A Note on Partial Numberings*

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The different behaviour of total and partial numberings with respect to the reducibility preorder is investigated. Partial numberings appear quite naturally in computability studies for topological spaces. The degrees of partial numberings form a distributive lattice which in the case of an infinite numbered set is neither complete nor contains a least element. Friedberg numberings are no longer minimal in this situation. Indeed, there is an infinite descending chain of non-equivalent Friedberg numberings below every given numbering, as well as an uncountable antichain.

Dedicated to Klaus Weihrauch on occasion of his 60th birthday.

1 Introduction

Numberings appear quite naturally in computability theory. In most cases they are obtained by an appropriate coding of programs. Doing so, destroys the distinction between programs (and/or the functions they compute) and data, thus allowing the derivation of very powerful results such as the recursion theorem.

The idea of a systematical study of numbered sets seems to have first been proposed by A. N. Kolmogorov in the mid-nineteen fifties. A great part of the nowadays theory of numberings has been developed by the Russian school of computability theory (cf. e.g. [1, 2, 3, 4, 5]).

Numberings can be used to lift computability notions from the natural numbers to more abstract structures. In nearly all studies on numberings only totally defined numberings have been considered. This is legitimate as long as one has only numberings of algebraic structures in mind. The situation is completely different in case of topological spaces such as the computable reals. Here, the canonical numberings are only partial maps and, as has been shown in [7], they are necessarily so, as long as the space does not have enough finite points. These are points the neighbourhood filter of which has a finite base. If one enlarges the space by adding sufficiently many such points, the numbering can be extended to a total numbering of the larger space [10].

It is the aim of this paper to study structural properties of the collection of all partial numberings of a given set. As will be seen, they behave quite differently from what is known for total numberings. Total numberings are usually compared by a preorder, the reducibility relation. The induced equivalence classes, called degrees, form an upper semilattice, in which the degrees generated by Friedberg (i.e. one-to-one) numberings are minimal.

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The reducibility relation can be generalised to partial numberings in a straightforward way. The corresponding degrees form a distributive lattice now. In the case of an infinite numbered set it is neither complete nor has it a least element. The degrees generated by partial one-to-one numberings are no longer minimal in this situation. We call such numberings partial Friedberg numberings.

The degree of any numbering, which now may contain also partial numberings, is uncountable. Moreover, if the numbered set is infinite then below every numbering there is both, an infinite descending chain as well as an uncountable antichain of non-equivalent partial Friedberg numberings. Note that at most countably many total numberings can be reduced to a given numbering.

As has already been mentioned above, one has to deal with partial numberings in computability studies for topological spaces. In this case there is a natural computability notion for numberings. A numbering is computable if one can effectively list all basic open sets containing a given point, uniformly in the index of the point. The set of degrees of computable numberings is an ideal in the lattice of all degrees of partial numberings. Moreover, there is a greatest among these degrees. It is generated by computable numberings that allow to pass uniformly from an effective listing of a normed base of a filter to the point the filter converges to. Such numberings have turned out to be very important in computability considerations for effectively given topological spaces [7, 8, 9, 10].

The paper is organised as follows. Section 2 contains basic definitions. Moreover, it is shown that the degrees of partial numberings form a distributive lattice. In Section 3 the degree structures of total and partial numberings, respectively, are compared and in Section 4 the special role of partial Friedberg numberings is studied. Then, in Section 5, computable numberings of effectively given topological spaces are considered. Concluding remarks will be found in Section 6.

2 Basic definitions and results

In what follows, let $\langle\ ,\ \rangle\colon \omega^2 \to \omega$ be a computable pairing function with corresponding projections π_1 and π_2 such that $\pi_i(\langle a_1, a_2\rangle) = a_i$. Furthermore, let $P^{(n)}$ denote the set of all n-ary partial computable functions and let φ be a Gödel numbering of $P^{(1)}$. For a function $f \in P^{(1)}$, we let $f(a)\!\downarrow \in C$ mean that the computation of $f(a)$ stops with value in C. Generally, if g is a partial function then $g(a) = b$ means that g is defined for the argument a and has value b.

The following, rather technical result is needed in the sequel.

Lemma 2.1 *There is an uncountable family $\mathcal{C}$ of subsets of ω such that for any $m \in \omega$ and any two distinct sets $A, B \in \mathcal{C}$, $\pi_1^{-1}(\{m\})$ intersects $A \setminus B$.*

Proof. As is well known, for each $m \in \omega$, $\pi_1^{-1}(\{m\})$ is infinite. Let $(v_i^m)_{i\in\omega}$ be an enumeration of $\pi_1^{-1}(\{m\})$ without repetitions and set $V_i = \{\, v_i^m \mid m \in \omega \,\}$ $(i \in \omega)$. Then all sets V_i are pairwise disjoint. Define $\mathcal{V}$ to be the family of these sets.

Now, let $\mathbb{C}$ be the collection of all families $\mathcal{Z}$ of subsets of ω such that the following two conditions hold:

1. $\mathcal{V} \subseteq \mathcal{Z}$.
2. For all $A \in \mathcal{Z}$, all $m \in \omega$ and all finite subsets $\mathcal{B}$ of $\mathcal{Z} \setminus \{A\}$, $\pi_1^{-1}(\{m\})$ intersects $A \setminus \bigcup \mathcal{B}$.

Order $\mathbb{C}$ by set inclusion. By construction, $\mathcal{V} \in \mathbb{C}$. Hence, $\mathbb{C}$ is not empty. Moreover, every chain in $\mathbb{C}$ has an upper bound. The union of all families in the chain is in $\mathbb{C}$ again. With Zorn's Lemma we therefore obtain that $\mathbb{C}$ contains a maximal element, say $\mathcal{C}$.

Because of Condition (1), $\mathcal{C}$ is infinite. We will show now that $\mathcal{C}$ must be uncountable. Assume to the contrary that it is countable and let it be enumerated by $A_0, A_1, \ldots$ without repetition. By Condition (2) we can find some $a_i^m \in \pi_1^{-1}(\{m\}) \cap (A_{i+1} \setminus \bigcup_{\nu=0}^{i} A_\nu)$, for all $i, m \in \omega$. Let $M = \{\, a_i^m \mid i, m \in \omega \,\}$ and suppose that $M \in \mathcal{C}$, say $M = A_j$. Then $a_j^0 \in M$, by the definition of M, but $a_j^0 \notin A_j$, by the choice of a_j^0, a contradiction. Thus $M \notin \mathcal{C}$.

Next, we prove that also $\mathcal{C} \cup \{M\} \in \mathbb{C}$, in contradiction to the maximality of $\mathcal{C}$. It follows that $\mathcal{C}$ must be uncountable.

Condition (1) is obviously satisfied. For the verification of Condition (2) let $B_0, \dots, B_n \in \mathcal{C}$, say $B_\nu = A_{i_\nu}$, for $\nu \le n$. Without restriction assume that $i_0 < \dots < i_n$. By the definition of M we have in this case that

$$a_{i_n}^m \in \pi_1^{-1}(\{m\}) \cap (M \setminus \bigcup_{\nu=0}^{i_n} A_\nu) \subseteq \pi_1^{-1}(\{m\}) \cap (M \setminus \bigcup_{\nu=0}^{n} B_\nu).$$

Next, let $A, B_0, \dots, B_{n-1}$ be pairwise distinct sets in $\mathcal{C}$. It remains to show that $\pi_1^{-1}(\{m\})$ intersects $A \setminus (\bigcup_{\nu=0}^{n-1} B_\nu \cup M)$, for every $m \in \omega$. Let $A = A_j$ and note that by the construction of M, $A \cap M$ is contained in $\{\, a_\nu^m \mid m \in \omega \wedge \nu < j \,\}$. Therefore, $A \setminus M \supseteq A_j \setminus \bigcup_{\nu=0}^{j-1} A_\nu$ and hence

$$A \setminus (\bigcup_{\mu=0}^{n-1} B_\mu \cup M) = (A \setminus M) \setminus \bigcup_{\mu=0}^{n-1} B_\mu \supseteq (A \setminus \bigcup_{\nu=0}^{j-1} A_\nu) \setminus \bigcup_{\mu=0}^{n-1} B_\mu = A \setminus (\bigcup_{\nu=0}^{j-1} A_\nu \cup \bigcup_{\mu=0}^{n-1} B_\mu).$$

By Condition (2) the last set in the preceding line intersects each $\pi_1^{-1}(\{m\})$ $(m \in \omega)$. □

Definition 2.2 Let S be a countable set. A partial mapping ν with domain $\mathrm{dom}(\nu) \subseteq \omega$ onto the set S is said to be a *(partial) numbering* of S.

For a given $s \in S$, any $n \in \mathrm{dom}(\nu)$ with $\nu(n) = s$ is called *index* of s. In case that $\mathrm{dom}(\nu) = \omega$ we say that ν is a *total numbering*. The set of all partial numberings of the set S is denoted by $\mathrm{Num}_p(S)$ and $\mathrm{Num}(S)$ stands for the set of all total numberings of S.

Definition 2.3 Let $\alpha, \beta \in \mathrm{Num}_p(S)$.

1. $\alpha \le \beta$, read α is *reducible* to β, if there is some *witness* function $f \in P^{(1)}$ such that $\mathrm{dom}(\alpha) \subseteq \mathrm{dom}(f)$, $f(\mathrm{dom}(\alpha)) \subseteq \mathrm{dom}(\beta)$, and $\alpha(a) = \beta(f(a))$, for all $a \in \mathrm{dom}(\alpha)$.
2. $\alpha \equiv \beta$, read α is *equivalent* to β, if $\alpha \le \beta$ and $\beta \le \alpha$.

Note that if $\alpha \le \beta$ and α is total, then the witness function will be total as well. Hence, in the case of total numberings the just defined reducibility notion coincides with the well known reducibility notion for such numberings.

Sometimes, it is useful to modify a given partial numbering in such a way that every element of S has infinitely many indices.

Definition 2.4 Let $\alpha \in \mathrm{Num}_p(S)$. The *cylindrification* of α is the numbering $c(\alpha)$ defined by

$$c(\alpha)(\langle i, j\rangle) = \begin{cases} \alpha(i) & \text{if } i \in \mathrm{dom}(\alpha), \\ \text{undefined} & \text{otherwise}, \end{cases} \qquad i, j \in \omega.$$

Obviously, $c(\alpha) \equiv \alpha$. Cylindrifications of total numberings were introduced by Ershov [1].

The relation $\le$ on $\mathrm{Num}_p(S)$ is reflexive and transitive. Therefore we can define degrees of numberings as follows:

$$\mathrm{deg}_p(\alpha) = \{\, \beta \in \mathrm{Num}_p(S) \mid \alpha \equiv \beta \,\}, \quad \alpha \in \mathrm{Num}_p(S),$$
$$\mathrm{deg}(\alpha) = \{\, \beta \in \mathrm{Num}(S) \mid \alpha \equiv \beta \,\}, \quad \alpha \in \mathrm{Num}(S).$$

As usual the reducibility $\le$ induces partial orderings on the sets of degrees which we also denote by $\le$. Thus, we have the two partial orders:

$$\mathcal{L}_p(S) = (\{\, \mathrm{deg}_p(\alpha) \mid \alpha \in \mathrm{Num}_p(S) \,\}, \le)$$
$$\mathcal{L}(S) = (\{\, \mathrm{deg}(\alpha) \mid \alpha \in \mathrm{Num}(S) \,\}, \le)$$

It is well known that these algebraic structures are upper semilattices in which the least upper bound of degrees of α and β is induced by the *join* $\alpha \oplus \beta$ defined as follows: For $a \in \omega$

$$\alpha \oplus \beta(2a) = \begin{cases} \alpha(a) & \text{if } a \in \mathrm{dom}(\alpha), \\ \text{undefined} & \text{otherwise,} \end{cases}$$

$$\alpha \oplus \beta(2a+1) = \begin{cases} \beta(a), & \text{if } a \in \mathrm{dom}(\beta), \\ \text{undefined} & \text{otherwise.} \end{cases}$$

In the case of $\mathcal{L}_p(S)$ also the greatest lower bound of two degrees exists. It is induced by the *meet* $\alpha \sqcap \beta$ of the numberings α and β. For $m, n \in \omega$

$$\alpha \sqcap \beta(\langle m, n\rangle) = \begin{cases} \alpha(m) & \text{if } m \in \mathrm{dom}(\alpha),\ n \in \mathrm{dom}(\beta) \text{ and } \alpha(m) = \beta(n), \\ \text{undefined} & \text{otherwise.} \end{cases}$$

It follows that $\mathcal{L}_p(S)$ is a lattice. Note that $\mathcal{L}(S)$ is not a lattice, if S contains more than one element [1, 4].

Proposition 2.5 *$\mathcal{L}_p(S)$ is a distributive lattice.*

P r o o f. For numberings $\alpha, \beta, \gamma \in \mathrm{Num}_p(S)$ we only have to show [6] that

$$(\alpha \oplus \beta) \sqcap \gamma \leq \alpha \oplus (\beta \sqcap \gamma).$$

As follows from the definitions,

$$((\alpha \oplus \beta) \sqcap \gamma)(\langle 2m, n\rangle) = \alpha(m),$$

if $m \in \mathrm{dom}(\alpha)$, $n \in \mathrm{dom}(\gamma)$ and $\alpha(m) = \gamma(n)$. Moreover

$$((\alpha \oplus \beta) \sqcap \gamma)(\langle 2m+1, n\rangle) = \beta(m),$$

if $m \in \mathrm{dom}(\beta)$, $n \in \mathrm{dom}(\gamma)$ and $\beta(m) = \gamma(n)$. In any other case $((\alpha \oplus \beta) \sqcap \gamma)(a)$ is undefined.

On the other hand,

$$(\alpha \oplus (\beta \sqcap \gamma))(2m) = \alpha(m),$$

if $m \in \mathrm{dom}(\alpha)$, and

$$(\alpha \oplus (\beta \sqcap \gamma))(2\langle i, j\rangle + 1) = \beta(i),$$

if $i \in \mathrm{dom}(\beta)$, $j \in \mathrm{dom}(\gamma)$ and $\beta(i) = \gamma(j)$. In all other cases $(\alpha \oplus (\beta \sqcap \gamma))(a)$ is undefined. It follows that the function which maps $\langle 2m, n\rangle$ to $2m$ and $\langle 2m+1, n\rangle$ to $2\langle m, n\rangle + 1$ is a witness function. □

Corollary 2.6 *A minimal element of $\mathcal{L}_p(S)$, if any, is its least element.*

3 A comparison of the degree structures

To avoid trivialities, we will assume that S contains at least two elements.

First of all, note that reducibility relation behaves more differently on $\mathrm{Num}(S)$ and $\mathrm{Num}_p(S)$ than it may seem at the first glance. For arbitrary $\alpha \in \mathrm{Num}(S)$ the set

$$\{\, \beta \in \mathrm{Num}(S) \mid \beta \leq \alpha \,\}$$

is infinite, but countable. In contrary, if $\alpha \in \mathrm{Num}_p(S)$ then the set

$$\{\, \beta \in \mathrm{Num}_p(S) \mid \beta \leq \alpha \,\}$$

is uncountable. Indeed, consider the cylindrification $c(\alpha)$ of α, then by identifying mappings with their graphs we have uncountably many partial numberings $\beta \subseteq c(\alpha)$ of the same set S. The identity function on ω reduces each of these numberings β to $c(\alpha)$.

It follows that for numberings $\alpha \in \mathrm{Num}(S)$, $\deg(\alpha)$ is properly contained in $\deg_p(\alpha)$. As is easily verified, the mapping that takes $\deg(\alpha)$ to $\deg_p(\alpha)$, for $\alpha \in \mathrm{Num}(S)$, is an order-monomorphism which preserves finite least upper bounds.

Proposition 3.1 *$\mathcal{L}(S)$ is embeddable into $\mathcal{L}_p(S)$ as upper subsemilattice*

Proposition 3.2 *If S is a finite nonempty set, then both $\mathcal{L}_p(S)$ and $\mathcal{L}(S)$ have a least element.*

Proof. Let α be a partial Friedberg numbering of S, i.e., let α be a partial one-to-one mapping onto S. Then $\deg_p(\alpha)$ is the least element of the semilattice $\mathcal{L}_p(S)$.

In order to generate the least element of $\mathcal{L}(S)$, we totalise α. Let $S = \{s_0, \ldots, s_{n-1}\}$ and without restriction assume that $\alpha(i) = s_i$, for $i < n$. Set $\beta(j) = \alpha(i)$, if $j \equiv i \mod n$. Then $\beta \in \mathrm{Num}(S)$ with $\beta \leq \alpha$. It follows that $\deg(\beta)$ is the least element of $\mathcal{L}(S)$. □

Proposition 3.3 *For every set S with at least two elements, both $\mathcal{L}_p(S)$ and $\mathcal{L}(S)$ have no maximal elements.*

Proof. Since $\mathcal{L}_p(S)$ and $\mathcal{L}(S)$ are upper semilattices, it follows that they can only have one maximal element or none at all. For sets S with at least two elements it is known that $\mathcal{L}(S)$ is uncountable [1]. Note that for every numbering $\nu \in \mathrm{Num}_p(S)$ and any total computable function f there is at most one total numbering of S which is reducible to ν by f. Hence, both $\mathcal{L}_p(S)$ and $\mathcal{L}(S)$ have no greatest element. □

4 Partial Friedberg numberings

Friedberg numberings play an important role in the theory of total numberings. As is well known, they are minimal with respect to the reducibility pre-order and if the numbered set is infinite, there are uncountably many of them which are pairwise incomparable. Moreover in this case, every decidable total numbering is equivalent to such a numbering [1]. Here, a total numbering α is *decidable* if the set of all $\langle i, j \rangle$ with $\alpha(i) = \alpha(j)$ is recursive.

Proposition 4.1 *Let S be an infinite set. Then the degree of any partial Friedberg numbering is not minimal in $\mathcal{L}_p(S)$.*

Proof. Let $\alpha \in \mathrm{Num}_p(S)$ be any partial Friedberg numbering. We will define a partial Friedberg numbering $\beta \in \mathrm{Num}_p(S)$ such that

$$\beta \leq \alpha, \quad \text{but} \quad \alpha \not\leq \beta.$$

Let $f \in P^{(1)}$ be a surjective function so that the pre-image $f^{-1}(\{i\})$ is infinite, for every $i \in \omega$. Any universal partial computable function may be used as f. If we take exactly one element $j_i \in f^{-1}(\{i\})$, for every $i \in \mathrm{dom}(\alpha)$, set $\beta(j_i) = \alpha(i)$ and let β be undefined in any other case, we obtain a partial Friedberg numbering of S which is reducible to α by the function f.

In order to ensure that also the second condition holds, we have to be more careful in our choice of the j's. We have to guarantee that no partial computable function φ_n reduces α to β. This can be achieved by a diagonal construction.

Let $a_0, a_1, a_2, \ldots$ be a listing of the elements of $\mathrm{dom}(\alpha)$. For every n choose a number b_n such that

$$f(b_n) = a_n, \quad \text{but} \quad b_n \neq \varphi_n(a_n)$$

and define β by

$$\beta(b_i) = \alpha(a_i),$$

for $i \in \omega$. In any other case let β be undefined. □

The above construction shows that for every partial numbering there is a partial Friedberg numbering reducible to it, but not vice versa. By iterating the construction we obtain an infinite descending chain of non-equivalent partial Friedberg numberings.

Corollary 4.2 *Let S be an infinite set. Then there is an infinite descending chain generated by partial Friedberg numberings below every degree in $\mathcal{L}_p(S)$.*

Proposition 4.3 *Let the set S have at least two elements and $\alpha \in \mathrm{Num}_p(S)$. Then $deg_p(\alpha)$ is uncountable, but contains at most countably many partial Friedberg numberings.*

P r o o f. Consider the following straightforward modification of cylindrification. For any nonempty subset Y of ω set $\beta_Y(\langle m, n\rangle) = \alpha(m)$, if $m \in \mathrm{dom}(\alpha)$ and $n \in Y$, and let $\beta_Y(\langle m, n\rangle)$ be undefined, otherwise. Then $\beta_Y \in \mathrm{Num}_p(S)$. Moreover, for any fixed $a \in Y$ the functions π_1 and f with $f(m) = \langle m, a\rangle$, for $m \in \omega$, reduce β_Y to α and α to β_Y, respectively. This shows that $\mathrm{deg}_p(\alpha)$ is uncountable.

For the verification of the remaining statement let $\gamma \in \mathrm{Num}_p(S)$ be a partial Friedberg numbering with $\alpha \leq \gamma$ and let $h \in P^{(1)}$ be a witness function. Then $h(\mathrm{dom}(\alpha)) = \mathrm{dom}(\gamma)$. Thus, if $\eta \in \mathrm{Num}_p(S)$ is a further partial Friedberg numbering so that $\alpha \leq \eta$ via h, then $\gamma = \eta$. Therefore, α can be reduced to at most countably many partial Friedberg numberings. □

In the proof of the last statement we showed the following fact which will be used again later.

Corollary 4.4 *Let $\alpha, \gamma, \eta \in \mathrm{Num}_p(S)$ so that η and γ are partial Friedberg numberings. If there is some partial computable function which witnesses that both $\alpha \leq \gamma$ and $\alpha \leq \eta$, then $\gamma = \eta$.*

Proposition 4.5 *Let S be an infinite set and $\alpha \in \mathrm{Num}_p(S)$. Then there are uncountably many pairwise incomparable partial Friedberg numberings which are reducible to α.*

P r o o f. Let $S = \{s_0, s_1, \dots\}$ and $i_n \in \alpha^{-1}(\{s_n\})$, for $n \in \omega$. We will show that there are uncountably many pairwise incomparable partial Friedberg numberings below the cylindrification of α.

Obviously, for all $n \in \omega$, there are distinct numbers $a_n, b_n \in \omega$ such that $\varphi_{\pi_1(n)}(\langle i_n, a_n\rangle) \neq \langle i_n, b_n\rangle$. Each sequence $(\langle i_n, c_n\rangle)_{n\in\omega}$ with $c_n \in \{a_n, b_n\}$, for $n \in \omega$, defines a partial Friedberg numbering β of S, which is reducible to $c(\alpha)$ by the identity function on ω: set $\beta(\langle i_n, c_n\rangle) = s_n$, for $n \in \omega$, and let $\beta(k)$ be undefined, otherwise.

Now, let $\mathcal{C}$ be an uncountable family of subsets of ω as in Lemma 2.1 and for every $A \in \mathcal{C}$ let $(j^A_n)_{n\in\omega}$ be the sequence with $j^A_n = \langle i_n, a_n\rangle$, if $n \in A$, and $j^A_n = \langle i_n, b_n\rangle$, otherwise. Then we obtain an uncountable family of sequences of the above kind. We consider the partial Friedberg numberings defined by these sequences.

It remains to show that they are pairwise incomparable. Let β, γ be two such numberings defined by sequences $(j^A_n)_{n\in\omega}$ and $(j^B_n)_{n\in\omega}$, respectively, and assume that β is reducible to γ by φ_m. Then there is some $n \in A \setminus B$ with $\pi_1(n) = m$. Let r be the least such n. It follows that $j^A_r = \langle i_r, a_r\rangle$ and $j^B_r = \langle i_r, b_r\rangle$. Since

$$\gamma(\langle i_r, b_r\rangle) = s_r = \beta(\langle i_r, a_r\rangle) = \gamma(\varphi_m(\langle i_r, a_r\rangle)),$$

we have that $\langle i_r, b_r\rangle = \varphi_m(\langle i_r, a_r\rangle)$, which is impossible by the choice of a_r and b_r. □

Corollary 4.4 implies that the partial degrees generated by the uncountable antichain of partial Friedberg numberings considered in the preceding proposition cannot have a lower bound. Thus, we obtain the following consequence.

Proposition 4.6 *Let S be an infinite set. Then the lattice $\mathcal{L}_p(S)$ is not complete and has no least element.*

The next theorem summarises what we have shown so far.

Theorem 4.7 *Let S be a nonempty set. Then $\mathcal{L}_p(S)$ is a distributive lattice without maximal elements. Moreover, the following two statements hold:*

1. *If S is finite then $\mathcal{L}_p(S)$ has a least element.*

2. *In case S is infinite, $\mathcal{L}_p(S)$ has no least element. In addition, it is not complete and below each degree there is an infinite descending chain as well as an uncountable antichain, both generated by partial Friedberg numberings.*

5 Computable numberings

Let $\mathcal{T} = (T, \tau)$ be a countable topological T_0-space with a countable basis $\mathcal{B}$ and let B be a total numbering of $\mathcal{B}$. In computer science applications of topology as well as in constructive approaches to topology the basic open sets are considered as finitely describable objects. They are basic properties which determine the points of the space. The numbering B can be thought of as being obtained by an encoding of the finite descriptions. A central aim of topology is making statements about approximation. In the applications and approaches just mentioned there is a canonical relation between the (code numbers of the) descriptions which is stronger than usual set inclusion between the described sets. This relation is computably enumerable (c.e.), which in general is not true for set inclusion. It is known from effective topology that one has to use this stronger relation (cf. e.g. [7, 8, 9]).

Definition 5.1 Let $\mathcal{T} = (T, \tau)$ be a countable topological T_0-space, $\mathcal{B}$ be a countable basis of τ, B be a numbering of $\mathcal{B}$, and $\prec_B$ be a transitive binary relation on ω. $(\mathcal{T}, \mathcal{B}, B, \prec_B)$ is an *effectively given* topological space, if the following four conditions hold:

1. B is total.
2. $\prec_B$ is c.e.
3. For all $m, n \in \omega$, from $m \prec_B n$ it follows that $B_m \subseteq B_n$.
4. For all $z \in T$ and $m, n \in \omega$ with $z \in B_m \cap B_n$ there is a number $a \in \omega$ such that $z \in B_a$, $a \prec_B m$ and $a \prec_B n$.

For the remainder of this section we assume that $(\mathcal{T}, \mathcal{B}, B, \prec_B)$ is an effectively given topological space.

Definition 5.2 Let $\alpha \in \mathrm{Num}_p(T)$. We say that α is *computable* if there is some c.e. set L such that for all $i \in \mathrm{dom}(\alpha)$ and $n \in \omega$, $\langle i, n\rangle \in L$ if and only if $\alpha(i) \in B_n$.

Thus, α is computable exactly if all basic open sets B_n are completely enumerable, uniformly in n. Here, a subset X of T is *completely enumerable*, if there is a c.e. set A such that $\alpha(i) \in X$ if and only if $i \in A$, for all $i \in \mathrm{dom}(\alpha)$. Let $\mathrm{Com}(\mathcal{T})$ denote the set of all computable numberings of T.

Lemma 5.3 *Let $\alpha, \beta \in \mathrm{Num}_p(T)$. Then the following two statements hold:*

1. *If $\beta \leq \alpha$ and α is computable, so is β.*
2. *If α and β are computable, so are both $\alpha \oplus \beta$ and $\alpha \sqcap \beta$.*

Proof. (1). Let $L \subseteq \omega$ and $f \in P^{(1)}$, respectively, witness that α is computable and β is reducible to α. Then the set $\{\, \langle i, n\rangle \mid i \in \mathrm{dom}(f) \wedge n \in \omega \wedge \langle f(i), n\rangle \in L \,\}$ witnesses the computability of β.

(2). Let $M, N \subseteq \omega$ witness the computability of α and β, respectively. Then the set

$$\{\, \langle 2i, n\rangle \mid \langle i, n\rangle \in M \,\} \cup \{\, \langle 2i+1, n\rangle \mid \langle i, n\rangle \in N \,\}$$

witnesses the computability of $\alpha \oplus \beta$. The computability of $\alpha \sqcap \beta$ is a consequence of (1). □

It follows that the partial degree of a computable numbering of T contains only computable numberings.

Definition 5.4 The algebraic structure

$$\mathcal{R}(\mathcal{T}) = (\mathrm{Com}(\mathcal{T})/_{\equiv}, \leq)$$

is called *Rogers lattice* of computable numberings of the set T of elements of $\mathcal{T}$.

The following result is a consequence of Lemma 5.3.

Proposition 5.5 *$\mathcal{R}(\mathcal{T})$ is an ideal of $\mathcal{L}_p(T)$. In particular, $\mathcal{R}(\mathcal{T})$ is a distributive lattice.*

As we have seen in Proposition 3.3, the lattice $\mathcal{L}_p(T)$ has no maximal elements, if T contains at least two points. In case of the sublattice $\mathcal{R}(\mathcal{T})$ the situation is different. There is a greatest element, which is generated by numberings that allow to pass uniformly from a normed computable enumeration of a strong base of the neighbourhood filter of some point to an index of this point [9].

It is well known that each point y of a T_0-space is uniquely determined by its neighbourhood filter $\mathcal{N}(y)$ and/or a base of it.

Definition 5.6 Let $\mathcal{H}$ be a filter. A nonempty subset $\mathcal{F}$ of $\mathcal{H}$ is called *strong base* of $\mathcal{H}$ if the following two conditions hold:

1. For all $m, n \in \omega$ with $B_m, B_n \in \mathcal{F}$ there is some index $a \in \omega$ such that $B_a \in \mathcal{F}$, $a \prec_B m$, and $a \prec_B n$.

2. For all $m \in \omega$ with $B_m \in \mathcal{H}$ there is some index $a \in \omega$ such that $B_a \in \mathcal{F}$ and $a \prec_B m$.

If α is computable, a strong base of basic open sets can effectively be enumerated for each neighbourhood filter. For effectively given spaces this can be done in a normed way (cf. [9]).

Definition 5.7 An enumeration $(B_{f(a)})_{a\in\omega}$ with $f\colon \omega \to \omega$ is said to be *normed* if f is decreasing with respect to $\prec_B$. If f is computable, it is also called *computable* and any Gödel number of f is said to be an *index* of it.

In case $(B_{f(a)})$ enumerates a strong base of the neighbourhood filter of some point, we say it *converges* to that point.

Definition 5.8 Let $\alpha \in \mathrm{Num}_p(T)$. We say that:

1. α *allows effective limit passing* if there is a function $\mathrm{pt} \in P^{(1)}$ such that, if m is an index of a normed computable enumeration of basic open sets which converges to some point $y \in T$, then $\mathrm{pt}(m)\!\downarrow \in \mathrm{dom}(\alpha)$ and $\alpha(\mathrm{pt}(m)) = y$.

2. α is *acceptable* if it allows effective limit passing and is computable.

It has been proved in [9] that the point set of every effectively given topological space $\mathcal{T}$ which has a computable numbering, also has an acceptable one. Moreover, it was shown for partial numberings α, β of T that $\alpha \le \beta$, if α is computable and β allows effective limit passing. In case that α is acceptable we have that β is acceptable as well, exactly if $\alpha \equiv \beta$. It follows that the acceptable numberings are maximal among the computable ones. In addition, they form a single degree consisting only of acceptable numberings.

Proposition 5.9 *If $\mathcal{R}(\mathcal{T})$ is nonempty then it has a greatest element.*

With Propositions 3.2, 4.5 and 4.6 as well as Corollary 4.2 we thus have the following result.

Theorem 5.10 *Let $(\mathcal{T}, \mathcal{B}, B, \prec_B)$ be an effectively given topological space. Then the following two statements hold:*

1. *If T is finite then Rogers lattice $\mathcal{R}(\mathcal{T})$ has both a greatest and a least element.*

2. *If T is infinite then Rogers lattice $\mathcal{R}(\mathcal{T})$ has a greatest, but no least element. Moreover, it is not complete and below each of its elements there is an infinite descending chain as well as an uncountable antichain, both generated by computable Friedberg numberings.*

6 Final remarks

In this paper the different behaviour of total and partial numberings with respect to reducibility was pointed out. Partial numberings appear quite naturally in computability studies for topological spaces. In the case of such numberings a computable function may reduce several numberings to one numbering.

This was used to show that each partial degree is uncountable, whereas a total degree is only countable. The collection of all partial degrees is a distributive lattice with respect to the order induced by the reducibility preorder. If the numbered set is infinite, the lattice is neither complete nor has it a least element. In addition, there is an infinite descending chain and an uncountable antichain below every degree, both generated by partial Friedberg numberings. Hence, these numberings are no longer minimal in the reducibility preorder, in contrary with the case of total numberings.

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References

[1] Yu. L. Ershov, Theorie der Numerierungen I. Zeitschr. f. math. Logik und Grundlagen d. Math. **19**, 289–388 (1973).
[2] Yu. L. Ershov, Theorie der Numerierungen II. Zeitschr. f. math. Logik und Grundlagen d. Math. **21**, 473–584 (1975).
[3] Yu. L. Ershov, Theorie der Numerierungen III. Zeitschr. f. math. Logik und Grundlagen d. Math. **23**, 289–371 (1977).
[4] Yu. L. Ershov, Theory of Numberings (in Russian) (Nauka, Moscow 1977).
[5] Yu. L. Ershov, Theory of numberings. In: Handbook of Computability Theory (E. R. Griffor, ed.), pp. 473–503 (Elsevier, Amsterdam 1999).
[6] G. Grätzer, General Lattice Theory (Birkhäuser Verlag, Basel 1978).
[7] D. Spreen, On some decision problems in programming. Inform. and Computation **122**, 120–139 (1995); Corrigendum **148**, 241–244 (1999).
[8] D. Spreen, Effective inseparability in a topological setting. Ann. Pure Appl. Logic **80**, 257–275 (1996).
[9] D. Spreen, On effective topological spaces. J. Symb. Logic **63**, 185–221 (1998); Corrigendum **65**, 1917–1918 (2000).
[10] D. Spreen, Can partial indexings be totalized? J. Symb. Logic **66**, 1157–1185 (2001).