textbooks of A.I. Mal'tsev [12], H. Rogers [15] and R. Soare [16]. For the main concepts and notions of the theory of numberings we refer to the book of Yu.L. Ershov [6].

A surjective mapping α of the set $\mathbb{N}$ of natural numbers onto a nonempty set $\mathcal{A}$ is called a *numbering* of $\mathcal{A}$. The collections of all numberings of $\mathcal{A}$ will be denoted by $\mathrm{Num}(\mathcal{A})$. Suppose that $\mathcal{A}$ is a family of objects that admit constructive descriptions. By this we mean that one can define a language $\mathcal{L}$ (henceforth identified with a corresponding set of "well formed formulas") and an interpretation of (fragments of) this language via an onto partial mapping $i\colon\ \mathcal{L} \longrightarrow \mathcal{A}$. For any object $a \in \mathcal{A}$, each formula Φ of $\mathcal{L}$ such that $i(\Phi) = a$ is interpreted as a "description" of a. Suppose further that $G\colon\ \mathbb{N} \longrightarrow \mathcal{L}$ is a Gödel numbering.

Following [7], we propose:

Definition 1.1. A numbering α of $\mathcal{A}$ is called *computable in* $\mathcal{L}$ *with respect to* i if there exists a computable mapping f such that $\alpha = i \circ G \circ f$.

An example to illustrate the previous definition may be in order at this point. A Turing machine can be viewed as a finite sequence of symbols from a certain alphabet Σ. Thus the set of Turing machines can be identified with a particular "set of formulas" $\mathcal{L} \subseteq \Sigma^*$. Let i be the mapping associating to each $M \in \mathcal{L}$ the partial function $i(M)$ computed by M, and let G be the standard Gödel numbering of the Turing machines. Let $G(x) = M_x$. Then a numbering α of the set of partial computable functions is computable in $\mathcal{L}$ with respect to i according to Definition 1.1 if for some computable function f, one has $\alpha(x) = i(M_{f(x)})$. This coincides with the familiar notion of a computable numbering of the family of partial computable functions, as studied in the classical theory of numberings, see e.g. [6].

It is immediate to see that Definition 1.1 does not depend on the choice of the Gödel numbering G. Hence, via identification of $\mathcal{L}$ with $\mathbb{N}$ through some fixed Gödel numbering, the above definition states that α is computable if there is some computable function f from $\mathbb{N}$ to $\mathcal{L}$ such that $\alpha = i \circ f$.

Definition 1.1 has a wide scope of applications, based upon suitable choices of $\mathcal{L}$ and i. Throughout this paper we will confine ourselves to families of arithmetical sets and relations. As language $\mathcal{L}$ we take in this case the collection of arithmetical first–order formulas, and i will be a mapping associating to each formula the corresponding set or relation defined by that formula in the standard model $\mathfrak{N}$ of Peano arithmetic. Restriction to this particular case leads to the following definition, see [7]:

Definition 1.2. A numbering α of a family $\mathcal{A}$ of Σ^0_{n+1}–sets, with $n \geq 0$, is called Σ^0_{n+1}– *computable* if there exists a computable function f from $\mathbb{N}$ into the Σ_{n+1}–formulas of Peano arithmetic such that, for every m,

$$\alpha(m) = \{x \in \mathbb{N} \mid \mathfrak{N} \models f(m)(\overline{x})\}$$

(where the symbol $\overline{x}$ stands for the numeral for x). The set of Σ^0_{n+1}–computable numberings of $\mathcal{A}$ will be denoted by $\mathrm{Com}^0_{n+1}(\mathcal{A})$.

In other words, a Σ^0_{n+1}–computable numbering is just a computable numbering in $\mathcal{L}$ with respect to i – in the sense of Definition 1.1, where $\mathcal{L}$ and i are as above specified – of a family of Σ^0_{n+1}–sets.

Computable numberings of families of sets which are first–order definable in the standard model of Peano arithmetic are called *arithmetical numberings*. A family $\mathcal{A}$ for which $\mathrm{Com}^0_{n+1}(\mathcal{A}) \neq \emptyset$ will be called Σ^0_{n+1}–*computable*.

We now revise some of the basic definitions of the theory of numberings. In order to avoid trivial cases, we will always deal in the following with families $\mathcal{A}$ that contain at least two elements, i.e. $|\mathcal{A}| > 1$.

We will use the symbols $\langle \cdot, \cdot \rangle$ to denote the usual computable pairing function between $\mathbb{N}^2$ and $\mathbb{N}$, with computable projections π_0, π_1 (hence $\pi_i(\langle x_0, x_1 \rangle) = x_i$ and $\langle \pi_0(x), \pi_1(x) \rangle = x$). Likewise, the symbol $\langle \cdot, \cdot, \cdot \rangle$ will denote the usual computable bijection of $\mathbb{N}^3$ onto $\mathbb{N}$, with computable projections $\pi^3_0, \pi^3_1, \pi^3_2$, etc.

Definition 1.3. Let $\alpha \in \mathrm{Num}(\mathcal{A})$, $\beta \in \mathrm{Num}(\mathcal{B})$, where $\mathcal{A}$ and $\mathcal{B}$ are any given families. We say that α is *reducible* to β (in symbols, $\alpha \leqslant \beta$) if $\alpha = \beta \circ f$ for some computable function f. If $\alpha \leqslant \beta$ and $\beta \leqslant \alpha$ then $\mathcal{A} = \mathcal{B}$

and we say that α and β are *equivalent* (in symbols, $\alpha \equiv \beta$)numberings of $\mathcal{A}$. The equivalence class of a numbering α is called the *degree* of α, denoted by $\deg(\alpha)$. The set of all degrees of the elements of $\mathrm{Num}(\mathcal{A})$ will be denoted by $L(\mathcal{A})$.

It is straightforward to see that this notion of reducibility on numberings of $\mathcal{A}$ is a pre-ordering relation on $\mathrm{Num}(\mathcal{A})$, and induces a partial ordering relation (still denoted by the symbol $\leqslant$) on $L(\mathcal{A})$, where $\deg(\alpha) \leqslant \deg(\beta)$ if $\alpha \leqslant \beta$. Further, $L(\mathcal{A})$ is an upper semilattice: If $\alpha, \beta \in \mathrm{Num}(\mathcal{A})$ then the least upper bound of $\deg(\alpha)$ and $\deg(\beta)$ is easily seen to be $\deg(\alpha \oplus \beta)$ where

$$\alpha \oplus \beta(x) \risingdotseq \begin{cases} \alpha(y) & \text{if } x = 2y, \\ \beta(y) & \text{if } x = 2y + 1. \end{cases}$$

The numbering $\alpha \oplus \beta$ is called the *join of* α and β.

We are interested in the substructure of $L(\mathcal{A})$ obtained by restricting the universe to the degrees of elements of $\mathrm{Com}^0_{n+1}(\mathcal{A})$, if $\mathcal{A}$ is $\Sigma^0_{n+1}-$ computable.

Definition 1.4. Let $\mathcal{R}^0_{n+1}(\mathcal{A}) \risingdotseq \langle \mathrm{Com}^0_{n+1}(\mathcal{A})/_{\equiv}, \leqslant \rangle$.

It is easy to see that $\mathcal{R}^0_{n+1}(\mathcal{A})$ is an upper semilattice with the least upper bound operation still given by $\deg(\alpha) \vee \deg(\beta) = \deg(\alpha \oplus \beta)$. In fact $\mathcal{R}^0_{n+1}(\mathcal{A})$ is an ideal of $L(\mathcal{A})$. $\mathcal{R}^0_{n+1}(\mathcal{A})$ is called the *Rogers semilattice* of the family $\mathcal{A}$.

1.1 Relativized reducibility

The above defined reducibility provides a way of "translating" numberings into other numberings. Different kinds of "translations" can be devised, though. For instance, when dealing with computable numberings of c.e. sets, it is often useful to compare numberings with respect to the reducibility provided by $\mathbf{0}'-$computable functions, i.e. α is reducible to β if there exists a $\mathbf{0}'-$computable function f such that $\alpha = \beta \circ f$. We now provide a general setting for some extended notions of reducibility between numberings.

Given a set X, let the corresponding boldface letter $\mathbf{X}$ denote the Turing degree of X, i.e. $\mathbf{X} = \deg_T(X)$. We will follow the usual notation $\mathbf{0}^{(n)} = \deg_T(\emptyset^{(n)})$, and we let $\emptyset^{(0)} = \emptyset$ and $\mathbf{0}^{(0)} = \mathbf{0}$. Since a partial function is $X-$computable if and only if is $Y-$computable for *all* $Y \in \mathbf{X}$, for a given Turing degree $\mathbf{X}$ we will say that a partial function φ is said to be $\mathbf{X}-$*computable*, if φ is $X-$computable for some $X \in \mathbf{X}$. Consequently, for every set X we will often write $\varphi^{\mathbf{X}}_e$ to denote φ^X_e, in the standard listing of the partial functions which are computable with oracle X.

Well known notions from classical computability theory are relativized in a obvious way. For instance, an $\mathbf{X}$– *maximal* set is a set M which is c.e. in X, and for every set $V \supseteq M$ which is c.e. in X we have that either $V \setminus M$ is finite or $\mathbb{N} \setminus V$ is finite. In these cases, one gets the classical notion by taking $X = \emptyset$.

Definition 1.5. Let $\alpha \in \mathrm{Num}(\mathcal{A})$, $\beta \in \mathrm{Num}(\mathcal{A})$, and let $X \subseteq \mathbb{N}$. We say that α is $\mathbf{X}$– *reducible* to β (in symbols, $\alpha \leqslant_{\mathbf{X}} \beta$) if there exists an $\mathbf{X}$–computable function f such that $\alpha = \beta \circ f$. We say that the numberings α and β of $\mathcal{A}$ are $\mathbf{X}$– *equivalent* (in symbols, $\alpha \equiv_{\mathbf{X}} \beta$) if $\alpha \leqslant_{\mathbf{X}} \beta$ and $\beta \leqslant_{\mathbf{X}} \alpha$. The set $\mathrm{Num}(\mathcal{A})$ of numberings of $\mathcal{A}$ is partitioned by $\equiv_{\mathbf{X}}$ into equivalence classes. The equivalence class of a numbering $\alpha \in \mathrm{Num}(\mathcal{A})$ under $\equiv_{\mathbf{X}}$ will be denoted by $\deg_{\mathbf{X}}(\alpha)$. The set $L^{\mathbf{X}}(\mathcal{A}) \leftrightharpoons \mathrm{Num}(\mathcal{A})/_{\equiv_{\mathbf{X}}}$ is a partially ordered set, in fact an upper semilattice, with the least upper bound operation still induced by the join of numberings. Its partial ordering relation will still be denoted by the symbol $\leqslant_{\mathbf{X}}$.

Restriction to $\mathrm{Com}^0_{n+1}(\mathcal{A})$ gives:

Definition 1.6. Let $\mathcal{A}$ be a Σ^0_{n+1}–computable family, and let X be any set. Define

$$\mathcal{R}^{0,\mathbf{X}}_{n+1}(\mathcal{A}) \leftrightharpoons \langle \mathrm{Com}^0_{n+1}(\mathcal{A})/\equiv_{\mathbf{X}}, \leqslant_{\mathbf{X}} \rangle.$$

The following definitions provide suitable relativizations of two important notions of the theory of numberings.

Definition 1.7. We say that a numbering α of a family $\mathcal{A}$ is $\mathbf{X}$– *universal* in $\mathrm{Com}^0_{n+1}(\mathcal{A})$ if $\alpha \in \mathrm{Com}^0_{n+1}(\mathcal{A})$ and $\beta \leqslant_{\mathbf{X}} \alpha$ for all numberings $\beta \in \mathrm{Com}^0_{n+1}(\mathcal{A})$.

Remark 1.1. The notion of a *principal numbering of* a family $\mathcal{A}$, [6], coincides with the notion of a numbering which is $\mathbf{0}$– universal in $\mathrm{Com}^0_1(\mathcal{A})$.

Definition 1.8. A numbering $\alpha \in \mathrm{Com}^0_{n+1}(\mathcal{A})$ is said to be $\mathbf{X}$– *minimal* if there exists no $\beta \in \mathrm{Com}^0_{n+1}(\mathcal{A})$ such that $\beta <_{\mathbf{X}} \alpha$. In other words, $\deg_{\mathbf{X}}(\alpha)$ is a minimal element in the poset $\mathcal{R}^{0,\mathbf{X}}_{n+1}(\mathcal{A})$.

We shall usually omit to mention X and $\mathbf{X}$ in our notations if X is a computable set.

For a Σ^0_{n+1}–computable family $\mathcal{A}$, the following are immediate:

- $\mathcal{R}^{0,\mathbf{X}}_{n+1}(\mathcal{A})$ is an upper semilattice, called the $\mathbf{X}$– *Rogers semilattice of* $\mathcal{A}$. Similarly to what happens for $\mathcal{R}^0_{n+1}(\mathcal{A})$, the least upper bound of $\deg_{\mathbf{X}}(\alpha)$ and $\deg_{\mathbf{X}}(\beta)$ is $\deg_{\mathbf{X}}(\alpha \oplus \beta)$.

- If $Y \leq_T X$, then every degree of $\mathcal{R}^{0,\mathbf{X}}_{n+1}(\mathcal{A})$ is union of degrees from $\mathcal{R}^{0,\mathbf{Y}}_{n+1}(\mathcal{A})$:

$$\deg_{\mathbf{X}}(\alpha) = \bigcup\{\deg_{\mathbf{Y}}(\beta) : \beta \in \deg_{\mathbf{X}}(\alpha)\}.$$

- If $Y \leq_T X$ and $|\mathcal{R}^{0,\mathbf{Y}}_{n+1}(\mathcal{A})| = 1$ then $|\mathcal{R}^{0,\mathbf{X}}_{n+1}(\mathcal{A})| = 1$.

- $|\mathcal{R}^{0,\mathbf{0}^{(n+2)}}_{n+1}(\mathcal{A})| = 1$, as the relation "$\alpha(m) = \alpha(n)$" is $\mathbf{0}^{(n+2)}-$decidable for $\alpha \in \mathrm{Com}^0_{n+1}(\mathcal{A})$.

- If α is a numbering which is $\mathbf{0}^{(m)}-$universal in $\mathrm{Com}^0_{n+1}(\mathcal{A})$, then it is $\mathbf{0}^{(k)}-$universal in $\mathrm{Com}^0_{n+1}(\mathcal{A})$ for every $k \geq m$.

1.2 Some results on $\Sigma^0_{n+1}-$computable numberings

Although straightforward, the following theorem and its immediate corollary give a useful characterization of $\Sigma^0_{n+1}-$computable numberings. The theorem is stated in the more general case of numberings of arithmetical relations rather than arithmetical sets.

Theorem 1.1 (S.Goncharov and A.Sorbi, [7]). *Let $\mathcal{A}$ be a family of $\Sigma^0_{n+1}-$subsets of $\mathbb{N}^k$, $k \geq 1$. Then a numbering $\alpha\colon \mathbb{N} \longrightarrow \mathcal{A}$ is $\Sigma^0_{n+1}-$computable if and only if the set $\{(\overrightarrow{x}, m) \mid \overrightarrow{x} \in \alpha(m)\}$ is $\mathbf{0}^{(n)}-$computably enumerable.*

Applied to families of subsets of $\mathbb{N}$, the theorem gives:

Corollary 1.1.1 ([7]). *A numbering α of a family $\mathcal{A}$ of $\Sigma^0_{n+1}-$subsets of $\mathbb{N}$, is $\Sigma^0_{n+1}-$computable if and only if the relation "$x \in \alpha(y)$" is Σ^0_{n+1}.*

For the case $n = 0$, this is exactly the classical notion of computable numbering of a family of c.e. sets.

The following result is an extension to the $\Sigma^0_{n+2}-$case of a useful criterion introduced for Σ^0_2-computable numberings by Goncharov and Sorbi, [7].

Lemma 1.1. *A numbering α of a family $\mathcal{A}$ of $\Sigma^0_{n+2}-$sets is $\Sigma^0_{n+2}-$computable if and only if there exists a $\mathbf{0}^{(n)}-$computable function $f(m, s)$ such that, letting $B_{m,s} = D_{f(m,s)}$ for every $m, s \in \mathbb{N}$ (where D_u denotes the finite set with canonical index u), the following holds, for every m:*

$$\alpha(m) = \underline{\lim}_{\,s} B_{m,s}.$$

(Where, given a family $\{X_s\}_{s\in\mathbb{N}}$ of subsets of $\mathbb{N}$, we let

$$\underline{\lim}_{\,s} X_s \leftrightharpoons \{x \mid \exists t \forall s \geq t(x \in X_s)\}).$$

Proof. Let $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$. Since the set $\{\langle x, m\rangle \mid x \in \alpha(m)\}$ is Σ^0_{n+2}, let $\{B^s \mid s \in \omega\}$ be a $\mathbf{0}^{(n)}$–computable sequence of finite sets such that

$$x \in \alpha(m) \Leftrightarrow (\exists t)(\forall s > t)(\langle x, m\rangle \in B^s).$$

Then take

$$B_{m,s} \leftrightharpoons \{x \mid \langle x, m\rangle \in B^s\}.$$

The converse is trivial. □

Theorem 1.2. *Let $\mathcal{A}$ be a Σ^0_{n+1}–computable family, $n \geq 0$, and let α be a numbering of $\mathcal{A}$. Then the following statements are equivalent:*

(i) α is Σ^0_{n+1}–computable;

(ii) α is reducible to some Σ^0_{n+1}–computable numbering of $\mathcal{A}$;

(iii) α is $\mathbf{0}^{(n)}$–reducible to some Σ^0_{n+1}–computable numbering of $\mathcal{A}$;

(iv) α is reducible to a numbering universal in $\mathrm{Com}^0_{n+1}(\Sigma^0_{n+1})$;

(v) α is $\mathbf{0}^{(n)}$–reducible to a numbering universal in $\mathrm{Com}^0_{n+1}(\Sigma^0_{n+1})$.

Proof. It is easy to see that

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(i)}:$$

To show that (iii) $\Rightarrow$ (i) just notice that if $\alpha = \beta \circ f$, where β is a Σ^0_{n+1}–computable numbering and f is some $\mathbf{0}^{(n)}$–computable function, then

$$x \in \alpha(y) \Leftrightarrow \exists z(f(y) = z \;\&\; x \in \beta(z)),$$

which gives that the relation "$x \in \alpha(y)$" is Σ^0_{n+1}, hence, by Corollary 1.1.1, α is Σ^0_{n+1}–computable.

Let us now show that

$$\text{(i)} \Rightarrow \text{(iv)} \Rightarrow \text{(v)}.$$

To show that (i) $\Rightarrow$ (iv), it is enough to show that there exists a numbering β universal in $\mathrm{Com}^0_{n+1}(\Sigma^0_{n+1})$. By the relativized Universal Function Theorem, let $K(x, y)$ be some partial $\mathbf{0}^{(n)}$–computable function that is universal for the unary partial $\mathbf{0}^{(n)}$–computable functions. Then the numbering

$$\beta(x) \leftrightharpoons \{y \mid K(x, y)\downarrow\}$$

is a $\Sigma^0_{n+1}-$computable numbering of the family of all $\Sigma^0_{n+1}-$sets. If now ν is any $\Sigma^0_{n+1}-$computable numbering, then by Post's Theorem, the relation "$x \in \alpha(y)$" is the domain of some partial $\mathbf{0}^{(n)}-$computable function $\varphi(x, y)$. By the relativized s^m_n-Theorem there exists a computable function φ such that $\varphi^{\emptyset^{(n)}}_{f(x)}(y) = \varphi(x, y)$. Hence $\nu = \beta \circ f$. Since ν is arbitrary, it follows that the numbering β is universal in $\mathrm{Com}^0_{n+1}(\Sigma^0_{n+1})$.

The rest of the proof is now trivial since, exactly as in the proof of (iii) $\Rightarrow$ (i), one can easily show that (v) $\Rightarrow$ (i) . □

Badaev and Goncharov, [1], have solved the problem of the cardinality of the set of minimal elements in $\mathcal{R}^0_{n+2}(\mathcal{A})$, for any infinite family $\mathcal{A}$. The following theorem will have several applications throughout the paper.

Theorem 1.3 (S.Badaev and S.Goncharov, [1]). *For every n, if $\mathcal{A}$ is an infinite $\Sigma^0_{n+2}-$computable family, then $\mathcal{R}^0_{n+2}(\mathcal{A})$ has infinitely many minimal elements.*

Proof. Let $\mathcal{A}$ be an infinite $\Sigma^0_{n+2}-$family, and let $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$. Take any maximal set M. Assume that

$$\overline{M} = \{m_0 < m_1 < \cdots\}.$$

For every $A \in \mathcal{A}$ define

$$\alpha_{M,A}(x) \leftrightharpoons \begin{cases} \alpha(i) & \text{if } x = m_i, \\ A & \text{if } x \in M. \end{cases}$$

It is possible to show:

- $\alpha_{M,A} \in \mathrm{Com}^0_{n+2}(\mathcal{A})$;
- $\alpha_{M,A}$ is minimal. To see this, assume that $\gamma = \alpha_{M,A} \circ f$, for some computable function f. Let $R \leftrightharpoons \mathrm{range}(f)$, and let g be a computable function such that $M = \mathrm{range}(g)$. Since $\mathcal{A}$ is infinite, $R \cap \overline{M}$ is infinite, and thus by maximality of M it follows that $\overline{R} \cap \overline{M}$ is finite. Finally, let k be such that $\gamma(k) = A$. Towards showing that $\alpha_{M,A} \leqslant \gamma$ define, for every $x \in R \cup M$,

$$h(x) \leftrightharpoons \begin{cases} k & \text{if } \exists y(g(y) = x \;\&\; \forall z < y(f(z) \neq x)), \\ \mu y\,(f(y) = x) & \text{otherwise.} \end{cases}$$

Then, for every $x \in R \cup M$ we have $\alpha_{M,A}(x) = \gamma(h(x))$. Since $\overline{R \cup M}$ is finite, one can suitably extend h to a computable function reducing $\alpha_{M,A}$ to γ, as desired.

- If $B \in \mathcal{A}$ and $A \neq B$ then $\alpha_{M,A} \not\leqslant \alpha_{M,B}$ (thus there are infinitely many minimal elements in $\mathcal{R}^0_{n+2}(\mathcal{A})$, as $\mathcal{A}$ is infinite). Suppose by contradiction that $\alpha_{M,A} = \alpha_{M,B} \circ h$, via some computable function h. Since $\alpha_{M,B}$ maps every element of M to B,we have that $h[\alpha^{-1}_{M,A}(A)] \subseteq \overline{M}$. On the other hand, if $x \in h[\alpha^{-1}_{M,A}(A)]$, then $\alpha_{M,B}(x) = A$, therefore $\overline{M} \setminus h[\alpha^{-1}_{M,A}(A)]$ is infinite. But $M \subseteq \alpha^{-1}_{M,A}(A)$, hence $h[M] \subseteq \overline{M}$ and thus $h[M]$ is finite by maximality of M. Finally if $y \in h^{-1}[h[M]]$ then $\alpha_{M,B}(h(y)) = A$, so $\overline{M} \setminus h^{-1}[h[M]]$ is infinite, and thus as before $h^{-1}[h[M]] \setminus M$ is finite. As $M \subseteq h^{-1}[h[M]]$ and $h[M]$ is finite, this would imply that M is computable, a contradiction.
- $\alpha_{M,A} \equiv_{\mathbf{0}'} \alpha$, for every A.

□

Remark 1.2. Inspection of the proof of the previous theorem shows that if we define $\alpha_{M,A}$ starting from a $\mathbf{0}^{(i)}-$maximal set M, with $i \leq n$, then for every A the numbering $\alpha_{M,A}$ is $\mathbf{0}^{(i)}-$minimal; if $A \neq B$ then $\alpha_{M,A} \not\leqslant_{\mathbf{0}^{(i)}} \alpha_{M,B}$; and finally $\alpha_{M,A} \equiv_{\mathbf{0}^{(i+1)}} \alpha$.

Remark 1.3. In the case of a finite family $\mathcal{A}$, the construction of the previous theorem gives us *atoms* in the Rogers semilattice $\mathcal{R}^0_{n+2}(\mathcal{A})$.

More precisely, if $\mathcal{A} \subseteq \Sigma^0_{n+2}$ is a finite family, and some element $B \in \mathcal{A}$ different from A has infinitely many $\alpha-$indices then the numbering $\alpha_{M,A}$ is not minimal and for every numbering $\beta \in \mathrm{Com}^0_{n+2}(\mathcal{A})$, the fact that $\beta \leqslant \alpha_{M,A}$ implies that either β is reducible to any numbering of $\mathcal{A}$ or $\beta \equiv \alpha_{M,A}$.

2. Complete numberings and completions

In this section we develop the theory of complete numberings in the context of arithmetical numberings. Both the classical notion of reducibility as well as some relativized versions are taken into account. We also introduce the notion of a uniformly complete numbering.

2.1 Complete numberings: The basic facts

Complete numberings were introduced by A.I. Mal'tsev, [10]. For a thorough investigation of complete numberings, see [6].

Definition 2.1. A numbering α of an abstract nonempty family $\mathcal{A}$ is called *complete with respect to* $a \in \mathcal{A}$, if for every partial computable function φ there exists a total computable function f such that, for every

x,

$$\alpha(f(x)) = \begin{cases} \alpha(\varphi(x)) & \text{if} \quad \varphi(x)\downarrow, \\ a & \text{otherwise.} \end{cases}$$

We say in this case that f *$\alpha-$extends φ with respect to a*. The element a is said to be a *special object for α*.

The following definition is a bit less demanding, [6]:

Definition 2.2. A numbering α is *precomplete* if for every partial computable function φ there is a computable function f such that, for every x, if $\varphi(x)\downarrow$ then $\alpha(\varphi(x)) = \alpha(f(x))$. We say that f *$\alpha-$extends φ*.

Remark 2.1. The previous definitions are given in terms of partial computable functions of one variable. Of course they could equivalently be given in terms of functions of several variables. If for instance α is precomplete and $\lambda x, y\, \varphi(x, y)$ is a partial computable function of two variables then there exists a computable function $\lambda x, y\, f(x, y)$ such that $\alpha(\varphi(x, y)) = \alpha(f(x, y))$, for all x, y such that $\varphi(x, y)$ is defined. To see this, given φ, just consider a computable function g that $\alpha-$extends the partial computable function $\lambda u\, \varphi(\pi_0(u), \pi_1(u))$. Then one can take $f(x, y) \leftrightharpoons g(\langle x, y\rangle)$.

It is immediate to see that every complete numbering is precomplete. We have:

Theorem 2.1 (Yu.L.Ershov, [6]). *Let α be a numbering of a family $\mathcal{A}$. Then the following statements are equivalent:*

(1) α is precomplete;

(2) there exists a computable function h such that for every e, $\varphi_{h(e)}$ is total and for all x,

$$\varphi_e(x)\downarrow \Rightarrow \alpha(\varphi_{h(e)}(x)) = \alpha(\varphi_e(x));$$

(3) (The Uniform Fixed Point Theorem) there exists computable function g such that for every e,

$$\varphi_e(g(e))\downarrow \Rightarrow \alpha(g(e)) = \alpha(\varphi_e(g(e))).$$

Proof. (1) $\Rightarrow$ (2). Let α be precomplete. To show (2), we can argue as follows. By the Universal Function Theorem, let U be a unary universal partial computable function, for instance let $U(\langle e, x\rangle) = \varphi_e(x)$. Let f be a computable function such that $\alpha(U(z)) = \alpha(f(z))$, for every

$z \in \mathrm{dom}(U)$. By the s_n^m–Theorem, let h be a computable function such that $\varphi_{h(e)}(x) = f(\langle e, x\rangle)$. It is easy to see that h is the desired function.

(2) $\Rightarrow$ (3). To show the Uniform Fixed Point property (3), by the s_n^m–Theorem let f be a computable function such that $\varphi_{f(e)}(x) = \varphi_e(\varphi_x(x))$ for all e, x, and by (2) let h be a computable function such that for every e $\varphi_{h(e)}$ is total and for every x,

$$\varphi_{f(e)}(x)\downarrow \Rightarrow \alpha(\varphi_{h(e)}(x)) = \alpha(\varphi_{f(e)}(x)).$$

Define $g(e) = \varphi_{h(e)}(h(e))$ for all e. It is easy to see that if $\varphi_e(g(e))\downarrow$ then $\alpha(g(e)) = \alpha(\varphi_e(g(e)))$.

(3) $\Rightarrow$ (1). Let g be a computable function such that for every e,

$$\varphi_e(g(e))\downarrow \Rightarrow \alpha(g(e)) = \alpha(\varphi_e(g(e))).$$

For a given partial computable function φ, define the function $\psi(x, y)$,

$$\psi(x, y) = \begin{cases} \varphi(x) & \text{if} \quad \varphi(x)\downarrow, \\ \uparrow & \text{otherwise.} \end{cases}$$

By the s_n^m–Theorem there exists a computable function f such that $\varphi_{f(x)} = \lambda y\, \psi(x, y)$, for all x. It is easy to verify that the computable function $g \circ f$ α–extends the partial function φ.

□

Remark 2.2. In [9], A.I. Mal'tsev considered numberings with the uniform fixed point property (3) and called them "complete" numberings. Later in [10], he defined "complete" numbering just as in Definition 2.1 and renamed "precomplete" numberings the numberings with the uniform fixed point property. The definition of "precomplete numberings" as in Definition 2.2 was introduced by Yu.L. Ershov in [6].

Theorem 2.2 (A.I.Mal'tsev, [10]). *Let α, β be numberings of a family $\mathcal{A}$. If $\alpha \equiv \beta$, and α is complete with respect to a special object a (precomplete), then β is complete as well with respect to a (precomplete, respectively). In fact, if α and β are precomplete and $\alpha \equiv \beta$ then α and β are computably isomorphic, i.e. there exists a computable permutation p of $\mathbb{N}$ such that $\beta = \alpha \circ p$.*

Proof. This is a well known fact of the theory of numberings, see e.g. [6]. We give however a sketch of the proof. We first show that if $\alpha \equiv \beta$, and α is complete with respect to the special object a, then β is complete with respect to a. (The same argument will show that if α is precomplete so is β.) If $\alpha = \beta \circ h$, $\beta = \alpha \circ k$, where h, k are computable functions, and α is complete with respect to a, then one can argue that β is complete with respect to a as follows. If φ is partial computable, then let f α–extend $k \circ \varphi$ with respect to a. It easily follows that $h \circ f$ β–extends φ with respect to A.

Now, if $\alpha = \beta \circ h$ and $\beta = \alpha \circ k$ and α and β are precomplete, then by a standard back–and–forth argument one can produce a computable permutation p such that $\alpha = \beta \circ p$. For this, we can define by stages a computable approximation $\{p^s\}_{s \in \mathbb{N}}$ to p, so that, for every s, p^s is a partial function with finite domain, and $p^s \subseteq p^{s+1}$. We start up with defining $p^0 \leftrightharpoons \emptyset$. At stage $s = 2t + 1$ we arrange, if $t \notin \mathrm{dom}(p^{s-1})$, that $t \in \mathrm{dom}(p^s)$, by defining $p^s \leftrightharpoons p^{s-1} \cup \{(t, y)\}$, for a suitable number y. At stage $s = 2t + 2$ we arrange, if not already achieved at any of the previous stages, that $t \in \mathrm{range}(p^s)$, by defining $p^s \leftrightharpoons p^{s-1} \cup \{(y, t)\}$, for suitable number y. In order to choose a correct number y at each stage so as to achieve that p is 1–1 and $\alpha = \beta \circ p$, one can use the fact that if ν is precomplete, then from any finite set $F = \{n_1, \dots, n_k\}$ of numbers such that $\nu(n_1) = \dots = \nu(n_k)$ we can uniformly find $n \notin F$ such that $\nu(n) = \nu(n_1)$. Indeed, if f is a computable function such that $\varphi_{f(e)}$ is total and ν–extends φ_e (such a function exists by (2) of Theorem 2.1), then by the Recursion Theorem let e be such that

$$\varphi_e(x) = \begin{cases} n_1 & \text{if } \varphi_{f(e)}(0) \notin F, \\ \max F + 1 & \text{otherwise.} \end{cases}$$

Then the number

$$n \leftrightharpoons \begin{cases} \varphi_{f(e)}(0) & \text{if } \varphi_{f(e)}(0) \notin F, \\ \max F + 1 & \text{otherwise.} \end{cases}$$

is the desired number. Using this, given any m one can uniformly enumerate an infinite set X of numbers such that $\nu(m) = \nu(n)$ for every $n \in X$. □

Finally, the degrees of complete numberings satisfy a distinguished structural property in both the semilattices $L(\mathcal{A})$ and $\mathcal{R}^0_{n+1}(\mathcal{A})$. We first need the following definition:

Definition 2.3. A numbering $\alpha \in \mathrm{Num}(\mathcal{A})$ is called *splittable* if it is equivalent to the join of some pair of incomparable numberings of subfamilies of $\mathcal{A}$. Likewise α is called $\mathbf{X}$–*splittable* if it is $\mathbf{X}$–equivalent to the join of some pair of numberings of subfamilies of $\mathcal{A}$ that are incomparable with respect to $\leqslant_{\mathbf{X}}$.

We recall that an element of an upper semilattice is called *splittable* if it is the least upper bound of two incomparable elements of that semilattice. Note that if $\alpha \in \mathrm{Num}(\mathcal{A})$ is not splittable then the degree $\deg(\alpha)$

is not a splittable element in any upper semilattice of numberings of $\mathcal{A}$, but the converse is not necessarily true.

Theorem 2.3 (Yu.L.Ershov, [5]). *Precomplete numberings are not splittable.*

Proof. Suppose that α is precomplete and splittable. By Theorem 2.2 we may assume that $\alpha = \alpha_0 \oplus \alpha_1$. Let φ and ψ be partial computable functions such that

(1) if $\varphi_e(x)$ is even, then $\varphi(e,x) = x$;

(2) if $\varphi_e(x)$ is odd then $\psi(e,u) = x$ and $\varphi(e,x) = u$ for some u, and, for every e, $\lambda u\ \psi(e,u)$ is total if there exist infinitely many numbers x such that $\varphi_e(x)$ is odd.

(To achieve this, one can use a 1–1 enumeration of the graph of the universal function $K(e,x)$. Whenever a triple $\langle e,x,y\rangle$ appears in this enumeration such that y is even, then we enumerate $\langle e,x,x\rangle$ in the graph of φ; whenever a triple $\langle e,x,y\rangle$ appears such that y is odd, then we enumerate $\langle e,u,x\rangle$ in the graph of ψ and $\langle e,x,u\rangle$ in the graph of φ, where u is the least number such that $\psi(e,u)\uparrow$.) By precompleteness of α let $f(e,x)$ be a computable function that $\alpha-$extends φ, and by the Recursion Theorem, let e be such that $\varphi_e(x) = f(e,x)$, so that φ_e is total. Finally let

$$A = \{x \mid \varphi_e(x) \text{ odd}\}.$$

If A is finite, then for almost all x we have $x = \varphi(e,x)$, thus

$$\begin{aligned}\alpha(x) &= \alpha(\varphi(e,x)) = \alpha(f(e,x))\\ &= \alpha(\varphi_e(x))\\ &= \alpha_0(\frac{\varphi_e(x)}{2})\end{aligned}$$

from which we get that $\alpha \leqslant \alpha_0$.

If, on the contrary, A is infinite, then for every u, $\psi(e,u)$ is defined, $u = \varphi(e,\psi(e,u))$ and $\varphi_e(\psi(e,u)) = \varphi_e(x)$, for some x such that $\varphi_e(x)$ is odd. Hence

$$\begin{aligned}\alpha(u) &= \alpha(\varphi(e,\psi(e,u)))\\ &= \alpha(f(e,\psi(e,u)))\\ &= \alpha(\varphi_e(\psi(e,u))).\end{aligned}$$

But for some x, $\psi(e,u) = x$ and $\varphi_e(x)$ is odd; then

$$\alpha(u) = \alpha_1(\frac{\varphi_e(\psi(e,u)) - 1}{2})$$

which implies that $\alpha \leqslant \alpha_1$. □

For more on precomplete numberings see [6].

2.2 The completion operator and its properties

In the theory of numberings there is a well known and powerful construction, due to Ershov [6], which allows, given any numbering α of a family $\mathcal{A}$, to find a complete numbering of $\mathcal{A}$ with respect to any special object a. This construction is defined as follows.

Definition 2.4. Let $U(x)$ be a unary universal partial computable function, for instance let $U(\langle e, x\rangle) = \varphi_e(x)$. Given numbering α of a family $\mathcal{A}$ and $a \in \mathcal{A}$, define

$$\alpha_a^{\mathbf{0}}(x) \leftrightharpoons \begin{cases} \alpha(U(x)) & \text{if} \quad U(x)\downarrow, \\ a & \text{otherwise.} \end{cases}$$

(The superscript **0** denotes that the completion is defined through a function which is universal for the class of all partial computable functions, no oracle needed. Suitable generalizations to the case of functions computable via oracles will be provided later.)

Clearly, since range$(U) = \mathbb{N}$, $\alpha_a^{\mathbf{0}}$ is a numbering of the whole family.

Theorem 2.4 (Yu.L. Ershov). *For every α, the numbering $\alpha_a^{\mathbf{0}}$ is complete with respect to a.*

Proof. (See [6]) Let φ_e be any partial computable function. By the s_n^m–Theorem let s be a computable function such that

$$U(\langle s(e), x\rangle) = \varphi_{s(e)}(x) = U(\varphi_e(x)).$$

We have

$$\alpha_a^{\mathbf{0}}(\varphi_e(x)) = \begin{cases} \alpha(U(\varphi_e(x))) & \text{if } U(\varphi_e(x))\downarrow, \\ a & \text{otherwise.} \end{cases}$$

On the other hand, $U(\varphi_e(x)) = U(\langle s(e), x\rangle)$, and as

$$\alpha_a^{\mathbf{0}}(\langle s(e), x\rangle) = \begin{cases} \alpha(U(\langle s(e), x\rangle)) & \text{if } U(\langle s(e), x\rangle)\downarrow, \\ a & \text{otherwise,} \end{cases}$$

it follows that $\alpha_a^{\mathbf{0}}(\varphi_e(x)) = \alpha_a^{\mathbf{0}}(\langle s(e), x\rangle)$. Hence the computable function $\lambda x\ \langle s(e), x\rangle$ $\alpha_a^{\mathbf{0}}$–extends φ_e with respect to a. □

Let us now turn to the case of Σ_{n+1}^0–computable families. We will go back to the general setting in Section 1.1, where we propose a relativized notion of completion. The basic question in which we are interested now is the following: If $\alpha \in \mathrm{Com}^0_{n+1}(\mathcal{A})$, is $\alpha^{\mathbf{0}}_a \in \mathrm{Com}^0_{n+1}(\mathcal{A})$? In order to avoid incomputability, in the case of families of Σ^0_1–sets we usually need to take a to be the least element $\perp$ of $\mathcal{A}$ under inclusion, if any such element exists. This is justified by the fact that if $\mathcal{A}$ is a Σ^0_1–computable family and $\mathcal{A}$ has the least element $\perp$ under $\subseteq$, then for every $\alpha \in \mathrm{Com}^0_1(\mathcal{A})$ we have that $\alpha^0_\perp \in \mathrm{Com}^0_1(\mathcal{A})$.

To see this, one just needs to observe that

$$x \in \alpha^{\mathbf{0}}_\perp(y) \Leftrightarrow x \in \perp \ \vee \ [U(y)\downarrow \ \& \ x \in \alpha(U(y))]$$

for every x, y. Thus the relation "$x \in \alpha^{\mathbf{0}}_\perp(y)$" is Σ^0_1.

On the other hand, the following theorem shows that for every Σ^0_{n+2}–computable numbering α of any family $\mathcal{A} \subseteq \Sigma^0_{n+2}$ and for *any* $A \in \mathcal{A}$, the numbering $\alpha^{\mathbf{0}}_A$ is still Σ^0_{n+2}–computable. This, together with the fact that the completion operator is well behaved with respect to the reducibility $\leqslant$ (see next theorem, item (2)), shows that the mapping $\alpha \mapsto \alpha^{\mathbf{0}}_A$ induces an operator on the semilattice $\mathcal{R}^0_{n+2}(\mathcal{A})$.

Theorem 2.5. *Let $\mathcal{A}$ be a family of Σ^0_{n+2} –sets; let $\alpha, \beta \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ and pick $A \in \mathcal{A}$. Then*

(1) $\alpha^{\mathbf{0}}_A \in \mathrm{Com}^0_{\mathrm{n}+2}(\mathcal{A})$*;*

(2) if $\alpha \leqslant \beta$ then $\alpha^{\mathbf{0}}_A \leqslant \beta^{\mathbf{0}}_A$.

(3) $\alpha \leqslant \alpha^{\mathbf{0}}_A$;

(4) $\alpha^{\mathbf{0}}_A \equiv_{\mathbf{0}'} \alpha$;

(5) $\alpha^{\mathbf{0}}_A \equiv \alpha$ if and only if α is complete with respect to A.

Proof. Let $\alpha, \beta \in \mathrm{Com}^0_{n+2}(\mathcal{A})$, and let $A \in \mathcal{A}$.

(1) For every x, y we have

$$x \in \alpha^{\mathbf{0}}_A(y) \Leftrightarrow \exists z(U(y) = z \ \& \ x \in \alpha(z)) \ \vee \ (U(y)\uparrow \ \& \ x \in A).$$

Since the relations "$x \in \alpha(z)$", "$x \in A$", are Σ^0_{n+2}, and the relations "$U(y) = z$", "$U(y)\uparrow$" are $\mathbf{0}'$–computable, it follows that the relation "$x \in \alpha^{\mathbf{0}}_A(y)$" is Σ^0_{n+2}.

Remark 2.3. The same argument shows also that the ternary relation, in x, i, y, "$x \in \alpha^{\mathbf{0}}_{\alpha(i)}(y)$" is Σ^0_{n+2}.

(2) Suppose that $\alpha \leqslant \beta$ via some computable function f, i.e. $\alpha = \beta \circ f$. Let e be such that $\varphi_e = f \circ U$. Then $f(U(x))) = U(\langle e, x\rangle)$. Define $g(x) = \langle e, x\rangle$. It easily follows that

$$\alpha_A^{\mathbf{0}} = \beta_A^{\mathbf{0}} \circ g.$$

Indeed, $U(x) \downarrow$ if and only if $U(\langle e, x\rangle) \downarrow$, thus if $U(x) \uparrow$ then $\alpha_A^{\mathbf{0}}(x) = \beta_A^{\mathbf{0}}(g(x)) = A$; otherwise

$$\begin{aligned}\alpha_A^{\mathbf{0}}(x) &= \alpha(U(x)) \\ &= \beta(f(U(x))) \\ &= \beta(U(g(x))) = \beta_A^{\mathbf{0}}(g(x)).\end{aligned}$$

Remark 2.4. Notice that the proof is uniform: One can effectively find, independently from A, an index for g starting from any index for f.

(3) Let e be an index of the identity function, i.e. $\varphi_e(x) = x$. Then

$$\alpha_A^{\mathbf{0}}(\langle e, x\rangle) = \alpha(U(\langle e, x\rangle)) = \alpha(x).$$

Hence $\alpha \leqslant \alpha_A^{\mathbf{0}}$ via the function $\lambda x\ \langle e, x\rangle$. (Notice that this function does not depend on either α or A.)

(4) By construction, $\alpha_A^{\mathbf{0}} \leqslant_{\mathbf{0}'} \alpha$, thus by (3) we have that $\alpha_A^{\mathbf{0}} \equiv_{\mathbf{0}'} \alpha$.

(5) As observed in Theorem 2.2, if $\beta \equiv \alpha$ and α is complete with respect to some object, then β is complete with respect to the same special object as α. It follows that if α is not complete with respect to A then $\alpha < \alpha_A^{\mathbf{0}}$.

On the other hand, suppose that α is complete with respect to A. Let f be a computable function that $\alpha-$extends U with respect to A. Therefore $\alpha_A^{\mathbf{0}}(x) = \alpha(f(x))$ for every x and therefore $\alpha_A^{\mathbf{0}} \leqslant \alpha$.

□

Notice that the previous theorem remains true for $\alpha_A^{\mathbf{0}}$ in the classical case when $\mathcal{A} \subseteq \Sigma_1^0$, provided that $\mathcal{A}$ has least set $\perp$ with respect to inclusion, and we take $A = \perp$.

Corollary 2.5.1. *If α is a numbering which is universal in $\mathrm{Com}_{n+2}^0(\mathcal{A})$, then α is complete with respect to any member of $\mathcal{A}$.*

Proof. If α is universal in $\mathrm{Com}_{n+2}^0(\mathcal{A})$ then, by (1) and (3) of Theorem 2.5, $\alpha \equiv \alpha_A^{\mathbf{0}}$, for every A. It follows that α is complete with respect to any $A \in \mathcal{A}$. □

Corollary 2.5.1 gives us easily natural examples of numberings which are complete with respect to every element. This should be compared with Denisov and Lavrov, [4], who showed for the first time that one can have numberings which are complete with respect to more than one element.

Corollary 2.5.2. *For every $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$, the numbering $\alpha^{\mathbf{0}}_A$ is not splittable. In particular, the greatest degree of $\mathcal{R}^0_{n+2}(\mathcal{A})$, if any, is not splittable.*

Proof. Immediate, by Theorem 2.3 and Corollary 2.5.1. □

Corollary 2.5.3. *Let $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$, and let $A \in \mathcal{A}$. Then any creative set is $m-$reducible both to the $\alpha^{\mathbf{0}}_A-$index set of any $B \in \mathcal{A}$ different from A and to the $\alpha^{\mathbf{0}}_A-$index set of all the elements of $\mathcal{A}$ different from A. In particular, the $\alpha^{\mathbf{0}}_A-$index set of A is productive.*

Proof. The properties of the numbering $\alpha^{\mathbf{0}}_A$ proposed in the corollary hold, indeed, for any complete numbering with the special object A (see A.I. Mal'tsev, [10]).

Let C be a creative set, and let b be such that $\alpha^{\mathbf{0}}_A(b) = B$ and $B \neq A$. Define a partial computable function φ as follows:

$$\varphi(x) \rightleftharpoons \begin{cases} b & \text{if } x \in C, \\ \uparrow & \text{otherwise.} \end{cases}$$

Let f be a computable function that $\alpha^{\mathbf{0}}_A-$extends φ with respect to A. Then

$$\alpha^{\mathbf{0}}_A(f(x)) = \begin{cases} B & \text{if } x \in C, \\ A & \text{otherwise.} \end{cases}$$

Therefore, for every x,

$$x \in C \Leftrightarrow \alpha^{\mathbf{0}}_A(f(x)) = B \Leftrightarrow \alpha^{\mathbf{0}}_A(f(x)) \neq A$$

i.e. the function f $m-$reduces the set C to the sets $\{x \mid \alpha^{\mathbf{0}}_A(x) = B\}$ and $\{x \mid \alpha^{\mathbf{0}}_A(x) \neq A\}$. □

Corollary 2.5.3 has a number of useful consequences. We recall that α is a *Friedberg numbering* if for every x, y we have that $\alpha(x) = \alpha(y)$ if and only if $x = y$, and α is a *positive* numbering if the relation in x, y "$\alpha(x) = \alpha(y)$" is c.e. Note that for Friedberg and positive numberings, any index set is c.e. Corollary 2.5.3 implies that Friedberg and positive numberings, as well as the minimal numberings which one can build by the method introduced by Badaev and Goncharov (see Theorem 1.3) are not complete with respect to any element. In the last case, the claim follows from the fact that one of the index sets contains a maximal set, while the other index sets are included in the complement of this maximal set; thus no index set can be productive.

The following theorem gives an elegant and rather unexpected proof of the fact that in $\mathcal{R}^0_{n+2}(\mathcal{A})$ any non complete numbering is meet–reducible.

Theorem 2.6. *If* $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ *and* $A, B \in \mathcal{A}$*, with* $A \neq B$*, then, in* $\mathcal{R}^0_{n+2}(\mathcal{A})$*,* $\deg(\alpha)$ *is the greatest lower bound of* $\deg(\alpha^{\mathbf{0}}_A)$ *and* $\deg(\alpha^{\mathbf{0}}_B)$*.*

Proof. By Theorem 2.5, we have that $\alpha \leqslant \alpha^{\mathbf{0}}_A$ and $\alpha \leqslant \alpha^{\mathbf{0}}_B$.

Let now f and g be computable functions which reduce a numbering γ to $\alpha^{\mathbf{0}}_A$ and $\alpha^{\mathbf{0}}_B$ respectively. Thus, for every x,

$$\gamma(x) = \alpha^{\mathbf{0}}_A(f(x)) = \begin{cases} \alpha(U(f(x))) & \text{if } U(f(x))\downarrow, \\ A & \text{otherwise} \end{cases}$$

and

$$\gamma(x) = \alpha^{\mathbf{0}}_B(g(x)) = \begin{cases} \alpha(U(g(x))) & \text{if } U(g(x))\downarrow, \\ B & \text{otherwise.} \end{cases}$$

Since $A \neq B$ then $U(f(x))\downarrow$ or $U(g(x))\downarrow$. Let $\varphi_v = \lambda x U(f(x))$ and let $\varphi_w = \lambda x U(g(x))$ and define

$$h(x) \leftrightharpoons \begin{cases} U(f(x)) & \text{if } \exists s(\varphi_{v,s}(x)\downarrow \ \&\ \varphi_{w,s}(x)\uparrow), \\ U(g(x)) & \text{otherwise.} \end{cases}$$

So h is computable and we have $\gamma = \alpha \circ h$, hence $\gamma \leqslant \alpha$. □

Notice that γ need not be a numbering of the whole family in the previous proof.

Corollary 2.6.1. *If* $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ *and there exists a pair of distinct objects of* $\mathcal{A}$ *with respect to which* α *is not complete, then* $\deg(\alpha)$ *is meet–reducible in* $\mathcal{R}^0_{n+2}(\mathcal{A})$*.*

Proof. Immediate by Theorem 2.5 and Theorem 2.6. □

Remark 2.5. Theorem 2.6 and Corollary 2.6.1 remain true if one works in the upper semilattice $\mathrm{Num}(\mathcal{A})$ instead of $\mathrm{Com}^0_{n+2}(\mathcal{A})$, as γ need not be a Σ^0_{n+2}–computable numbering of $\mathcal{A}$ in the proofs of Theorem 2.6 and Corollary 2.6.1.

2.3 Relativization of the completion operator

We now go back to the relativized version $\leqslant_{\mathbf{X}}$ of reducibility on numberings, introduced in Section 1.1, and provide a suitable generalization of the completion operator. We generalize the notion of a complete numbering, by allowing functions that are computable relatively to oracles.

We begin with some general definitions, relative to numberings of any abstract family.

Definition 2.5. Let $X \subseteq \mathbb{N}$. A numbering α of a set $\mathcal{A}$ is called $\mathbf{X}$–*complete with respect to the special object* $a \in \mathcal{A}$ if for every partial $\mathbf{X}$–computable function $\varphi(x)$ there exists some total $\mathbf{X}$–computable function $f(x)$ such that

$$\alpha(f(x)) = \begin{cases} \alpha(\varphi(x)) & \text{if} \quad \varphi(x)\downarrow, \\ a & \text{otherwise.} \end{cases}$$

We say that f α–*extends* φ *with respect to* a.

The definition of an $\mathbf{X}$–precomplete numbering is given accordingly.

We begin with noting that $\mathbf{X}$–complete numberings satisfy a relativized version of the fixed point property stated in Theorem 2.1, which now reads as follows:

Theorem 2.7. *If* α *is an* $\mathbf{X}$–*precomplete numbering, then for every partial* $\mathbf{X}$–*computable function* φ *one can uniformly find a number* n *such that if* $\varphi(n)\downarrow$ *then* $\alpha(n) = \alpha(\varphi(n))$.

Proof. Assume that α is an $\mathbf{X}$–precomplete numbering. Given any index of a partial $\mathbf{X}$–computable function φ, one can uniformly find an index z of an $\mathbf{X}$–computable function f such that $\alpha(f(x)) = \alpha(\varphi(\varphi_x^{\mathbf{X}}(x)))$, for every x such that $\varphi(\varphi_x^{\mathbf{X}}(x))\downarrow$. Take now $n = f(z)$, and argue as in the proof of Theorem 2.1. □

The relativized version of Theorem 2.2 reads as follows:

Theorem 2.8. *For every set* $X \subseteq \mathbb{N}$ *and any pair of numberings* α, β, *if* α *and* β *are* $\mathbf{X}$–*equivalent and* α *is* $\mathbf{X}$–*complete with respect to* a *then* β *is* $\mathbf{X}$–*complete with respect to* a. *In fact in this case* α *and* β *are* $\mathbf{X}$–*computably isomorphic (i.e.* $\alpha = \beta \circ p$, *for some* $\mathbf{X}$–*computable permutation* p. *This holds also if* α *and* β *are just both precomplete).*

Proof. The proof is a direct relativization of the proof of Theorem 2.2. To show for instance that if α and β are $\mathbf{X}$–equivalent and α is $\mathbf{X}$–complete with respect to a then β is $\mathbf{X}$–complete with respect to a, one can argue as follows. Let $\alpha = \beta \circ h$ and $\beta = \alpha \circ k$ where h, k are $\mathbf{X}$–

computable, and let φ be any partial $\mathbf{X}$– computable function. Then $k\circ\varphi$ is partial $\mathbf{X}$– computable, and there exists an $\mathbf{X}$– computable function f that α– extends $k \circ \varphi$ with respect to a. Then $h \circ f$ is $\mathbf{X}$– computable and β– extends φ with respect to a. $\square$

Finally we give the relativized version of Theorem 2.3.

Theorem 2.9. *If α is $\mathbf{X}$-precomplete, then α is not $\mathbf{X}$– splittable.*

Proof. The proof of this fact is a straightforward relativization of the proof of Theorem 2.3. Let $\alpha = \alpha_0 \oplus \alpha_1$ be $\mathbf{X}$– precomplete and let $K^{\mathbf{X}}(e,x) \rightleftharpoons \varphi_e^{\mathbf{X}}(x)$ be the universal partial $\mathbf{X}$– computable function for the unary partial $\mathbf{X}$– computable functions. Starting from $K^{\mathbf{X}}$, one defines suitable partial $\mathbf{X}$– computable functions $\varphi(e,x)$ and $\psi(e,s)$ satisfying properties (1) and (2) stated in the proof of Theorem 2.3. Then we apply the relativized Recursion Theorem to find e such that $\varphi_e^{\mathbf{X}}(x) = f(e,x)$, where f is an $\mathbf{X}$– computable function that α– extends φ. As in the proof of Theorem 2.3, it follows that either $\alpha \leqslant_{\mathbf{X}} \alpha_0$ or $\alpha \leqslant_{\mathbf{X}} \alpha_1$. $\square$

We now want to relativize the completion operator which we have introduced in the previous section. We need for this to fix some preliminary notations and terminology. First of all, for every $X \subseteq \mathbb{N}$, let $\{\varphi_e^X \mid e \in \omega\}$ be the standard Kleene–numbering of the partial functions which are computable with oracle X. (As anticipated in Section 1.1 we will write $\varphi_e^{\mathbf{X}}$ for φ_e^X.) The relativized Universal Function Theorem justifies the following definition.

Definition 2.6. For every arbitrary subset X of $\mathbb{N}$ define

$$U^{\mathbf{X}}(\langle e,x\rangle) \leftrightharpoons \varphi_e^{\mathbf{X}}(x)$$

for all $e,x \in \mathbb{N}$. The function $U^{\mathbf{X}}$ is partial $\mathbf{X}$– computable.

Let now α be a numbering of a family $\mathcal{A}$, and let $a \in \mathcal{A}$.

Definition 2.7. Define

$$\alpha_a^{\mathbf{X}}(x) \leftrightharpoons \begin{cases} \alpha(U^{\mathbf{X}}(x)) & \text{if} \quad U^{\mathbf{X}}(x)\downarrow, \\ a & \text{otherwise.} \end{cases}$$

Since range$(U^{\mathbf{X}}) = \mathbb{N}$, $\alpha_a^{\mathbf{X}}$ is a numbering of the whole family.

Notice that if X is computable, then we may assume $U^{\mathbf{X}}$ to be the function U of Definition 2.4, and thus in this case $\alpha_a^{\mathbf{X}}$ is exactly the numbering $\alpha_a^{\mathbf{0}}$, which we have defined in Definition 2.4.

We would like now to point out some properties of the completion operator which hold for abstract numberings. We remind the reader that we always consider numbered families that contain at least two elements. Moreover, if $\mathcal{A}$ is a family of sets with least element under inclusion then this element will be denoted by $\perp$.

The following theorem follows along the lines of Theorem 2.5 and Theorem 2.6.

Theorem 2.10. *Let $\mathcal{A}$ be any family, let $a, b \in \mathcal{A}$, and let α, β be any numberings of $\mathcal{A}$. Then the following statements hold for every subsets $X, Y \subseteq \mathbb{N}$:*

(1) if $Y \leq_T X$, then the numbering $\alpha_a^{\mathbf{X}}$ is $\mathbf{Y}$–complete with respect to the special object a;

(2) if $\alpha \leqslant_{\mathbf{X}} \beta$ then $\alpha_a^{\mathbf{X}} \leqslant \beta_a^{\mathbf{X}}$.

(3) $\alpha \leqslant \alpha_a^{\mathbf{X}}$;

(4) $\alpha_a^{\mathbf{X}} \equiv_{\mathbf{X}'} \alpha$;

(5) $\alpha \equiv_{\mathbf{X}} \alpha_a^{\mathbf{X}}$ if and only if α is $\mathbf{X}$–complete with respect to a;

(6) if $\beta \leqslant_{\mathbf{X}} \alpha_a^{\mathbf{X}}$ then $\beta \leqslant \alpha_a^{\mathbf{X}}$;

(7) if $Y \leq_T X$ then $\alpha_a^{\mathbf{X}'} \equiv (\alpha_a^{\mathbf{X}'})_b^{\mathbf{Y}}$. In particular $\alpha_a^{\mathbf{X}'}$ is $\mathbf{Y}$–complete with respect to any special element b;

(8) if $a \neq b$ then for every γ such that $\gamma \leqslant_{\mathbf{X}} \alpha_a^{\mathbf{X}}$ and $\gamma \leqslant_{\mathbf{X}} \alpha_b^{\mathbf{X}}$ one has $\gamma \leqslant_{\mathbf{X}} \alpha$.

Proof. Let $\mathcal{A}$ and a be given. The proof of this theorem consists for the most part a straightforward relativization of Theorems 2.5 and 2.6. We will be however a bit more detailed than necessary, because this will enable us to make some interesting remarks on the proof. Finally, let $Y \leq_T X$.

(1) Let $\varphi = \varphi_e^{\mathbf{Y}}$ be any partial $\mathbf{Y}$– computable function. Then the function $U^{\mathbf{X}}(\varphi(x))$ is partial $\mathbf{X}$–computable. By the relativized s_n^m–Theorem let s be a computable function such that

$$U^{\mathbf{X}}(\langle s(e), x\rangle) = \varphi_{s(e)}^{\mathbf{X}}(x) = U^{\mathbf{X}}(\varphi_e^{\mathbf{Y}}(x)).$$

We have

$$\alpha_a^{\mathbf{X}}(\varphi_e^{\mathbf{Y}}(x)) = \begin{cases} \alpha(U^{\mathbf{X}}(\varphi_e^{\mathbf{Y}}(x))) & \text{if } U^{\mathbf{X}}(\varphi_e^{\mathbf{Y}}(x)) \downarrow, \\ a & \text{otherwise.} \end{cases}$$

On the other hand, $U^{\mathbf{X}}(\varphi_e^{\mathbf{Y}}(x)) = U^{\mathbf{X}}(\langle s(e), x\rangle)$, and as

$$\alpha_a^{\mathbf{X}}(\langle s(e), x\rangle) = \begin{cases} \alpha(U^{\mathbf{X}}(\langle s(e), x\rangle)) & \text{if } U^{\mathbf{X}}(\langle s(e), x\rangle) \downarrow, \\ a & \text{otherwise,} \end{cases}$$

it follows that $\alpha_a^{\mathbf{X}}(\varphi_e^{\mathbf{Y}}(x)) = \alpha_a^{\mathbf{X}}(\langle s(e), x\rangle)$.

Remark 2.6. Notice that the function that $\alpha_a^{\mathbf{X}}$–extends φ with respect to a is the function $g(x) = \langle s(e), x\rangle$, i.e. a computable function! Furthermore, the proof is uniform in any index of φ.

(2) Let $\alpha \leqslant_{\mathbf{X}} \beta$ via some $\mathbf{X}$–computable function f, i.e. $\alpha = \beta \circ f$. Let e be such that $\varphi_e^{\mathbf{X}} = f \circ U^{\mathbf{X}}$. Then $f(U^{\mathbf{X}}(x))) = U^{\mathbf{X}}(\langle e, x\rangle)$. Define $g(x) = \langle e, x\rangle$. As in the proof of (2) of Theorem 2.5, it easily follows that $\alpha_a^{\mathbf{X}} = \beta_a^{\mathbf{X}} \circ g$.

Remark 2.7. Notice that the function g is computable! As in the previous item, the proof is uniform, i.e. an index for g can be uniformly found from any index of f.

(3) Let e be an index of the identity function, i.e. $\varphi_e^{\mathbf{X}}(x) = x$. Then, as $U^{\mathbf{X}}(\langle e, x\rangle)$ is always defined,

$$\alpha_a^{\mathbf{X}}(\langle e, x\rangle) = \alpha(U^{\mathbf{X}}(\langle e, x\rangle)) = \alpha(x).$$

Hence $\alpha \leqslant \alpha_a^{\mathbf{X}}$ via the function $\lambda x\ \langle e, x\rangle$.

(4) By definition, we immediately have $\alpha_a^{\mathbf{X}} \leqslant_{\mathbf{X}'} \alpha$. Since $\alpha \leqslant \alpha_a^{\mathbf{X}}$ by the previous item, we get $\alpha_a^{\mathbf{X}} \equiv_{\mathbf{X}'} \alpha$.

(5) Due to (3), claim (5) can be equivalently formulated as follows: $\alpha_a^{\mathbf{X}} \leqslant_{\mathbf{X}} \alpha$ if and only if α is $\mathbf{X}$–compete with respect to a. To see this, first assume that $\alpha_a^{\mathbf{X}} \leqslant_{\mathbf{X}} \alpha$. Then, by (3), $\alpha_a^{\mathbf{X}} \equiv_{\mathbf{X}} \alpha$, and thus by Theorem 2.8 α is $\mathbf{X}$–complete with respect to a. Conversely, suppose that α is $\mathbf{X}$–complete with respect to a. Let f be an $\mathbf{X}$–computable function that α–extends $U^{\mathbf{X}}$ with respect to a. Then $\alpha_a^{\mathbf{X}} = \alpha \circ f$, thus $\alpha_a^{\mathbf{X}} \leqslant_{\mathbf{X}} \alpha$.

(6) Let f be an $\mathbf{X}$–computable function such that $\beta = \alpha_a^{\mathbf{X}} \circ f$. Then, by definition of $\alpha_a^{\mathbf{X}}$, we have, for every x,

$$\beta(x) = \alpha_a^{\mathbf{X}}(f(x)) = \begin{cases} \alpha(U^{\mathbf{X}}(f(x))) & \text{if } U^{\mathbf{X}}(f(x)) \downarrow \\ a & \text{otherwise.} \end{cases}$$

Let e be such that $U^{\mathbf{X}} \circ f = \varphi_e^{\mathbf{X}}$. Then $U^{\mathbf{X}}(f(x)) = U^{\mathbf{X}}(\langle e, x\rangle)$ and thus, for every x,

$$\beta(x) = \alpha_a^{\mathbf{X}}(\langle e, x\rangle).$$

Then the function $g(x) = \langle e, x\rangle$ reduces β to $\alpha_a^{\mathbf{X}}$.

Remark 2.8. Notice that the proof is uniform, i.e. there is some computable function g such that if $\varphi_u^{\mathbf{X}}$ reduces β to $\alpha_a^{\mathbf{X}}$, then the computable function (with no oracle!) $\varphi_{g(u)}$ reduces β to $\alpha_a^{\mathbf{X}}$.

(7) By definition of completion we have

$$(\alpha_a^{\mathbf{X}'})_b^{\mathbf{Y}}(x) = \begin{cases} \alpha_a^{\mathbf{X}'}(U^{\mathbf{Y}}(x)) & \text{if } U^{\mathbf{Y}}(x)\downarrow, \\ b & \text{otherwise} \end{cases}$$
$$= \begin{cases} \alpha(U^{\mathbf{X}'}(U^{\mathbf{Y}}(x))) & \text{if } U^{\mathbf{X}'}(U^{\mathbf{Y}}(x))\downarrow, \\ a & \text{if } U^{\mathbf{Y}}(x)\downarrow \text{ and } U^{\mathbf{X}'}(U^{\mathbf{Y}}(x))\uparrow \\ b & \text{otherwise.} \end{cases}$$

Since $\mathrm{range}(U^{\mathbf{X}'}) = \mathbb{N}$, there exists x_0 such that $U^{\mathbf{X}'}(x_0)\downarrow$ and $\alpha(U^{\mathbf{X}'}(x_0)) = b$. Consider the total $\mathbf{X}'$–computable function $h(x)$ defined as follows

$$h(x) = \begin{cases} U^{\mathbf{Y}}(x) & \text{if } U^{\mathbf{Y}}(x)\downarrow, \\ x_0 & \text{if } U^{\mathbf{Y}}(x)\uparrow. \end{cases}$$

Obviously, $(\alpha_a^{\mathbf{X}'})_b^{\mathbf{Y}}(x) = \alpha_a^{\mathbf{X}'}(h(x))$. Therefore $(\alpha_a^{\mathbf{X}'})_b^{\mathbf{Y}} \leqslant_{\mathbf{X}'} \alpha_a^{\mathbf{X}'}$. On the other hand, $\alpha_a^{\mathbf{X}'} \leqslant (\alpha_a^{\mathbf{X}'})_b^{\mathbf{Y}}$ by (3), thus $(\alpha_a^{\mathbf{X}'})_b^{\mathbf{Y}} \equiv_{\mathbf{X}'} \alpha_a^{\mathbf{X}'}$. By (6) it follows that $(\alpha_a^{\mathbf{X}'})_b^{\mathbf{Y}} \leqslant \alpha_a^{\mathbf{X}'}$. On the other hand, by (3) we have $\alpha_a^{\mathbf{X}'} \leqslant (\alpha_a^{\mathbf{X}'})_b^{\mathbf{Y}}$. Hence $(\alpha_a^{\mathbf{X}'})_b^{\mathbf{Y}} \equiv \alpha_a^{\mathbf{X}'}$. The claim now follows by item (5).

(8) Let us show that if $\gamma \leqslant_{\mathbf{X}} \alpha_a^{\mathbf{X}}$ and $\gamma \leqslant_{\mathbf{X}} \alpha_b^{\mathbf{X}}$ (via, say, $\mathbf{X}$–computable functions f and g) then $\gamma \leqslant_{\mathbf{X}} \alpha$. We have, for every x,

$$\gamma(x) = \alpha_a^{\mathbf{X}}(f(x)) = \begin{cases} \alpha(U^{\mathbf{X}}(f(x))) & \text{if } U^{\mathbf{X}}(f(x))\downarrow, \\ a & \text{otherwise} \end{cases}$$

and

$$\gamma(x) = \alpha_b^{\mathbf{X}}(g(x)) = \begin{cases} \alpha(U^{\mathbf{X}}(g(x))) & \text{if } U^{\mathbf{X}}(g(x))\downarrow, \\ b & \text{otherwise.} \end{cases}$$

Since $a \neq b$, obviously we have that $U^{\mathbf{X}}(f(x))\downarrow$ or $U^{\mathbf{X}}(g(x))\downarrow$. Let $\varphi_v^{\mathbf{X}} = \lambda x U^{\mathbf{X}}(f(x))$ and $\varphi_w^{\mathbf{X}} = \lambda x U^{\mathbf{X}}(g(x))$ and define

$$h(x) \leftrightharpoons \begin{cases} U^{\mathbf{X}}(f(x)) & \text{if } \exists s(\varphi_{v,s}^{\mathbf{X}}(x)\downarrow \ \& \ \varphi_{w,s}^{\mathbf{X}}(x)\uparrow), \\ U^{\mathbf{X}}(g(x)) & \text{otherwise.} \end{cases}$$

So we have $\gamma = \alpha \circ h$, where h is an $\mathbf{X}$–computable function. Hence $\gamma \leqslant_{\mathbf{X}} \alpha$. (As remarked for Theorem 2.6, γ need not be a numbering of the whole family in the previous argument.)

□

We now list several useful corollaries of the previous theorem, for numberings of some fixed family.

Corollary 2.10.1. *If $Y \leq_T X$ then $(\alpha_a^{\mathbf{X}})_a^{\mathbf{Y}} \equiv \alpha_a^{\mathbf{X}}$.*

Proof. By (1), (3), (5) and (6) of Theorem 2.10. □

Corollary 2.10.2. *$\alpha \leqslant \alpha_a^{\mathbf{0}^{(n)}}$. Moreover, α is $\mathbf{0}^{(n)}$–complete with respect to a if and only if $\alpha_a^{\mathbf{0}^{(n)}} \equiv_{\mathbf{0}^{(n)}} \alpha$.*

Proof. By (3) and (5) of Theorem 2.10. □

Corollary 2.10.3. *The numbering $\alpha_a^{\mathbf{0}^{(n)}}$ is $\mathbf{0}^{(i)}$–complete with respect to a for all $i \leq n$.*

Proof. By (1) of Theorem 2.10. □

Corollary 2.10.4. *If $\beta \leqslant_{\mathbf{0}^{(n)}} \alpha_a^{\mathbf{0}^{(n)}}$ then $\beta \leqslant \alpha_a^{\mathbf{0}^{(n)}}$.*

Proof. By (6) of Theorem 2.10. □

Corollary 2.10.5. *$(\alpha_a^{\mathbf{0}^{(n+1)}})_b^{\mathbf{0}^{(n)}} \equiv \alpha_a^{\mathbf{0}^{(n+1)}}$, all a, b. In particular, $\alpha_a^{\mathbf{0}^{(n+1)}}$ is $\mathbf{0}^{(i)}$–complete with respect to each element of the family and for all $i \leq n$.*

Proof. By (1) and (7) of Theorem 2.10. □

Finally we give the relativized analogues of Corollary 2.5.2 and Corollary 2.5.3.

Corollary 2.10.6. *For every numbering α and every set X, the numbering $\alpha_a^{\mathbf{X}}$ is not $\mathbf{X}$–splittable.*

Proof. It follows from Theorem 2.9. □

Corollary 2.10.7. *The set of all* $\alpha_a^{\mathbf{X}}$*–indices of the special object* a *is* $\mathbf{X}$*–productive.*

Proof. The proof is a straightforward relativization of the proof of Corollary 2.5.3. Let e be such that $\alpha_a^{\mathbf{X}}(e) \neq a$. Choose any $\mathbf{X}$–creative set C, and define

$$\varphi(x) \leftrightharpoons \begin{cases} e & \text{if } x \in C, \\ \uparrow & \text{otherwise.} \end{cases}$$

Then φ is partial $\mathbf{X}$–computable, and by $\mathbf{X}$–completion, there is some $\mathbf{X}$–computable function f that $\alpha_a^{\mathbf{X}}$–extends φ with respect to a. Then it is easy to see that the function f $\mathbf{m}$–reduces C to the complement of the $\alpha_a^{\mathbf{X}}$–index set $\{x \mid \alpha_a^{\mathbf{X}}(x) = a\}$. □

2.4 Computability of completions

We now go back to consider in details Σ^0_{n+1}–computable families. Let $\alpha \in \mathrm{Com}^0_{n+1}(\mathcal{A})$. We are interested in completions of α which are still in $\mathrm{Com}^0_{n+1}(\mathcal{A})$.

Theorem 2.11. *Let* $\alpha \in \mathrm{Com}^0_{n+1}(\mathcal{A})$. *Then*

(1) $\alpha_A^{\mathbf{0}^{(m)}} \in \mathrm{Com}^0_{\mathrm{n+1}}(\mathcal{A})$ *for every* $m < n$ *and each* $A \in \mathcal{A}$;

(2) $\alpha_A^{\mathbf{0}^{(n)}} \in \mathrm{Com}^0_{\mathrm{n+1}}(\mathcal{A})$ *if and only if* $\mathcal{A}$ *has the least set* $\perp$ *and* $A = \perp$;

(3) $\alpha_A^{\mathbf{0}^{(m)}} \notin \mathrm{Com}^0_{\mathrm{n+1}}(\mathcal{A})$ *for every* $m > n$ *and each* $A \in \mathcal{A}$.

Proof. Let $\alpha \in \mathrm{Com}^0_{n+1}(\mathcal{A})$.

(1) Let $A \in \mathcal{A}$ and $m < n$. By definition of completion it follows that $x \in \alpha_A^{\mathbf{0}^{(m)}}(y)$ if and only if

$$\exists z (U^{\mathbf{0}^{(m)}}(y) = z \;\&\; x \in \alpha(y)) \;\vee\; (U^{\mathbf{0}^{(m)}}(y)\uparrow \;\&\; x \in A).$$

Since $U^{\mathbf{0}^{(m)}}$ is a partial $\mathbf{0}^{(m)}$–computable function, it is immediate to check that the relation "$x \in \alpha_A^{\mathbf{0}^{(n)}}(y)$" is Σ^0_{n+1}.

(2) Let $A = \perp \in \mathcal{A}$. In this case

$$x \in \alpha_\perp^{\mathbf{0}^{(n)}}(y) \Leftrightarrow x \in \perp \;\vee\; \exists z (U^{\mathbf{0}^{(n)}}(y) = z \;\&\; x \in \alpha(y)).$$

Hence $\alpha_\perp^{\mathbf{0}^{(n)}} \in \mathrm{Com}^0_{n+1}(\mathcal{A})$.

Conversely, let $\alpha_A^{\mathbf{0}^{(n)}} \in \mathrm{Com}_{n+1}^0(\mathcal{A})$, and suppose that A is not the least element of $\mathcal{A}$. Hence there is some $B \in \mathcal{A}$ such that $A \nsubseteq B$. Let $a \in A \setminus B$, and let b be such that $\alpha_A^{\mathbf{0}^{(n)}}(b) = B$. Let

$$C \leftrightharpoons \{x \mid a \in \alpha_A^{\mathbf{0}^{(n)}}(x)\}.$$

Obviously, $C \in \Sigma_{n+1}^0$, and thus the partial function φ defined as follows,

$$\varphi(x) = \begin{cases} b & \text{if } x \in C, \\ \uparrow & \text{otherwise,} \end{cases}$$

is $\mathbf{0}^{(n)}$–computable. Since $\alpha_A^{\mathbf{0}^{(n)}}$ is $\mathbf{0}^{(n)}$–complete with respect to A, let f be a $\mathbf{0}^{(n)}$–computable function that $\alpha_A^{\mathbf{0}^{(n)}}$–extends φ with respect to A, i.e.

$$\alpha_A^{\mathbf{0}^{(n)}}(f(x)) = \begin{cases} B & \text{if } x \in C, \\ A & \text{otherwise.} \end{cases}$$

By the relativized version of the Fixed Point Theorem (see Theorem 2.7) let e be such that $\alpha_A^{\mathbf{0}^{(n)}}(e) = \alpha_A^{\mathbf{0}^{(n)}}(f(e))$. Hence

$$e \in C \Leftrightarrow a \in \alpha_A^{\mathbf{0}^{(n)}}(e)$$

and on the other hand, by properties of f,

$$e \in C \Leftrightarrow \alpha_A^{\mathbf{0}^{(n)}}(e) = B.$$

This yields a contradiction.

(3) Let $A \in \mathcal{A}$ and $m > n$. Recall that we consider only nontrivial families, i.e. $|\mathcal{A}| > 1$. Let $B, C \in \mathcal{A}$, $B \nsubseteq C$ and choose a number $b \in B \setminus C$.

Assume now that $\alpha_A^{\mathbf{0}^{(m)}} \in \mathrm{Com}_{n+1}^0(\mathcal{A})$ and consider the set $X \leftrightharpoons \{x \mid b \in \alpha_A^{\mathbf{0}^{(m)}}(x)\}$. X is clearly $\mathbf{0}^{(n+1)}$–computable. Inequality $m > n$ implies that X is also $\mathbf{0}^{(m)}$–computable. The subfamilies $\alpha_A^{\mathbf{0}^{(m)}}[X]$ and $\alpha_A^{\mathbf{0}^{(m)}}[\mathbb{N} \setminus X]$ are disjoint and contain B and C respectively. Denote by x_0 and x_1 $\alpha_A^{\mathbf{0}^{(m)}}$–indices of B and C respectively, and define a function $f(x)$ as follows

$$f(x) \leftrightharpoons \begin{cases} x_1 & \text{if } x \in X, \\ x_0 & \text{otherwise.} \end{cases}$$

By Theorem 2.7, $\alpha_A^{\mathbf{0}^{(m)}}(f(x_2)) = \alpha_A^{\mathbf{0}^{(m)}}(x_2)$ for some x_2. We have

$$b \in \alpha_A^{\mathbf{0}^{(m)}}(x_2) \Leftrightarrow x_2 \in X$$

and

$$b \in \alpha_A^{\mathbf{0}^{(m)}}(f(x_2)) \Leftrightarrow x_2 \in \mathbb{N} \setminus X,$$

a contradiction. Therefore, $\alpha_A^{\mathbf{0}^{(m)}}$ is not a $\Sigma^0_{n+1}-$computable numbering.

□

Corollary 2.11.1. *Let α be $\mathbf{0}^{(m)}-$universal in $\mathrm{Com}^0_{n+1}(\mathcal{A})$. Then:*

(1) if $m < n$ then α is $\mathbf{0}^{(m)}-$complete with respect to every element of $\mathcal{A}$;

(2) if $m \leqslant n$ and $\mathcal{A}$ has the least element $\perp$ then α is $\mathbf{0}^{(m)}-$complete with respect to $\perp$;

(3) α is not $\mathbf{0}^{(n)}-$complete with respect to any non-least element of $\mathcal{A}$;

(4) α is not $\mathbf{0}^{(m)}-$complete with respect to any $A \in \mathcal{A}$ if $m > n$.

Proof. Let A be any element of $\mathcal{A}$. By Theorem 2.10, $\alpha_A^{\mathbf{0}^{(m)}}$ is Σ^0_{n+1}-computable numbering if $m < n$ or if $m \leq n$ and $A = \perp$. Since α is $\mathbf{0}^{(m)}-$universal, it follows that $\alpha_A^{\mathbf{0}^{(m)}} \leqslant_{\mathbf{0}^{(m)}} \alpha$. Then by Corollary 2.10.2, α is $\mathbf{0}^{(m)}-$complete with respect to A. So, claims (1) and (2) are proved.

Let now either $m = n$ and $A \neq \perp$ or $m > n$. By Theorem 2.11, numbering $\alpha_A^{\mathbf{0}^{(m)}}$ is not Σ^0_{n+1}- computable. Therefore, $\alpha_A^{\mathbf{0}^{(m)}} \nleqslant_{\mathbf{0}^{(m)}} \alpha$ because all Σ^0_{n+1}- computable numberings of $\mathcal{A}$ are $\mathbf{0}^{(m)}-$reducible to α. Corollary 2.10.2 in this case implies that α is not $\mathbf{0}^{(m)}-$complete with respect to A. □

We now ask whether α is $\mathbf{Y}-$complete with respect to a if $Y \leq_T X$ and α is $\mathbf{X}-$complete with respect to a. This question naturally arises from (1) of Theorem 2.10. In particular, we have seen that $\alpha_A^{\mathbf{0}^{(m)}} \in \mathrm{Com}^0_{n+1}(\mathcal{A})$ if $m < n$, and $\alpha_A^{\mathbf{0}^{(m)}}$ is $\mathbf{0}^{(i)}-$complete with respect to A, for all $i \leq m$. It is then natural to ask, given any object of the family, whether it is possible to find numberings that are $\mathbf{0}^{(i)}-$complete, for some numbers $i < n$, with respect to that object, but not $\mathbf{0}^{(j)}-$complete, for different numbers $j < n$. The following theorem gives an answer to these questions.

Theorem 2.12. *Let $\mathcal{A}$ be any non-trivial $\Sigma^0_{n+1}-$computable family with $n \geq 1$. Then*

(1) for every $A \in \mathcal{A}$ and every set $I \subseteq \{0, 1, \ldots, n-1\}$, there exists a numbering $\alpha \in \mathrm{Com}^0_{n+1}(\mathcal{A})$ such that α is $\mathbf{0}^{(i)}$–complete with respect to A if and only if $i \in I$;

(2) if $\mathcal{A}$ has least set $\perp$, and if it has a Σ^0_{n+1}–computable numbering which is not $\mathbf{0}^{(n)}$–complete with respect to $\perp$ then for every set $I \subseteq \{0, 1, \ldots, n\}$, there exists a numbering $\alpha \in \mathrm{Com}^0_{n+1}(\mathcal{A})$ such that α is $\mathbf{0}^{(i)}$–complete with respect to $\perp$ if and only if $i \in I$.

Proof. We will consider only the case when $\mathcal{A}$ is an infinite family. The case with a finite $\mathcal{A}$ can be proved almost by the same way due to Remark 1.3. Let us prove (1). So, let $\mathcal{A}$ be any infinite Σ^0_{n+1}–computable family, $A \in \mathcal{A}$, and $I \subseteq \{0, 1, \ldots, n-1\}$. We prove the claim by induction on the cardinality k of the set $C_I = \{0, 1, \ldots, n-1\} \setminus I$.

For $k = 0$, i.e. $I = \{0, 1, \ldots, n-1\}$, by Corollary 2.10.3 and (1) of Theorem 2.11 it suffices to take $\alpha = \beta_A^{\mathbf{0}^{(n-1)}}$, for any numbering $\beta \in \mathrm{Com}^0_{n+1}(\mathcal{A})$.

Suppose that the claim is true of k, and let $I \subseteq \{0, 1, \ldots, n-1\}$ be such that the cardinality of C_I is $k+1$. Let j_0 be the least element of C_I. By induction, there exists a numbering $\beta \in \mathrm{Com}^0_{n+1}(\mathcal{A})$ such that β is $\mathbf{0}^{(i)}$–complete with respect to A if and only if $i \in I \cup \{j_0\}$.

We distinguish two cases:

Case 1. If $j_0 = 0$ then take any maximal set M, choose $B \in \mathcal{A}$ different from A, and apply the Badaev–Goncharov construction ([1] or see here Theorem 1.3 and Remark 1.2) to produce a minimal numbering $\gamma \leftrightharpoons \beta_{M,B}$. It follows from Theorem 1.3 that $\beta_{M,B} \equiv_{\mathbf{0}'} \beta$. Thus by Theorem 2.8 γ is $\mathbf{0}^{(i)}$–complete with respect to A if and only if β is $\mathbf{0}^{(i)}$–complete with respect to A for every $i \neq 0$. On the other hand, γ is not complete with respect to A, since by Corollary 2.5.3 in this case the γ–index set of A should be productive, whereas it is contained in the complement of the maximal set M.

Case 2. Assume now that $j_0 > 0$. In this case take a $\mathbf{0}^{(j_0)}$–maximal set M; choose again $B \in \mathcal{A}$. Apply the Badaev–Goncharov construction (see Theorem 1.3) to produce a numbering $\gamma' \leftrightharpoons \beta_{M,B}$. We have that $\beta_{M,B} \equiv_{\mathbf{0}^{(j_0+1)}} \beta$.

By Theorem 2.10(5) it follows that for every $i > j_0$ the numbering γ' is $\mathbf{0}^{(i)}$–complete with respect to A if and only if β is $\mathbf{0}^{(i)}$–complete with respect to A. Moreover, since M is $\mathbf{0}^{(j_0)}$–maximal, it follows also that γ' is not $\mathbf{0}^{(j_0)}$–complete since in this case, by Corollary 2.10.7, the γ'–index set for A should be $\mathbf{0}^{(j_0)}$–productive.

Now, define $\gamma \leftrightharpoons (\gamma')_A^{\mathbf{0}^{(j_0-1)}}$. By Corollary 2.10.3, we have that γ is $\mathbf{0}^{(i)}$–complete with respect to A, for every $i < j_0$. But $\gamma \equiv_{\mathbf{0}^{(j_0)}} \gamma'$ and then for every $i \geq j_0$, γ is $\mathbf{0}^{(i)}$– complete with respect to A if and only if γ' is $\mathbf{0}^{(i)}$– complete with respect to A.

Putting things together, we have that γ is $\mathbf{0}^{(i)}$– complete with respect to A if and only if $i \in I$, as desired.

The proof for the case $\perp \in \mathcal{A}$ is similar, starting from the observation (see (2) of Theorem 2.11) that $\alpha_{\perp}^{\mathbf{0}^{(n)}}$ is $\mathbf{0}^{(i)}$– complete with respect to $\perp$ for every $i \leq n$. □

Remark 2.9. For every $n \geq 2$, there exists an infinite family $\mathcal{A}_n \subseteq \Sigma_n^0$ with $\perp \in \mathcal{A}_n$ such that all the numberings of $\mathrm{Com}_n^0(\mathcal{A})$ are $\mathbf{0}^{(n)}$– equivalent. In particular, all these numberings are $\mathbf{0}^{(n)}$– complete with respect to $\perp$.

The existence of such families $\mathcal{A}_n$, $n \geq 2$, follows by relativized versions of the construction of S.Badaev and S.Goncharov, [3], of an infinite family $\mathcal{A}_1$ of c.e. sets with least set under inclusion such that $|\mathcal{R}_1^0(\mathcal{A}_1)| = 1$.

2.5 Uniformly X– complete numberings

We now consider a useful although natural strengthening of the notion of completeness.

Definition 2.8. We say that a numbering β of a set $\mathcal{A}$ is *uniformly* **X**–*complete* if there exists a total **X**– computable function $h(i, m, x)$ such that for every i, m, x

$$\beta(h(i,m,x)) = \begin{cases} \beta(\varphi_i^{\mathbf{X}}(x)) & \text{if } \varphi_i^{\mathbf{X}}(x)\downarrow, \\ \beta(m) & \text{otherwise.} \end{cases}$$

We already know (see Theorem 2.10 (7)) that for every family $\mathcal{A}$, for every numbering α of $\mathcal{A}$ and for every $a \in \mathcal{A}$, the numbering $\alpha_a^{\mathbf{X}'}$ is complete with respect to any object of the family. In fact, one can prove:

Theorem 2.13. *For every set X and for every numbering α of some family $\mathcal{A}$, and for each $a \in \mathcal{A}$, the numbering $\alpha_a^{\mathbf{X}'}$ is uniformly* **X**–*complete.*

Proof. Let us fix a set X, a family $\mathcal{A}$, a numbering $\alpha \in \mathrm{Num}(\mathcal{A})$, and $a \in \mathcal{A}$. Define the partial $\mathbf{X}'$– computable function $H(i, m, x)$ as follows

$$H(i,m,x) = \begin{cases} U^{\mathbf{X}'}(\varphi_i^{\mathbf{X}}(x)) & \text{if } \varphi_i^{\mathbf{X}}(x)\downarrow, \\ U^{\mathbf{X}'}(m) & \text{otherwise.} \end{cases}$$

By the s_n^m-Theorem let $s(i,m)$ be a computable function such that for every i,m,x, $H(i,m,x)=\varphi_{s(i,m)}^{\mathbf{X}'}(x)$. Let $h(i,m,x)=\langle s(i,m),x\rangle$ for all i,m,x. Now we show that the function h satisfies Definition 2.8 for the numbering $\alpha_a^{\mathbf{X}'}$, i.e.

$$\alpha_a^{\mathbf{X}'}(h(i,m,x)) = \begin{cases} \alpha_a^{\mathbf{X}'}(\varphi_i^{\mathbf{X}}(x)) & \text{if } \varphi_i^{\mathbf{X}}(x)\downarrow, \\ \alpha_a^{\mathbf{X}'}(m) & \text{otherwise.} \end{cases}$$

Recall that $U^{\mathbf{X}'}(\langle e,y\rangle)=\varphi_e^{\mathbf{X}'}(y)$ for all e,y and

$$\alpha_a^{\mathbf{X}'}(h(i,m,x)) = \begin{cases} \alpha(U^{\mathbf{X}'}(h(i,m,x))) & \text{if } U^{\mathbf{X}'}(h(i,m,x))\downarrow, \\ a & \text{otherwise} \end{cases}$$

for all i,m,x.

Case 1. Assume that $\varphi_i^{\mathbf{X}}(x)\downarrow$ and $U^{\mathbf{X}'}(\varphi_i^{\mathbf{X}}(x))\downarrow$. In this case $H(i,m,x)$ is defined, $H(i,m,x)=U^{\mathbf{X}'}(\varphi_i^{\mathbf{X}}(x))$, and

$$\begin{aligned} \alpha_a^{\mathbf{X}'}(h(i,m,x)) &= \alpha(U^{\mathbf{X}'}(h(i,m,x))) = \alpha(\varphi_{s(i,m)}^{\mathbf{X}'}(x)) \\ &= \alpha(H(i,m,x)) = \alpha(U^{\mathbf{X}'}(\varphi_i^{\mathbf{X}}(x))) \\ &= \alpha_a^{\mathbf{X}'}(\varphi_i^{\mathbf{X}}(x)). \end{aligned}$$

Case 2. Next assume that $\varphi_i^{\mathbf{X}}(x)\downarrow$ and $U^{\mathbf{X}'}(\varphi_i^{\mathbf{X}}(x))\uparrow$. In this case $H(i,m,x)\uparrow$ and, hence, $U^{\mathbf{X}'}(h(i,m,x))\uparrow$ since

$$U^{\mathbf{X}'}(h(i,m,x)) = U^{\mathbf{X}'}(\varphi_{s(i,m)}^{\mathbf{X}}(x)) = U^{\mathbf{X}'}(H(i,m,x)).$$

This implies that $\alpha_a^{\mathbf{X}'}(h(i,m,x))=a$.

On the other hand, from

$$\alpha_a^{\mathbf{X}'}(\varphi_i^{\mathbf{X}}(x)) = \begin{cases} \alpha(U^{\mathbf{X}'}(\varphi_i^{\mathbf{X}}(x))) & \text{if } U^{\mathbf{X}'}(\varphi_i^{\mathbf{X}}(x))\downarrow, \\ a & \text{otherwise} \end{cases}$$

we have that $\alpha_a^{\mathbf{X}'}(\varphi_i^{\mathbf{X}}(x))=a$.

Notice that Cases 1 and 2 show that if $\varphi_i^{\mathbf{X}}(x)\downarrow$ then $\alpha_a^{\mathbf{X}'}(\varphi_i^{\mathbf{X}}(x)) = \alpha_a^{\mathbf{X}'}(h(i,m,x))$.

Case 3. Now assume that $\varphi_i^{\mathbf{X}}(x)\uparrow$. Let us show that $\alpha_a^{\mathbf{X}'}(h(i,m,x)) = \alpha_a^{\mathbf{X}'}(m)$. Consider the following two subcases.

Subcase 3a. $U^{\mathbf{X}'}(m)\uparrow$.

This subcase immediately yields $\alpha_a^{\mathbf{X}'}(m)=a$. On the other hand, $U^{\mathbf{X}'}(m)\uparrow$ implies $H(i,m,x)\uparrow$ and, just as in Case 2, we have that $U^{\mathbf{X}'}(h(i,m,x))\uparrow$. Therefore, $\alpha_a^{\mathbf{X}'}(h(i,m,x))=a$.

Subcase 3b. $U^{\mathbf{X}'}(m)\downarrow$.
In this subcase we have

$$\alpha_a^{\mathbf{X}'}(m) = \alpha(U^{\mathbf{X}'}(m)) = \alpha(H(i,m,x))$$

since $H(i,m,x) = U^{\mathbf{X}'}(m)$. Moreover,

$$\alpha(H(i,m,x)) = \alpha(U^{\mathbf{X}'}(h(i,m,x))) = \alpha_a^{\mathbf{X}'}(h(i,m,x)).$$

□

Restriction to arithmetical families gives:

Corollary 2.13.1. *For every numbering* $\alpha \in \mathrm{Com}^0_{n+3}(\mathcal{A})$ *there exists a* Σ^0_{n+3}*–computable uniformly complete numbering* β *such that* $\alpha \leqslant \beta$.

Proof. Let $A \in \mathcal{A}$. A simple calculation shows that $\alpha_A^{\mathbf{0}'} \in \mathrm{Com}^0_{n+3}(\mathcal{A})$. Thus, by the previous theorem, it suffices to take $\beta = \alpha_A^{\mathbf{0}'}$. □

This implies also:

Corollary 2.13.2. *For every numbering* $\alpha \in \mathrm{Com}^0_{n+3}(\mathcal{A})$ *there exists a numbering* $\beta \in \mathrm{Com}^0_{n+3}(\mathcal{A})$ *such that* $\alpha \leqslant \beta$ *and* β *is complete with respect to every element* $B \in \mathcal{A}$.

Proof. Immediate by the previous corollary. □

3. Universal numberings of finite families

We now take a closer look at numberings of finite arithmetical families. Clearly, for every n each finite family $\mathcal{A}$ of Σ^0_{n+1}–sets has Σ^0_m–computable numberings for all $m \geq n+1$.

It is a well known fact of the theory of numberings that every finite family of c.e. sets has a principal numbering, see e.g. [6]. By relativizing this result we obtain:

Theorem 3.1. *For every* n*, each finite family* $\mathcal{A}$ *of* Σ^0_{n+1}*–sets has a numbering* α *which is* $\mathbf{0}^{(n)}$*–universal in* $\mathrm{Com}^0_{n+1}(\mathcal{A})$.

Proof. The following proof is based on ideas of Lachlan, [8]. Let $\mathcal{A}$ be a finite family of Σ^0_{n+1}–sets, say $\mathcal{A} = \{A_1, \dots, A_n\}$. Then it is possible to find finite sets $F_1, \dots, F_n$ such that, for all $1 \leq i, j \leq n$,

$$F_i \subseteq F_j \Leftrightarrow A_i \subseteq A_j;$$
$$F_i \subseteq A_j \Leftrightarrow A_i \subseteq A_j.$$

Assume first that $\mathcal{A}$ has a least element $\perp$, say $A_1 = \perp$. Assume $F_1 = \emptyset$. Let $\mathfrak{C}$ be the set of all chains, i.e. strictly increasing sequences $F_{i_1} \subset \cdots \subset F_{i_k}$, of finite sets from the list $F_1, \ldots, F_n$, and for every chain $C \in \mathfrak{C}$ let F_C denote the maximal element in the chain, and let A_C denote the set of the family corresponding to F_C (i.e. $A_C = A_i$ if and only if $F_C = F_i$). It is not difficult (see [6] for details) to construct a Σ^0_{n+1}–computable numbering α such that, for every e,

- there exists $C \in \mathfrak{C}$ such that $F_C \subseteq W_e^{\mathbf{0}^{(n)}}$ and $\alpha(e) = A_C$;
- if $W_e^{\mathbf{0}^{(n)}} \in \mathcal{A}$ then $W_e^{\mathbf{0}^{(n)}} = \alpha(e)$.

Let now $\rho(e) = W_e^{\mathbf{0}^{(n)}}$. If $\beta \in \mathrm{Com}^0_{n+1}(\mathcal{A})$ then since ρ is a universal numbering in the class $\mathrm{Com}^0_{n+1}(\Sigma^0_{n+1})$ (consisting of the Σ^0_{n+1}–computable numberings of the family of all Σ^0_{n+1}–sets), there exists a computable function f such that $\beta = \rho \circ f$, but then $\beta = \alpha \circ f$, thus giving that α is universal. So in this case there exists a numbering which is universal in $\mathrm{Com}^0_{n+1}(\mathcal{A})$.

Suppose now that $\mathcal{A}$ does not have a least member. Let

$$\mathcal{A}_0 \leftrightharpoons \mathcal{A} \cup \{\emptyset\}$$

and by the above argument let α_0 be universal in $\mathrm{Com}^0_{n+1}(\mathcal{A}_0)$. Let f be a $\mathbf{0}^{(n)}$–computable function such that

$$\mathrm{range}(f) = \{x \mid \alpha_0(x) \neq \emptyset\}.$$

Then $\alpha = \alpha_0 \circ f \in \mathrm{Com}^0_{n+1}(\mathcal{A})$. Let now $\beta \in \mathrm{Com}^0_{n+1}(\mathcal{A})$, and define

$$\beta_0(x) \leftrightharpoons \begin{cases} \emptyset & \text{if } x = 0, \\ \beta(x-1) & \text{otherwise} \end{cases}$$

and let h be a computable function such that $\beta_0 = \alpha_0 \circ h$. Hence, for every x

$$\begin{aligned} \beta(x) &= \beta_0(x+1) \\ &= \alpha_0(h(x+1)). \end{aligned}$$

But $h(x+1) \in \mathrm{range}(f)$. Let

$$k(x) \leftrightharpoons \mu\, y\, (f(y) = h(x+1)).$$

It follows that $\beta(x) = \alpha_0(f(k(x)))$, i.e. $\beta = \alpha \circ k$. Since k is $\mathbf{0}^{(n)}$–computable, it follows that $\beta \leqslant_{\mathbf{0}^{(n)}} \alpha$, hence α is $\mathbf{0}^{(n)}$–universal in $\mathrm{Com}^0_{n+1}(\mathcal{A})$. □

Thus, as already remarked, every finite family $\mathcal{A}$ of Σ^0_1–sets has a numbering which is universal in $\mathrm{Com}^0_1(\mathcal{A})$. The situation is different if we consider Σ^0_{n+2}–computable finite families. We are able in fact to give a characterization of the Σ^0_{n+2}–computable finite families $\mathcal{A}$ possessing a numbering which is $\mathbf{0}^{(n)}$–universal in $\mathrm{Com}^0_{n+2}(\mathcal{A})$.

Theorem 3.2. *Let $\mathcal{A} \subseteq \Sigma^0_{n+2}$ be a finite family. Then the following statements are equivalent:*

(1) there exists a numbering of $\mathcal{A}$ which is universal in $\mathrm{Com}^0_{n+2}(\mathcal{A})$;

(2) $\mathcal{A}$ has a numbering which is $\mathbf{0}^{(n)}$–universal in $\mathrm{Com}^0_{n+2}(\mathcal{A})$;

(3) $\mathcal{A}$ contains a least element $\perp$ under inclusion.

Proof. (1) $\Rightarrow$ (2) is evident.

(2) $\Rightarrow$ (3). Let α be a $\mathbf{0}^{(n)}$–universal numbering in $\mathrm{Com}^0_{n+2}(\mathcal{A})$. Assume that the partially ordered set $\langle \mathcal{A}, \subseteq \rangle$ has no least element, and denote by $A_0, A_1, \ldots, A_k$ all its minimal elements. Clearly $k \geq 1$. For every $i \leq k$, let a_i stand for some α–index of A_i, i.e. $\alpha(a_i) = A_i$. As in the proof of the previous theorem, for every $i \leq k$ choose a finite set F_i such that $F_i \subseteq A_i$ and $F_i \nsubseteq A_j$ for all $j \neq i, j \leq k$. We will assume that $i+1$ denotes 0 if $i = k$. It is evident that for every $i \leq k$ the set $Q_i \leftrightharpoons \{x \mid F_i \subseteq \alpha(x)\}$ is $\mathbf{0}^{(n+1)}$–computably enumerable. By the relativized Reduction Theorem, there exist $\mathbf{0}^{(n+1)}$–computably enumerable pairwise disjoint sets $R_0, R_1, \ldots, R_k$ such that $R_i \subseteq Q_i$ for all $i \leq k$ and $\bigcup_{i \leq k} R_i = \bigcup_{i \leq k} Q_i$. It is clear that the sets $R_0, R_1, \ldots, R_k$ form a partition of $\mathbb{N}$ into $\mathbf{0}^{(n+1)}$–computable sets. For every x, find i such that $x \in R_i$ and define

$$f(x) \leftrightharpoons \begin{cases} a_{i+1} & \text{if } \varphi^{\mathbf{0}^{(n)}}_x(x)\downarrow \text{ and } \varphi^{\mathbf{0}^{(n)}}_x(x) \in R_i, \\ a_i & \text{otherwise} \end{cases}$$

The function f is clearly $\mathbf{0}^{(n+1)}$–computable.

Define a numbering β of the family $\{A_0, A_1, \ldots, A_k\}$ by $\beta \leftrightharpoons \alpha \circ f$. By Theorem 1.2, β is Σ^0_{n+2}–computable. By construction of f, we have that $\beta \not\leqslant_{\mathbf{0}^{(n)}} \alpha$. Define a numbering γ as follows. Let $b_0, b_1, \ldots, b_m$ be any list of α–indices of the elements of $\mathcal{A}$. Let $\gamma(x) = \alpha(b_x)$ if $x \leq m$, and let $\gamma(m+y+1) = \beta(y)$ for all y.

Obviously, $\gamma \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ and $\gamma \not\leqslant_{\mathbf{0}^{(n)}} \alpha$. Therefore, the numbering α can not be $\mathbf{0}^{(n)}$–universal in $\mathrm{Com}^0_{n+2}(\mathcal{A})$. Contradiction.

(3) $\Rightarrow$ (1). Let $\mathcal{A}$ contain a least element $\perp$ under inclusion. By Theorem 3.1, the family $\mathcal{A}$ has a numbering α which is $\mathbf{0}^{(n+1)}$–universal in $\mathrm{Com}^0_{n+2}(\mathcal{A})$. We will prove that $\alpha^{\mathbf{0}^{(n+1)}}_{\perp}$ is universal in $\mathrm{Com}^0_{n+2}(\mathcal{A})$.

By Theorem 2.11(2), the completion $\alpha_{\perp}^{\mathbf{0}^{(n+1)}}$ of α is $\Sigma^0_{n+2}-$computable. By Theorem 2.10(3), $\alpha \leqslant \alpha_{\perp}^{\mathbf{0}^{(n+1)}}$, and since α is $\mathbf{0}^{(n+1)}-$universal in $\mathrm{Com}^0_{n+2}(\mathcal{A})$ it follows that $\alpha \equiv_{\mathbf{0}^{(n+1)}} \alpha_{\perp}^{\mathbf{0}^{(n+1)}}$.

Therefore, for every $\beta \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ we have $\beta \leqslant_{\mathbf{0}^{(n+1)}} \alpha_{\perp}^{\mathbf{0}^{(n+1)}}$, and hence, by Theorem 2.10(6), $\beta \leqslant \alpha_{\perp}^{\mathbf{0}^{(n+1)}}$. So, $\alpha_{\perp}^{\mathbf{0}^{(n+1)}}$ is a universal in $\mathrm{Com}^0_{n+2}(\mathcal{A})$. □

Corollary 3.2.1. *Let $\mathcal{A}$ be a non-trivial finite family of $\Sigma^0_{n+2}-$sets. Then*

(1) if $\mathcal{A}$ has the least set $\perp$ then for all m, $\mathcal{A}$ possesses numberings which are $\mathbf{0}^{(m)}-$universal in $\mathrm{Com}^0_{n+2}(\mathcal{A})$;

(2) if $\mathcal{A}$ does not contain the least set under inclusion then $\mathcal{A}$ has a numbering $\mathbf{0}^{(m)}-$universal in $\mathrm{Com}^0_{n+2}(\mathcal{A})$ if and only if $m \geq n+1$.

Proof. It immediately follows from Theorems 3.1, 3.2. □

Corollary 3.2.2. *If $\mathcal{A}$ is a finite family of Σ^0_{n+2} sets, then for every $i \leq n$, $\mathcal{R}^{0,\mathbf{0}^{(i)}}_{n+2}(\mathcal{A})$ has greatest element if and only if $\mathcal{A}$ has a least element with respect to inclusion.*

Proof. By Theorem 3.2. □

Corollary 3.2.3. *If a finite family $\mathcal{A} \subseteq \Sigma^0_{n+1}$ contains $\perp$ then the numberings of $\mathcal{A}$ which are universal in $\mathrm{Com}^0_{n+1}(\mathcal{A})$ are all $\mathbf{0}^{(i)}-$complete with respect to $\perp$ for every $i \leq n$.*

Proof. Immediate from Theorem 3.2 and Corollary 3.2.1. □

As to infinite families, for every n there are infinite $\Sigma^0_{n+1}-$computable families $\mathcal{A}$ without numberings which are universal (even $\mathbf{0}^{(n)}-$universal) in $\mathrm{Com}^0_{n+1}(\mathcal{A})$, and there exist infinite $\Sigma^0_{n+1}-$computable families $\mathcal{A}$ with numberings which are universal in $\mathrm{Com}^0_{n+1}(\mathcal{A})$. This will be shown in more details in Theorem 1.4 and Theorem 1.5, [2].

4. Open questions

We conclude this overview of complete numberings and completions by listing some open problems.

We know from Remark 1.2 that the minimal numberings constructed by Badaev and Goncharov in Theorem 1.3 can not be complete. So the following question is quite natural:

Question 1. *Prove or disprove that no Σ^0_{n+2}– computable minimal numberings of any non–trivial family $\mathcal{A}$ can be complete.*

As a consequence of Theorem 2.10 and Corollary 2.10.1, we may ask:

Question 2. *Is it true that*

$$((\alpha_a^{\mathbf{0}^{(n)}})_b^{\mathbf{0}^{(n)}})_a^{\mathbf{0}^{(n)}} \equiv (\alpha_a^{\mathbf{0}^{(n)}})_b^{\mathbf{0}^{(n)}}.$$

In particular, is it true that

$$((\alpha_a^{\mathbf{0}})_b^{\mathbf{0}})_a^{\mathbf{0}} \equiv (\alpha_a^{\mathbf{0}})_b^{\mathbf{0}})?$$

We know that no complete numbering is splittable (see Theorem 2.3). One may ask:

Question 3. *Let $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ be a numbering of a non–trivial family $\mathcal{A}$ and suppose that α is not complete with respect to $A \in \mathcal{A}$. Does there exist a numbering β such that*

$$\alpha < \beta < \alpha_A^{\mathbf{0}}?$$

Question 4. *Let $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ be a numbering of a non–trivial family $\mathcal{A}$, and assume that α is not complete with respect to $A \in \mathcal{A}$. Does there exist a non–splittable numbering β such that*

$$\alpha < \beta < \alpha_A^{\mathbf{0}}?$$

References

[1] S.A. Badaev, S.S. Goncharov, *On Rogers semilattices of families of arithmetical sets.* Algebra i Logika, 2001, vol. 40, no. 5, pp. 507–522 (Russian); Algebra and Logic, 2001, vol. 40, no. 5, pp. 283–291 (English translation).

[2] S.A. Badaev, S.S. Goncharov, A. Sorbi, *Arithmetical numberings: completeness and universality.* This volume.

[3] S.S. Goncharov, S.A. Badaev, *Families with one-element Rogers semilattice.* Algebra i Logika, 1998, vol. 37, no. 1, pp. 36–62 (Russian); Algebra and Logic, 1998, vol. 37, no. 1, pp. 21–34 (English translation).

[4] S.D. Denisov, I.A. Lavrov, *Complete numerations with an infinite number of special elements.* Algebra i Logika, 1970, vol. 9, pp. 503–509 (Russian); Algebra and Logic, 1970, vol. 9, pp. 301–304 (English translation).

[5] Yu.L. Ershov, *On inseparable pairs.* Algebra i Logika, 1970, vol. 9, no. 6, pp. 661–666 (Russian); Algebra and Logic, 1970, vol. 9, no. 6, pp. 396–399 (English translation).

[6] Yu.L. Ershov, *Theory of Numberings.* Nauka, Moscow, 1977 (Russian).

[7] S.S. Goncharov, A. Sorbi, *Generalized computable numerations and non-trivial Rogers semilattices.* Algebra i Logika, 1997, vol. 36, no. 6, pp. 621–641 (Russian); Algebra and Logic, 1997, vol. 36, no. 6, pp. 359–369 (English translation).

[8] A.H. Lachlan, *Standard classes of recursively enumerable sets.* Zeit. Mat. Log. Grund. Math., 1964, vol. 10, pp. 23–42.

[9] A.I. Mal'tsev, *Constructive algebras, I* Uspekhi Mat. Nauk, 1961, vol. 16, no. 3, pp. 3–60 (Russian); in [13], pp. 248–214 (English translation).

[10] A.I. Mal'tsev, *Sets with complete numberings.* Algebra i Logika, 1963, vol. 2, no. 2, pp. 4–29 (Russian); in [13], pp. 287–312 (English translation).

[11] A.I. Mal'tsev, *Towards a theory of computable families of objects.* Algebra i Logika, 1963, vol. 3, no. 4, pp. 5–31 (Russian); in [13], pp. 353–378 (English translation).

[12] A.I. Mal'tsev, *Algorithms and Recursive Functions.* Nauka, Moscow, 1965 (Russian) ; Wolters-Noordoff Publishing, Groningen, 1970 (English translation).

[13] A.I. Mal'tsev, *The Metamathematics of Algebraic Systems.* North—Holland, Amsterdam, 1971.

[14] H. Rogers, *Gödel numberings of partial computable functions.* J. Symbolic Logic, 1958, v.23, n.3, pp.49–57.

[15] H. Rogers, Jr. *Theory of Recursive Functions and Effective Computability.* McGraw-Hill, New York, 1967.

[16] R.I. Soare *Recursively Enumerable Sets and Degrees.* Springer-Verlag, Berlin Heidelberg, 1987.