Numberings in the theory of computable models.

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The problems in the theory of computable models

In this my talk I will discuss some application of results of the theory of numberings and open problems in the theory of Computable models.

We will use some basic facts and concepts from model theory, universal algebra, and computability theory. We will follow in these knowledge of the first several chapters of the textbooks by Chang and Keisler on model theory, by C.Ash and J.Knight, by Ershov and Goncharov, and Handbook of recursive mathematics on theory of computable models will suffice to follow this talk.

Model Theory + Computability Theory

At present, the notion of computability is extremely important in mathematics. The active development of mathematical logic motivated the development of the mathematical theory of computability. The study of the computability phenomenon leads to a number of very interesting directions in mathematics and applications (cf. *Handbook on Recursive Mathematics* [1]).

One of such directions is the theory of constructive (computable) models presented in this talk. This theory was initiated by A.Fröhlich, J.Shepherdson, A.Mal'tsev, A.Kuznetsov, O.Rabin, and R.Vaught in the 1950's. Within the framework of this theory, the dependence of algorithmic properties of abstract models is studied by constructing representations of models on natural numbers. Relationships between algorithmic properties and structural properties of such models are also the subject of this theory. The development of this general approach began with the pioneering works of Markov and Novikov concerning algorithmic problems in algebra for finitely defined semigroups and groups.

The systematic studies of constructive and computable algebraic systems were initiated by A. Mal'tsev and was further developed by his pupils and colleagues. Mal'tsev's approach is based on the notion of a numbering.

Owing to a numbering of the universe of an algebraic system (i.e., a mapping of the set of all natural numbers onto the universe of this system), we can formulate algorithmic questions on properties of this algebraic system and express such questions in terms of numbers (names) of elements of the system. Moreover, fundamental algorithmic problems over abstract structures can be reduced to the study of algorithms on natural numbers or on words of some finite alphabet. Such numberings can be regarded as some system of coordinates (which is “effective” provided that the numbering is a constructivization). The use of numberings (constructivizations) permits one to study algorithmic properties of algebraic systems and to understand how these properties depend on the choice of a constructivization.

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1. Yu. L. Ershov,S. S. Goncharov, A. Nerode, J. Remmel, Handbook of recursive mathematics, in Studies in Logic and The Foundations of Mathematics, v.138-139, Elsevier, Amsterdam, 1998.

2. S. S. Goncharov, Countable Boolean Algebras and Decidability, in Siberian school of algebra and logic, v 3,Consultants bureau, New York,1997.

3. Yu. L. Ershov, S. S. Goncharov, Constructive Models, in Siberian school of algebra and logic, v 6, Consultants bureau, New York, 2000.

4. C.Ash, J.Knight, Computable Structures and the Hyperarithmetical Hierarchy, in Studies in Logic and the Foundations of Mathematics, v 144, Elsevier, Amsterdam, 2000.

5. S. S. Goncharov, B. Khoussainov,Problems in the theory of constructive algebraic systems, Contemporary Mathematics, v 257,American Mathematical Society,Providence, Rhode Island, 2000.

We mention some fundamental problems in this direction [1-4] such as

- the existence of computable representations

- (non)uniqueness of computable representations and algorithmic dimension (with special properties),

- the classification of computable models constructed structures connected with computable models,

- the computability of families of computable representations and classes of computable models,

- the classification of algorithmic problems with respect to complexity in different sense,

- Connection between definability and complexity in some sense.

We fix a language

$L = < f_0^{n_0}, f_1^{n_1}, \ldots, P_0^{m_0}, P_1^{m_1}, \ldots, c_0, c_1, \ldots >$

for which the functions $i \to n_i$ and $j \to m_j$ are computable. Such languages are called **computable languages**. The symbols $f_i^{n_i}$ and $P_j^{m_j}$ are operation and predicate symbols, respectively.

The full diagram of **A** is the set

$FD(\mathbf{A}) = \{\phi(a_1, \ldots, a_n) \mid \phi(x_1, \ldots, x_n)$ is a formula, $\mathbf{A} \models \phi(a_1, \ldots, a_n), a_1, \ldots, a_n \in A\}$.

From algebraic point of view, then it is natural to consider the atomic diagram of **A**,

that is the set

$AD(\mathbf{A}) = \{\phi(a_1, \ldots, a_n) \mid \phi(x_1, \ldots, x_n)$ is an atomic formula or a negation of an atomic formula, $\mathbf{A} \models \phi(a_1, \ldots, a_n), a_1, \ldots, a_n \in A\}$.

In the literature there is an equivalent terminology for constructive models and constructivizations that does not refer to numberings. These are computable models and computable presentations.

An model is called **computable** if the domain of the system is ω and the atomic diagram is a computable set. A model is called **decidable** if the domain of the system is ω and the full diagram is a computable set.

Notions related to computable categoricity

Let $\mathcal{A}$ be a computable structure.

We say that $\mathcal{A}$ is *computably categorical* if for all computable $\mathcal{B} \cong \mathcal{A}$, there is a computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

Similarly, $\mathcal{A}$ is Δ^0_α *categorical* if for all computable $\mathcal{B} \cong \mathcal{A}$, there is a Δ^0_α isomorphism.

We say that $\mathcal{A}$ is *relatively computably categorical* if for all $\mathcal{B} \cong \mathcal{B}$, there is an isomorphism that is computable relative to $\mathcal{B}$, and

$\mathcal{A}$ is *relatively* Δ^0_α *categorical* if for all $\mathcal{B} \cong \mathcal{A}$, there is a $\Delta^0_\alpha(\mathcal{B})$ isomorphism.

Kleene's $\mathcal{O}$

The system consists of a set $\mathcal{O}$ of notations, together with a partial ordering $<_{\mathcal{O}}$.

The ordinal 0 gets notation 1.

If a is a notation for α, then 2^a is a notation for $\alpha + 1$.

Then $a <_{\mathcal{O}} 2^a$, and also, if $b <_{\mathcal{O}} a$, then $b <_{\mathcal{O}} 2^a$.

Suppose α is a limit ordinal. If φ_e is a total function, giving notations for an increasing sequence ordinals with limit α, then $3 \cdot 5^e$ is a notation for α.

For all n, $\varphi_e(n) <_{\mathcal{O}} 3 \cdot 5^e$, and if $b <_{\mathcal{O}} \varphi_e(n)$, then $b <_{\mathcal{O}} 3 \cdot 5^e$.

The set $\mathcal{O}$ is Π^1_1 complete.

Computable infinitary formulas

Next, we say a little about computable infinitary formulas. Roughly speaking, the computable infinitary formulas are infinitary formulas with disjunctions and conjunctions over c.e. sets.

Taken all together, the computable infinitary formulas have the same expressive power as the formulas in the least admissible fragment of $L_{\omega_1 \omega}$.

1. A computable Σ_0, or Π_0 formula is a finitary open formula.

2. Suppose $\alpha > 0$, where α is a computable ordinal.

(a) A computable Σ_α formula is a c.e. disjunction of formulas $\exists\overline{u}\,\psi(\overline{x},\overline{u})$, where ψ is computable Π_β for some $\beta < \alpha$.

(b) A computable Π_α formula is a c.e. conjunction of formulas $\forall\overline{u}\,\psi(\overline{x},\overline{u})$, where ψ is computable Σ_β for some $\beta < \alpha$.

A *Scott family for $\mathcal{A}$* is a set Φ of formulas, with a fixed tuple of parameters $\overline{c}$ in $\mathcal{A}$, such that

1. each tuple in $\mathcal{A}$ satisfies some $\varphi \in \Phi$, and

2. if $\overline{a}$, $\overline{b}$ are tuples in $\mathcal{A}$ satisfying the *same* formula $\varphi \in \Phi$, then there is an automorphism of $\mathcal{A}$ taking $\overline{a}$ to $\overline{b}$.

A *formally c.e. Scott family* is a c.e. Scott family made up of finitary existential formulas.

A *formally Σ^0_α Scott family* is a Σ^0_α Scott family made up of "computable Σ_α" formulas.

Theorem 1 *For a structure $\mathcal{A}$, the set $\{\overline{a} : \mathcal{A} \models \varphi(\overline{a})\}$ is $\Sigma^0_\alpha(\mathcal{A})$, if φ is computable Σ_α, and $\Pi^0_\alpha(\mathcal{A})$, if φ is computable Π_α. Moreover, this is so with all imaginable uniformity, over structures and formulas.*

It is easy to see that if $\mathcal{A}$ has a formally c.e. Scott family, then it is relatively computably categorical, so it is computably categorical.

More generally, if $\mathcal{A}$ has a formally Σ^0_α Scott family, then we can see, using Theorem 1, that it is relatively Δ^0_α categorical, so it is Δ^0_α categorical.

Goncharov showed that, under some added effectiveness conditions (on a single copy), if $\mathcal{A}$ is computably categorical, then it has a formally c.e. Scott family.

Ash showed that, under some effectiveness conditions (on a single copy), if $\mathcal{A}$ is Δ^0_α categorical, then it has a formally Σ^0_α Scott family.

For the relative notions, the effectiveness conditions disappear. Ash-Knight-Manasse-Slaman, Chisholm proved

Theorem 2 *A computable structure $\mathcal{A}$ is relatively Δ^0_α categorical iff it has a formally Σ^0_α Scott family. In particular, $\mathcal{A}$ is relatively computably categorical iff it has a formally c.e. Scott family.*

It would be pleasant if computable categoricity and relative computable categoricity were the same, and the effectiveness conditions could be dropped from these results.

However, It was showed by me that this is not the case, using an Selivanov result from numbering theory.

Later Cholak, Goncharov, Khoussainov, and Shore gave an example of a structure that is computably categorical, but ceases to be after naming a constant. It follows from Theorem 2 that such a structure is not relatively computably categorical.

Notions related to computable dimension

The *computable dimension* of a structure $\mathcal{A}$ is the number of computable copies, up to computable isomorphism. Similarly, the Δ^0_α *dimension* is the number of computable copies, up to Δ^0_α isomorphism.

Basic results from Numbering theory

For $\mathcal{S} \subseteq P(\omega)$ an *numbering* is a binary relation ν such that $\mathcal{S} = \{\nu(i) : i \in \omega\}$, where $\nu(i) = \{x : (i, x) \in \nu\}$.

A *Friedberg numbering* of $\mathcal{S}$ is an numbering ν that is $1 - 1$, in the sense that if $i \neq j$, then $\nu(i) \neq \nu(j)$.

Suppose ν, μ are two numberings of the same family $\mathcal{S}$.

We write $\nu \leq \mu$ if there is a computable function f such that for all i, $\nu(i) = \mu(f(i))$—we can effectively pass from a ν-index to a μ-index for the same set.

We say that ν and μ are *computably equivalent* if $\mu \leq \nu$ and $\nu \leq \mu$. Note that if μ and ν are Friedberg numberings of $\mathcal{S}$, then $\mu \leq \nu$ implies $\nu \leq \mu$.

A family $\mathcal{S} \subseteq P(\omega)$ is *discrete* if for each $A \in \mathcal{S}$, there exists $\sigma \in 2^{<\omega}$ such that for all $B \in \mathcal{S}$, $\sigma \subseteq \chi_B$ iff $B = A$.

The family is *effectively discrete* if there is a c.e. set $E \subseteq 2^{<\omega}$ such that

(a) for each $A \in \mathcal{S}$, there is $\sigma \in E$ such that $\sigma \subseteq \chi_A$, and

(b) for all $\sigma \in E$ and all $A, B \in \mathcal{S}$, if $\sigma \subseteq \chi_A, \chi_B$, then $A = B$.

Theorem 3 (Selivanov) *There exists a family $\mathcal{S} \subseteq P(\omega)$ which has a unique computable Friedberg numbering, up to computable equivalence, and is discrete but not effectively discrete.*

Theorem 4 (Goncharov) *For each finite* $n \geq 1$ *there is a family of sets with just* n *computable Friedberg numberings, up to computable equivalence.*

Theorem 5 (Wehner) *There is a family* $\mathcal{S} \subseteq P(\omega)$ *with numberings in all non-computable degrees but no computable numbering.*

The numbering results of Selivanov, Goncharov, and Wehner all relativize.

Turning a family of sets into a graph

Let $\mathcal{S}$ be a family of sets. For each $A \in \mathcal{S}$, a *daisy graph* $\mathcal{G}_A$ consists of one *index* point a at the center, with $a \to a$, and, for each $n \in \omega$, a *petal* of the form $a \to a_0 \to a_m \to a$, where $m = 2n+1$ if $n \in A$. In all, $\mathcal{G}_A$ has one cycle of length 1, and, for each n, one cycle of length either $2n+1$if $n \in A$. Now, let $\mathcal{G}(\mathcal{S})$ be the union of a disjoint family of daisy graphs, one for each $A \in \mathcal{S}$.

Lemma 1 *Let* $\mathcal{S} \subseteq P(\omega)$.

(a) $\mathcal{G}(\mathcal{S})$ *is a rigid graph,*

(b) If $\mathcal{S}$ *has a unique computable Friedberg numbering, then* $\mathcal{G}(\mathcal{S})$ *is computably categorical.*

(c) If $\mathcal{S}$ *has just* n *computable Friedberg numberings, up to computable equivalence, then* $\mathcal{G}(\mathcal{S})$ *has computable dimension* n.

(d) If $\mathcal{S}$ *is discrete, then every element of* $\mathcal{G}(\mathcal{S})$ *has a finitary existential definition with no parameters.*

(e) Suppose $\mathcal{S}$ *has a computable Friedberg numbering, and is discrete but not effectively discrete. Then* $\mathcal{G}(\mathcal{S})$ *does not have a formally c.e. defining family.*

For $\mathcal{S} \subseteq P(\omega)$, we may also form a graph structure $\mathcal{G}^\infty(\mathcal{S})$ made up of infinitely many copies of $\mathcal{G}_A$ for each $A \in \mathcal{S}$. This structure is not rigid. Copies of the structure correspond to arbitrary numberings of $\mathcal{S}$—not Friedberg numberings.

Lemma 2 *Let $\mathcal{S} \subseteq P(\omega)$. Then the degrees of numberings of $\mathcal{S}$ are the same as the degrees of copies of $\mathcal{G}^\infty(\mathcal{S})$.*

Proof: If ν is an numbering of $\mathcal{S}$, then there is a copy of $\mathcal{G}^\infty(\mathcal{S})$ computable in ν. If $\mathcal{H}$ is a copy of $\mathcal{G}^\infty(\mathcal{S})$, then there is an numbering of $\mathcal{S}$ computable in $\mathcal{H}$, with indices corresponding to index points.

Results of Goncharov, Manasse, Slaman, and Wehner

Here we shall re-work the basic results that we plan to lift.

Theorem 6 (Goncharov) *There is a rigid graph structure $\mathcal{G}$ that is computably categorical with no formally c.e. defining family.*

Theorem 7 (Manasse) *There is a computable structure $\mathcal{A}$ with a relation R that is intrinsically c.e. but not relatively intrinsically c.e.*

Proof: Consider the cardinal sum of disjoint computable copies of the graph structure $\mathcal{G}$ from Theorem 6, and let R be the unique isomorphism.

Theorem 8 (Goncharov) *For each finite* n*, there is a rigid graph structure* $\mathcal{G}$ *with computable dimension* n*.*

Theorem 9 (Slaman, Wehner) *There is a structure* $\mathcal{A}$ *with copies in just the non-computable degrees.*

Coding a Δ^0_α structure in a computable one

To lift the basic results of Goncharov and Manasse, we first relativize, producing a Δ^0_α graph.

We then pass to a computable structure, using a pair of structures to code the arrow relation.

Theorem 10 *For a graph $\mathcal{G}$, and a some pair of structures $\mathcal{B}_1$, $\mathcal{B}_2$, for the same relational language, let $\mathcal{G}^* = (G \cup U, G, U, Q, \ldots)$, where*

1. *G is the universe of $\mathcal{G}$,*

2. *G and U are disjoint,*

3. *Q is a ternary relation assigning to each pair $a, b \in G$ an infinite set $U_{(a,b)}$,*

4. *the sets $U_{(a,b)}$ form a partition of U,*

5. *each relation in $\ldots$ is the union of its restrictions to the sets $U_{(a,b)}$, and*

6. *for each pair $a, b \in G$,*

$$(U_{(a,b)}, \ldots) \cong \begin{cases} \mathcal{B}_1 & \textit{if } \mathcal{G} \models a \to b \\ \mathcal{B}_2 & \textit{otherwise .} \end{cases}$$

There are conditions on the pair of structures $\mathcal{B}_i$ *under which a* Δ^0_α *graph structure* $\mathcal{G}$ *gives rise to a computable structure* $\mathcal{G}^*$*.*

The *standard back-and-forth relations* $\leq_\beta$ on the set of pairs $\{(i, \overline{b}) : \overline{b} \in \mathcal{B}_i\}$, are defined inductively as follows:

(i) $(i, \overline{b}) \leq_1 (j, \overline{c})$ if the existential formulas true of $\overline{c}$ in $\mathcal{B}_j$ are true of $\overline{b}$ in $\mathcal{B}_i$.

(ii) for $\beta > 1$, $(i, \overline{b}) \leq_\beta (j, \overline{c})$ if for all $\overline{c}'$ in $\mathcal{B}_j$, and all γ such that $1 \leq \gamma < \beta$, there exists $\overline{b}'$ in $\mathcal{B}_i$ such that $(j, \overline{c}, \overline{c}') \leq_\gamma (i, \overline{b}, \overline{b}')$.

Remark: By a result of Karp, $(i, \overline{b}) \leq_\beta (j, \overline{c})$ iff all Π_β formulas of $L_{\omega_1 \omega}$ formulas true of $\overline{b}$ in $\mathcal{B}_i$ are true of $\overline{c}$ in $\mathcal{B}_j$ (not just the computable Π_β formulas).

A pair of structures $(\mathcal{B}_1, \mathcal{B}_2)$ is α-*friendly* if the structures are computable and the standard back-and-forth relations $\leq_\beta$ are c.e., uniformly in $\beta < \alpha$.

(To make this precise, we fix a notation a for α in O and identify each ordinal $\beta < \alpha$ with its unique notation $b <_O a$.)

Lemma 3 *Let α be a computable successor ordinal. Let $\mathcal{B}_1, \mathcal{B}_2$ be a pair of structures such that*

1. *the pair $(\mathcal{B}_1, \mathcal{B}_2)$ is α-friendly,*

2. *$\mathcal{B}_1$, $\mathcal{B}_2$ satisfy the same Π_β sentences (of $L_{\omega_1\omega}$) for $\beta < \alpha$,*

3. *each $\mathcal{B}_i$ satisfies some computable Π_α sentence not true in the other,*

Then for any Δ^0_α set S, there is a uniformly computable sequence $(C_n)_{n\in\omega}$ such that

$$\mathcal{C}_n \cong \begin{cases} \mathcal{B}_1 & \text{if } n \in S \\ \mathcal{B}_2 & \text{otherwise .} \end{cases}$$

We need pairs of structures $\mathcal{B}_i$ satisfying the hypotheses of Lemma 3.

In addition, each $\mathcal{B}_i$ will be *uniformly relatively Δ^0_α categorical*; i.e., given an X-computable index for $\mathcal{C} \cong \mathcal{B}_i$, we can find a $\Delta^0_\alpha(X)$ index for an isomorphism from $\mathcal{B}_i$ onto $\mathcal{C}$.

We need some notation to describe some of the structures. If $\mathcal{C}_1$, $\mathcal{C}_2$ are structures for the same relational language, we write $\mathcal{C}_1|\mathcal{C}_2$ for the cardinal sum, where this includes unary predicates for the two universes.

Lemma 4 *For each computable successor ordinal α, there is an α-friendly pair of structures $\mathcal{B}_1$, $\mathcal{B}_2$ such that*

1. *$\mathcal{B}_1$ and $\mathcal{B}_2$ satisfy the same Π_β sentences (of $L_{\omega_1\omega}$) for $\beta < \alpha$,*

2. *each $\mathcal{B}_i$ satisfies some computable Π_α sentence not true in the other,satisfying the conditions of Lemma 3,*

3. *each $\mathcal{B}_i$ is uniformly relatively Δ^0_α-categorical.*

Proof:

For $\alpha = 1$, we let $\mathcal{B}_1$, $\mathcal{B}_2$ be orderings of types ω and ω^*.

For $\alpha = 2$, we use $\omega|\omega^2$ and $\omega^2|\omega$.

For $\alpha = 3$, we use $Z \cdot \omega$ and $Z \cdot \omega^2$.

For $\alpha = 4$, we use $\omega^2|\omega^3$ and $\omega^3|\omega^2$.

For $\alpha = 2n + 1$, we use $Z^n \cdot \omega$ and $Z^n \cdot \omega^*$.

For $\alpha = 2n + 2$, we use $\omega^n|\omega^{n+1}$ and $\omega^{n+1}|\omega^n$.

For $\alpha = \omega$, we use $\omega^\omega|\omega^{\omega+1}$ and $\omega^{\omega+1}|\omega^\omega$.

For $\alpha = \omega + 1$, we use $Z^\omega \cdot \omega$ and $Z^\omega \cdot \omega^*$.

For limit α, we use $\omega^\alpha|\omega^{\alpha+1}$ and $\omega^{\alpha+1}|\omega^\alpha$.

For $\alpha + 1$, where α is a limit ordinal, we use $Z^{\alpha} \cdot \omega$ and $Z^{\omega} \cdot \omega^*$.

For $\alpha + 2n$, where α is a limit ordinal, we use $\omega^{\alpha+n}|\omega^{\alpha+n+1}$ and $\omega^{\alpha+n+1}|\omega^{\alpha+n}$.

For $\alpha + 2n + 1$, where α is a limit ordinal, we use $Z^{\omega+n} \cdot \omega$ and $Z^{\omega+n} \cdot \omega^*$.

We note that if α is a limit ordinal, then structures that satisfy the same Π_β formulas for all $\beta < \alpha$ also satisfy the same Π_α formulas.

Lemma 5 *Suppose $\mathcal{G}$ is a graph structure, and $\mathcal{G}^*$ is constructed from $\mathcal{G}$, $\mathcal{B}_i$ in the way that was described at the beginning of this section.*

Then $\mathcal{G}$ has a Δ^0_α copy iff $\mathcal{G}^*$ has a computable copy. More generally, for any X, $\mathcal{G}$ has a $\Delta^0_\alpha(X)$ copy iff $\mathcal{G}^*$ has an X-computable copy.

In addition,

(a) if $\mathcal{G}$ has just one Δ^0_α copy, up to Δ^0_α isomorphism, then $\mathcal{G}^*$ is Δ^0_α categorical,

(b) if $\mathcal{G}$ has just n Δ^0_α copies, up to Δ^0_α isomorphism, then $\mathcal{G}^*$ has Δ^0_α dimension n.

(c) if $\mathcal{G}$ has no Σ^0_α Scott family made up of finitary existential formulas, then $\mathcal{G}^*$ has no formally Σ^0_α Scott family.

Lifting the basic results

Here is our lifting of the result of Goncharov on structures that are computably categorical but not relatively computably categorical.

Theorem 11 *For each computable successor ordinal α, there is a structure that is Δ^0_α categorical but not relatively Δ^0_α categorical(and without Σ^0_α–Scott family).*

Here is our lifting of the result of Manasse on relations that are intrinsically c.e. but not relatively intrinsically c.e.

Theorem 12 *For each computable successor ordinal* α*, there is a computable structure with a relation that is intrinsically* Σ^0_α *but not relatively intrinsically* Σ^0_α*.*

Here is our lifting of the result of Goncharov on structures with finite computable dimension.

Theorem 13 *For each computable successor ordinal α and each finite n, there is a computable structure with Δ^0_α dimension n.*

Here is our lifting of the result of Slaman and Wehner.

Theorem 14 *For each computable successor ordinal α, there is a structure with copies in just the degrees of sets X such that $\Delta^0_\alpha(X)$ is not Δ^0_α. In particular, for each finite n, there is a structure with copies in just the non-low_n degrees.*

Problems

Problem 1 *For a computable limit ordinal* α*, is there a computable structure which is* Δ^0_α *categorical but not relatively* Δ^0_α *categorical?*

Problem 2 *For a computable limit ordinal* α*, is there a computable structure* $\mathcal{A}$ *with a relation* R *that is intrinsically* Σ^0_α *but not relatively intrinsically* Σ^0_α*.*

Problem 3 *If* $\mathcal{A}$ *is* Δ^1_1 *categorical, must it be relatively* Δ^1_1 *categorical?*

Problem 4 *For a computable limit ordinal α and finite n, is there a structure with Δ^0_α dimension n?*

Problem 5 *Is it true that for any computable limit ordinal α, there is a rigid computable structure which is Δ^0_α categorical but not relatively Δ^0_α categorical?*

For certain computable successor ordinals α, we have a rigid structure which is Δ^0_α categorical but not relatively Δ^0_α categorical, because we have a rigid pair of structures to use in coding a Δ^0_α graph.

Intrinsically c.e. and intrinsically Σ^0_α relations

Definition 1 *Let $\mathcal{A}$ be a computable structure, and let R be a relation on $\mathcal{A}$.*

1. *R is* intrinsically c.e. on $\mathcal{A}$ *if in all computable copies of $\mathcal{A}$, the image of R is c.e.*

2. *R is* relatively intrinsically c.e. on $\mathcal{A}$ *if in* all *copies $\mathcal{B}$ of $\mathcal{A}$ (not just computable copies), the image of R is c.e. relative to $\mathcal{B}$.*

If, in the previous definitions, we replace *c.e.* by Σ^0_α, Δ^1_1, Π^1_1, then we obtain definitions of *intrinsically*, and *relatively intrinsically*, Σ^0_α, Δ^1_1, Π^1_1.

Ash and Nerode gave a syntactical condition sufficient for a relation to be intrinsically c.e. on a structure $\mathcal{A}$. They showed that, with some added effectiveness, on a single copy of $\mathcal{A}$, the condition is also necessary. The syntactical condition is in the following definition.

Definition 2 *A relation R is* formally c.e. *on a structure $\mathcal{A}$ if it is defined by a computable Σ_1 formula; i.e., a c.e. disjunction of existential formulas, with finitely many parameters in $\mathcal{A}$.*

Theorem 15 (Ash-Nerode) *For a relation R on a computable structure $\mathcal{A}$, under some effectiveness conditions*, R is intrinsically c.e. on $\mathcal{A}$ iff it is formally c.e. on $\mathcal{A}$.*

*It is enough to suppose that the existential diagram of $(\mathcal{A}, R)$ is computable.

By Ash-Knight-Manasse-Slaman, Chisholm it is shown that the syntactical condition by itself, with no added effectiveness, is necessary and sufficient for a relation to be relatively intrinsically c.e. on $\mathcal{A}$.

It would be pleasing if the intrinsically c.e. and relatively c.e. relations coincided. Goncharov and Manasse gave examples of relations R on computable structures $\mathcal{A}$ such that R is intrinsically c.e. but not formally c.e., so by Theorem 4, R is not relatively intrinsically c.e. on $\mathcal{A}$.

Harizanov considered the *degree spectrum* of R on $\mathcal{A}$, where this is the set of Turing degrees of images of R in computable copies of $\mathcal{A}$. The following is just one of her results.

Theorem 16 (Harizanov) *Let R be a relation on a structure $\mathcal{A}$, and suppose R is intrinsically c.e., while $\neg R$ is not. Then, under some extra effectiveness conditions*, for any c.e. degree* **d**, *there is a computable copy of $\mathcal{A}$ in which the image of R has degree* **d**.

Example: Let $\mathcal{A}$ be an algebraically closed field of infinite transcendence degree—the characteristic may be either 0 or p. Let R be the set of algebraic elements. Then R is defined by a c.e. disjunction of polynomial equations, with no parameters, so it is (relatively) intrinsically

*Again, it is enough to suppose that the existential diagram of $(\mathcal{A}, R)$ is computable.

c.e. There is a copy of $\mathcal{A}$ satisfying the effectiveness conditions of Theorem 16. Applying the theorem, we can produce computable copies of $\mathcal{A}$ in which the set of algebraic elements has any desired c.e. degree.

There are simple examples in which the spectrum consists of a single c.e. degree.

Example: If $\mathcal{A}$ is the standard model of arithmetic, and R is a c.e. set, then in all computable $\mathcal{B} \cong \mathcal{A}$, the image of R is always c.e., with the same Turing degree as R.

There are now many deep and interesting results, due to Harizanov, Goncharov and Khoussainov, Khoussainov and Shore, Hirschfeldt, Khoussainov, Shore, Slinko, and others, illustrating further possible spectra for intrinsically c.e. relations.

Barker lifted the Ash-Nerode Theorem to arbitrary levels in the hyperarithmetical hierarchy. Here is the natural extension of the syntactical condition *formally c.e.*

Definition 3 *A relation R on a structure $\mathcal{A}$ is* formally Σ^0_α *if it is definable by a computable Σ_α formula, with finitely many parameters.*

Theorem 17 (Barker) *For a structure $\mathcal{A}$ and relation R, under some effectiveness conditions, R is intrinsically Σ^0_α iff it is formally Σ^0_α on $\mathcal{A}$.*

Ash-Knight-Manasse-Slaman, Chisholm proved

Theorem 18 *For a relation R on a computable structure $\mathcal{A}$, R is relatively intrinsically Σ^0_α on $\mathcal{A}$ iff it is formally Σ^0_α on $\mathcal{A}$.*

Intrinsically Δ^1_1 and intrinsically Π^1_1 relations

Soskov gave results characterizing the intrinsically Δ^1_1 relations and the relatively intrinsically Π^1_1 relations. In this section, we first rework Soskov's results, and then give our characterization of intrinsically Π^1_1 relations.

Intrinsically Δ^1_1 relations

Theorem 19 (Soskov) *Suppose $\mathcal{A}$ is computable, and R is a Δ^1_1 relation that is invariant under automorphisms of $\mathcal{A}$. Then R is definable in $\mathcal{A}$ by a computable infinitary formula, with no parameters.*

Corollary 1 *For a computable structure $\mathcal{A}$, and a relation R on $\mathcal{A}$, the following are equivalent:*

1. *R is intrinsically Δ^1_1 on $\mathcal{A}$,*

2. *R is* relatively *intrinsically Δ^1_1 on $\mathcal{A}$,*

3. *R is definable in $\mathcal{A}$ by a computable infinitary formula, with finitely many parameters.*

Intrinsically Π^1_1 relations

Definition 4 *A relation* R *on* $\mathcal{A}$ *is* formally Π^1_1 on $\mathcal{A}$ *if it is defined in* $\mathcal{A}$ *by a* Π^1_1 *disjunction of computable infinitary formulas, with finitely many parameters.*

I. Soskov proved a result that may be restated as follows.

Theorem 20 (Soskov) *For a computable (or hyperarithmetical) structure* $\mathcal{A}$ *and relation* R *on* $\mathcal{A}$*, the following are equivalent:*

1. R *is relatively intrinsically* Π^1_1 *on* $\mathcal{A}$*,*

2. R *is formally* Π^1_1 *on* $\mathcal{A}$*.*

Theorem 21 *Suppose $\mathcal{A}$ is a computable structure, and let R be a relation on $\mathcal{A}$ that is Π^1_1 and invariant under automorphisms of $\mathcal{A}$. Then R is formally Π^1_1. Moreover, there is a definition with no parameters.*

Corollary 2 *For a computable structure $\mathcal{A}$ and relation R, the following are equivalent:*

1. *R is intrinsically Π^1_1 on $\mathcal{A}$,*

2. *R is relatively intrinsically Π^1_1 on $\mathcal{A}$,*

3. *R is formally Π^1_1 on $\mathcal{A}$.*

A relation is *properly* Π^1_1 if it is Π^1_1 and not Σ^1_1.

We have seen that if a relation R on a computable structure $\mathcal{A}$ is invariant and Π^1_1, then it is intrinsically Π^1_1.

Moreover, if there is some computable copy of $\mathcal{A}$ in which the image of R is Δ^1_1, then it is intrinsically Δ^1_1. This shows the following.

Corollary 3 *If a relation R on a computable structure $\mathcal{A}$ is invariant and properly Π^1_1, then the image of R in any computable copy is also properly Π^1_1.*

The next result produces computable copies of a given structure $\mathcal{A}$ with the *same* intrinsically Π^1_1 relation, but with no hyperarithmetical isomorphism. Again, for $\mathcal{B} \cong \mathcal{A}$, we write $R^{\mathcal{B}}$ for the image of R in $\mathcal{B}$.

Theorem 22 *Let $\mathcal{A}$ be a computable structure, with an invariant Π^1_1 unary relation R. Suppose that for any invariant Δ^1_1 relation $R' \subseteq R$ and any Δ^1_1 set Γ_0 of computable infinitary formulas, there is a computable structure, not isomorphic to $\mathcal{A}$, but with the same universe A, such that the identity function on R' preserves satisfaction of all formulas in Γ_0. Then the identity function on R extends to an isomorphism from $\mathcal{A}$ onto a computable copy $\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are not hyperarithmetically isomorphic.*

Proof: Since R is Π^1_1 and invariant, there is a formally Π^1_1 definition, with no parameters—say P is the Π^1_1 set of disjuncts. We use the fact that $\mathcal{O}$ is m-complete Π^1_1 . Let f be a computable function witnessing that $P \leq_m \mathcal{O}$. For each $a \in \mathcal{O}$, let P_a be the set of all n such that $f(n) <_{\mathcal{O}} a$. We can pass effectively from $a \in \mathcal{O}$ to a computable infinitary formula $\psi_a(x)$ equivalent to the disjunction of the formulas in P_a.

We describe $\mathcal{B}$ by a Π^1_1 set Γ of computable infinitary sentences, in a language with added constants for the elements of A.

1. We include a sentence saying that $\mathcal{B}$ is a computable structure with universe A.

2. To guarantee that $\mathcal{B} \cong \mathcal{A}$, we include all computable infinitary sentences true in $\mathcal{A}$

(in the language without the constants from A).

3. To guarantee that $R^{\mathcal{B}} = R^{\mathcal{A}}$, we include, for each $a \in A$, sentences $\psi_a(c)$ if $\mathcal{A} \models \psi_a(c)$, and $\neg\psi_a(c)$ if $\mathcal{A} \models \neg\psi_a(c)$.

4. To guarantee that there is an isomorphism that acts as the identity on R, we include $\varphi(\overline{c})$, for each tuple $\overline{c}$ in R and each computable infinitary formula $\varphi(\overline{x})$ such that $\mathcal{A} \models \varphi(\overline{c})$.

The hypotheses yield a model for any Δ^1_1 subset of Γ. Then, by Barwise-Kreisel Compactness, Γ has a model. We have a computable structure $\mathcal{B}$ that is isomorphic to $\mathcal{A}$, such that for all $a \in \mathcal{O}$, $\psi_a^{\mathcal{B}} = \psi_a^{\mathcal{A}}$, and the identity function on R preserves satisfaction of all computable infinitary formulas. For each $a \in \mathcal{O}$, if we expand $\mathcal{A}$ and $\mathcal{B}$ by constants for all elements satisfying ψ_a, the resulting structures are hyperarithmetical, and since they satisfy the same computable infinitary sentences, they are isomorphic.

We must show that there is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$ that acts as the identity on all of R. Let $\mathcal{A}^*$ be the structure $(A, \mathcal{A}, \mathcal{B})$, with separate relations for the two structures. Then $\mathcal{A}^*$ is computable. We form a Π^1_1 set Λ of computable infinitary sentences, in a language with a new binary relation symbol F, in addition to the symbols of the language of $\mathcal{A}^*$.

We include a sentence saying that F is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$, and sentences for all $a \in \mathcal{O}$ saying that F acts as the identity on the set of elements satisfying $\psi_a(x)$. It follows from the previous paragraph that for any Δ^1_1 set $\Lambda' \subseteq \Lambda$, there is an expansion of $\mathcal{A}^*$ satisfying Λ'. Then there is an expansion of $\mathcal{A}^*$ satisfying all of Λ.

Examples

Here are some examples of computable structures with intrinsically Π^1_1 relations.

Example 1. A *Harrison ordering* is a computable ordering of type $\omega_1^{CK}(1+\eta)$. Recall that η is the order type of the rationals, and for orderings $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}\cdot\mathcal{B}$ is the result of replacing each element of $\mathcal{B}$ by a copy of $\mathcal{A}$. Harrison showed the existence of such orderings. In fact, he showed that for any computable tree $T\subseteq\omega^{<\omega}$, if T has paths but no hyperarithmetical paths, then the Kleene-Brouwer ordering on T is a computable ordering of type $\omega_1^{CK}(1+\eta)+\alpha$, for some computable ordinal α. Let $\mathcal{A}$ be a Harrison ordering, and let R be the initial segment of type ω_1^{CK}. This set is intrinsically Π^1_1, since it is defined by the disjunction of computable infinitary formulas saying that the interval to the left of x has order type β, for computable ordinals β.

Example 2. A *Harrison Boolean algebra* is a computable Boolean algebra of type $I(\omega_1^{CK}(1+\eta))$. Recall that for an ordering $\mathcal{C}$, the interval algebra $I(\mathcal{C})$ is the algebra generated, under finite union, by the half-open intervals $[a,b)$, $(-\infty,b)$, $[a,\infty)$, with endpoints in $\mathcal{C}$. Let $\mathcal{A}$ be a Harrison Boolean algebra, and let R be the set of superatomic elements—those contained in one of the Frechet ideals. This is intrinsically Π^1_1, since it is defined by the disjunction of computable infinitary formulas saying that x is a finite join of α-atoms, for computable ordinals α.

Example 3. Recall that a countable Abelian p-group $\mathcal{G}$ is determined up to isomorphism by its Ulm sequence $(u_\alpha(\mathcal{G}))_{\alpha<\lambda(\mathcal{G})}$, and the dimension of the divisible part. A *Harrison p-group* is a computable Abelian p-group $\mathcal{G}$ such that $\lambda(\mathcal{G}) = \omega_1^{CK}$, $u_{\mathcal{G}}(\alpha) = \infty$, for all $\alpha < \omega_1^{CK}$, and the divisible part D has infinite dimension. A *Harrison group* is a Harrison p-group for some p. Let $\mathcal{A}$ be a Harrison group, and let R be the set of elements that have computable ordinal height—the complement of the divisible part. Then R is intrinsically Π_1^1 on $\mathcal{A}$, since it is defined by the disjunction of computable infinitary formulas saying that x has height α, for computable ordinals α.

The Scott Isomorphism Theorem says that for any countable structure $\mathcal{A}$ (for a countable language), there is an $L_{\omega_1,\omega}$ sentence σ such that the countable models of σ are exactly the copies of $\mathcal{A}$. In the proof, Scott assigned an ordinal to the structure. There is more than one definition of "Scott rank".

We can involve a sequence of expansions of $\mathcal{A}$. Let $\mathcal{A}_0 = \mathcal{A}$, let $\mathcal{A}_{\alpha+1}$ be the result of adding to $\mathcal{A}_\alpha$ predicates for the types realized in $\mathcal{A}_\alpha$, and for limit α, let $\mathcal{A}_\alpha$ be the limit of the expansions $\mathcal{A}_\beta$, for $\beta < \alpha$. For some countable ordinal α, $\mathcal{A}_\alpha$ is atomic. The least such α is the *rank*. For a hyperarithmetical structure $\mathcal{A}$, the maximum possible rank is $\omega_1^{CK} + 1$.

Another possible rank, for a hyperarithmetical structure $\mathcal{A}$, is the least ordinal α such that for each tuple $\overline{a}$ in $\mathcal{A}$, there is some $\beta < \alpha$ such that the set of all computable Π_γ formulas true of $\overline{a}$, for $\gamma < \beta$, defines the orbit of $\overline{a}$ under automorphisms. These definitions are not equivalent, but they agree to the extent that if the one rank is a computable ordinal, or ω_1^{CK}, or $\omega_1^{CK} + 1$, then so is the other.

A hyperarithmetical structure $\mathcal{A}$ has rank $\omega_1^{CK}+1$ just in case there is a tuple $\overline{a}$ in $\mathcal{A}$ whose orbit under automorphisms is not defined by any computable infinitary formula.

In the three examples of intrinsically Π^1_1 relations described above, the structures have Scott rank $\omega_1^{CK}+1$. Below, we describe a general class of examples arising in computable structures of this rank.

Proposition 1 *Let $\mathcal{A}$ be a computable structure of Scott rank $\omega_1^{CK}+1$. Let $\overline{a}$ be a tuple in $\mathcal{A}$ whose orbit is not defined by any computable infinitary formula, and let R be the complementary relation. Then R is intrinsically Π^1_1, and not Δ^1_1.*

The structures in Examples 1, 2, and 3 above all have Scott rank $\omega_1^{CK}+1$, but the intrinsically Π^1_1 relations that we described above are not complements of single orbits.

We can apply Proposition 1 to obtain further intrinsically Π^1_1 relations on these same structures. In particular, in the Harrison ordering, if a is an element outside the well-ordered initial segment, then the orbit of a is not defined by any computable infinitary formula.

Say a is first in its copy of ω_1^{CK}. Then the orbit of a consists of the elements that are first in their copy of ω_1^{CK}, but not first over-all. By Proposition 1, the complement of this orbit is intrinsically Π^1_1. It is not Δ^1_1.