Computability in Hierarchies and Topological Spaces (INTAS-00-499)

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Complete Numberings I

Serikzhan Badaev
(jointly with Sergey Goncharov and Andrea Sorbi)

Kazakh National University, Almaty(Kazakhstan)

e-mail: badaev@kazsu.kz

0. Basic notions

Any surjective mapping α of the set N of natural numbers onto a nonempty set $\mathcal{A}$ is called a *numbering* of $\mathcal{A}$.

Let α and β be numberings of $\mathcal{A}$ and let $X \subseteq N$. We say that numbering α is X-*reducible* to numbering β ($\alpha \leqslant_X \beta$) if there exists a X-computable function f such that $\alpha(n) = \beta f(n)$ for any $n \in N$.

We say that the numberings α and β are X-*equivalent* (in symbols, $\alpha \equiv_X \beta$) if $\alpha \leqslant_X \beta$ and $\beta \leqslant_X \alpha$.

Definition 1.(S.Goncharov and A.Sorbi) Numbering α of a family $\mathcal{A} \subseteq \Sigma^0_n$ is called Σ^0_n–*computable* if $x \in \alpha y$ is Σ^0_n–relation.

The set of all Σ^0_{n+1}–computable numberings of a family $\mathcal{A}$ is denoted by $\mathrm{Com}^0_{n+1}(\mathcal{A})$.

Reducibility of numberings is a pre-ordering relation on $\mathrm{Com}^0_{n+1}(\mathcal{A})$ which induces in the usual way a quotient structure $\mathcal{R}^0_{n+1}(\mathcal{A})$ which is an upper semilattice called *Rogers semilattice* of Σ^0_{n+1}−computable numberings of the family $\mathcal{A}$.

Every numbering $\alpha \in \mathrm{Com}^0_{n+1}(\mathcal{A})$ induces a *degree* $\deg(\alpha) = \{\beta \mid \beta \equiv \alpha\}$ in $\mathcal{R}^0_{n+1}(\mathcal{A})$.

Numbering α of $\mathcal{A}$ is called X-*universal in* $\mathrm{Com}^0_{n+1}(\mathcal{A})$ if
(i) $\alpha \in \mathrm{Com}^0_{n+1}(\mathcal{A})$,
(ii) $\beta \leqslant_X \alpha$ for every $\beta \in \mathrm{Com}^0_{n+1}(\mathcal{A})$.

We usually omit X in our notations if X is computable set.

Example 1. The family $\mathbf{\Sigma}^0_{n+1}$ has a universal numbering in $\mathrm{Com}^0_{n+1}(\mathbf{\Sigma}^0_{n+1})$, namely the relativization $W^{\mathbf{0}^{(n)}}$ of the classical Post numbering W of the family of all c.e. sets.

Example 2. For every n, the set $\mathcal{F}$ of all finite sets is obviously Σ^0_{n+1}-computable and has no universal numbering in $\mathrm{Com}^0_{n+1}(\mathcal{F})$. The latter holds by the relativized version of Lachlan's condition: if any Σ^0_{n+1}-computable family has a universal numbering then it is closed under unions of increasing Σ^0_{n+1}-computable sequences of its members.

1. Complete numberings: preliminaries

Definition 2. Numbering α of an abstract set $\mathcal{A}$ is called *complete w.r.t. special object* $a \in \mathcal{A}$ if for every partial computable function $f(x)$ there exists total computable function $g(x)$ s.t.

$$\alpha g(x) = \begin{cases} \alpha f(x) & \text{if} \quad f(x)\downarrow, \\ a & \text{otherwise.} \end{cases}$$

Recursion theorem and fixed point theorem both hold for any complete numbering.

Theorem 1 [Yu.L. Ershov]. Degrees of complete numberings are not splittable.

Corollary [A. Lachlan]. $\mathbf{m}$-degree of creative set is not splittable.

2. Completion operator and its properties

Definition 3. Let $K(x)$ be an unary universal partial computable function, for instance, $K(< e, x >) = \varphi_e(x)$. Define

$$\alpha_a^K(x) = \begin{cases} \alpha K(x) & \text{if} \quad K(x)\downarrow, \\ a & \text{otherwise.} \end{cases}$$

Theorem 2 [Yu.L. Ershov]. For every numbering α, numbering α_a^K is complete w.r.t. a.

Important point: For every Σ^0_{n+2}-computable numbering α of any family $\mathcal{A} \in \Sigma^0_{n+2}$ and for **arbitrary** $A \in \mathcal{A}$, numbering α_A^K is also Σ^0_{n+2}-computable numbering of $\mathcal{A}$.

Therefore, mapping $\alpha \to \alpha_A^K$ induces an operator on $\mathcal{R}^0_{n+2}(\mathcal{A})$.

To avoid incomputability in the case of the families of Σ^0_1-sets we have to choose $A = \perp$ if $\mathcal{A}$ has the least element $\perp$ under inclusion.

Theorem 3. Let $\mathcal{A}$ be a family of Σ^0_{n+2}-sets, let $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$, and let A, B be any sets of $\mathcal{A}$. Then

1) $\alpha^K_A \equiv_{\mathbf{0}'} \alpha$;

2) $\alpha \leqslant \alpha^K_A$;

3) $\alpha < \alpha^K_A$ iff α is not complete w.r.t. A;

4) if $A \neq B$ then

$$\mathbf{inf}(\mathrm{deg}(\alpha^K_A), \mathrm{deg}(\alpha^K_B)) = \mathrm{deg}(\alpha).$$

Remark. Statements 1-4 remain true for $\alpha^K_\perp$ in the classical case $\mathcal{A} \subseteq \Sigma^0_1$ provided $\mathcal{A}$ has the least set $\perp$.

Corollary 1. Every universal numbering in $\mathrm{Com}^0_{n+2}(\mathcal{A})$ is complete w.r.t. each element of $\mathcal{A}$.

Corollary 2. For every $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$, the degree of α^K_A is non-splittable in $\mathcal{R}^0_{n+2}(\mathcal{A})$. In particular, the degree of universal numbering in $\mathrm{Com}^0_{n+2}(\mathcal{A})$, if any, is never splittable.

Corollary 3. Index set of the special object A relative to α^K_A is productive set.

Σ^0_{n+2}-computable Friedberg and positive numberings as well as all Σ^0_{n+2}-computable minimal numberings which are built by method of Badaev-Goncharov are all incomplete.

3. Relativization of the completion operator

Definition 3. Let $X \subseteq N$. Numbering α of a set $\mathcal{A}$ is called *X-complete w.r.t. special object* $a \in \mathcal{A}$ if for every partial X-computable function $f(x)$ there exists total X-computable function $g(x)$ s.t.

$$\alpha g(x) = \begin{cases} \alpha f(x) & \text{if} \quad f(x)\downarrow, \\ a & \text{otherwise.} \end{cases}$$

Theorem 4. For every $Y \leqslant_T X$, if numberings α, β are Y-equivalent and α is X-complete w.r.t. a then β is X-complete w.r.t. a.

Definition 4. Let X be an arbitrary subset of N and let $K^X(< e, x >) \leftrightharpoons \varphi_e^X(x)$ for all $e, x \in N$. Define

$$\alpha_a^X = \begin{cases} \alpha K^X(x) & \text{if} \quad K^X(x)\downarrow, \\ a & \text{otherwise.} \end{cases}$$

Computability of completions

Theorem 5. Let $\mathcal{A}$ be non-trivial family of Σ^0_{n+1}-sets, and let $\alpha \in \mathrm{Com}^0_{n+1}(\mathcal{A})$. Then

1) $\alpha_A^{\mathbf{0^{(m)}}} \in \mathrm{Com}^0_{\mathrm{n+1}}(\mathcal{A})$ for every $m < n$ and each $A \in \mathcal{A}$;

2) $\alpha_A^{\mathbf{0^{(n)}}} \in \mathrm{Com}^0_{\mathrm{n+1}}(\mathcal{A})$ iff $\mathcal{A}$ has the least set $\perp$ and $A = \perp$;

3) $\alpha_A^{\mathbf{0^{(m)}}} \notin \mathrm{Com}^0_{\mathrm{n+1}}(\mathcal{A})$ for every $m > n$ and each $A \in \mathcal{A}$.

Properties of the completion operator

Theorem 6. Let $\mathcal{A}$ be any set, $a, b \in \mathcal{A}$, and let α, β be any numberings of $\mathcal{A}$. The following statements hold for every subsets $X, Y \subseteq N$:

1) $\alpha_a^X \equiv_{X'} \alpha$;

2) $\alpha \leqslant \alpha_a^X$;

3) $\alpha_a^X <_X \alpha$ iff α is not complete w.r.t. a;

4) for every γ and every $a \neq b$, if $\gamma \leqslant \alpha_a^X$ and $\gamma \leqslant \alpha_b^X$ then $\gamma \leqslant \alpha$;

5) if $Y \leqslant_T X$, then numbering α_a^X is Y-complete w.r.t. special object a;

6) if $\alpha \leqslant_{\mathbf{X}} \beta$ then $\alpha_a^{\mathbf{X}} \leqslant \beta_a^{\mathbf{X}}$;

7) if $\beta \leqslant_{\mathbf{X}} \alpha_a^{\mathbf{X}}$ then $\beta \leqslant \alpha_a^{\mathbf{X}}$;

8) if $Y \leq_T X$ then $\alpha_a^{\mathbf{X}'} \equiv (\alpha_a^{\mathbf{X}'})_b^{\mathbf{Y}}$. In particular $\alpha_a^{\mathbf{X}'}$ is $\mathbf{Y}-$complete with respect to any special element b.

Corollary 1. If $Y \leq_T X$ then $(\alpha_a^{\mathbf{X}})_a^{\mathbf{Y}} \equiv \alpha_a^{\mathbf{X}}$. In particular, the numbering $\alpha_a^{\mathbf{0}^{(n)}}$ is $\mathbf{0}^{(i)}-$complete with respect to a for all $i \leq n$.

Corollary 2. $(\alpha_a^{\mathbf{0}^{(n+1)}})_b^{\mathbf{0}^{(n)}} \equiv \alpha_a^{\mathbf{0}^{(n+1)}}$, for all a, b. In particular, $\alpha_a^{\mathbf{0}^{(n+1)}}$ is $\mathbf{0}^{(i)}-$complete with respect to each element of the family and for all $i \leq n$.

Theorem 7. Let $\mathcal{A}$ be any non-trivial Σ^0_{n+1}–computable family with $n \geq 1$. Then

(i) for every $A \in \mathcal{A}$ and each $I \subseteq \{0, 1, \ldots, n-1\}$ there exists a numbering $\alpha \in \mathrm{Com}^0_{n+1}(\mathcal{A})$ such that α is $\mathbf{0}^{(i)}$–complete with respect to A if and only if $i \in I$;

(ii) if $\mathcal{A}$ has least set $\perp$, and if it has a Σ^0_{n+1}–computable numbering which is not $\mathbf{0}^{(n)}$–complete with respect to $\perp$ then
for every set $I \subseteq \{0, 1, \ldots, n\}$, there exists a numbering $\alpha \in \mathrm{Com}^0_{n+1}(\mathcal{A})$ such that α is $\mathbf{0}^{(i)}$–complete with respect to $\perp$ if and only if $i \in I$.

4. Uniformly complete numberings

Definition 5. We say that a numbering β of a set $\mathcal{A}$ is *uniformly* $\mathbf{X}$*-complete* if there exists a total $\mathbf{X}$-computable function $h(i,m,x)$ such that for every i,m,x

$$\beta(h(i,m,x)) = \begin{cases} \beta(\varphi_i^{\mathbf{X}}(x)) & \text{if } \varphi_i^{\mathbf{X}}(x)\downarrow, \\ \beta(m) & \text{otherwise.} \end{cases}$$

Theorem 8. For every set X and for every numbering α of a family $\mathcal{A}$, and for each $a \in \mathcal{A}$, the numbering $\alpha_a^{\mathbf{X}'}$ is uniformly $\mathbf{X}$-complete.

Corollary 1. For every $\alpha \in \mathrm{Com}^0_{n+3}(\mathcal{A})$ there exists a Σ^0_{n+3}-computable uniformly complete numbering β such that $\alpha \leqslant \beta$.

Corollary 2. For every $\alpha \in \mathrm{Com}^0_{n+3}(\mathcal{A})$ there exists a numbering $\beta \in \mathrm{Com}^0_{n+3}(\mathcal{A})$ such that $\alpha \leqslant \beta$ and β is complete with respect to every element $B \in \mathcal{A}$.

5. Interconnections between complete numberings and universal numberings

Theorem 9. Let α be $\mathbf{0}^{(m)}$-universal numbering in $\mathrm{Com}^0_{n+1}(\mathcal{A})$. Then:

(1) if $m < n$ then α is $\mathbf{0}^{(m)}$–complete with respect to every element of $\mathcal{A}$;

(2) if $m \leqslant n$ and $\mathcal{A}$ has the least element $\bot$ then α is $\mathbf{0}^{(m)}$-complete with respect to $\bot$;

(3) α is not $\mathbf{0}^{(n)}$-complete with respect to any non-least element of $\mathcal{A}$;

(4) α is not $\mathbf{0}^{(m)}$-complete with respect to any $A \in \mathcal{A}$ if $m > n$.

Theorem 10. For every n, each finite family $\mathcal{A}$ of Σ^0_{n+1}-sets has a numbering α which is $\mathbf{0}^{(n)}$-universal in $\mathrm{Com}^0_{n+1}(\mathcal{A})$.

Theorem 11. Let $\mathcal{A} \subseteq \Sigma^0_{n+2}$ be a finite family. Then the following statements are equivalent:
(1) there exists a numbering of $\mathcal{A}$ which is universal in $\mathrm{Com}^0_{n+2}(\mathcal{A})$;
(2) $\mathcal{A}$ has a numbering which is $\mathbf{0}^{(n)}$-universal in $\mathrm{Com}^0_{n+2}(\mathcal{A})$;
(3) $\mathcal{A}$ contains a least element $\perp$ under inclusion.

Corollary. Let $\mathcal{A}$ be a non-trivial finite family of Σ^0_{n+2}-sets. Then
(1) if $\mathcal{A}$ has the least set $\perp$ then for all m, $\mathcal{A}$ possesses numberings which are $\mathbf{0}^{(m)}$-universal in $\mathrm{Com}^0_{n+2}(\mathcal{A})$;
(2) if $\mathcal{A}$ does not contain the least set under inclusion then $\mathcal{A}$ has a numbering $\mathbf{0}^{(m)}$-universal in $\mathrm{Com}^0_{n+2}(\mathcal{A})$ if and only if $m \geq n+1$.

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Complete Numberings II

Serikzhan Badaev
(jointly with Sergey Goncharov, Zarif Khisamiev, Sergey Podzorov and Andrea Sorbi)

Kazakh National University, Almaty(Kazakhstan)

e-mail: badaev@kazsu.kz

Definition 2. Numbering α of an abstract set $\mathcal{A}$ is called *complete w.r.t. special object* $a \in \mathcal{A}$ if for every partial computable function $f(x)$ there exists total computable function $g(x)$ s.t.

$$\alpha g(x) = \begin{cases} \alpha f(x) & \text{if} \quad f(x)\downarrow, \\ a & \text{otherwise.} \end{cases}$$

Definition 3. Let $K(x)$ be an unary universal partial computable function, for instance, $K(< e, x >) = \varphi_e(x)$. Define

$$\alpha_a^K(x) = \begin{cases} \alpha K(x) & \text{if} \quad K(x)\downarrow, \\ a & \text{otherwise.} \end{cases}$$

Definition 4. Let X be an arbitrary subset of N and let $K^X(< e, x >) \leftrightharpoons \varphi_e^X(x)$ for all $e, x \in N$. Define

$$\alpha_a^X = \begin{cases} \alpha K^X(x) & \text{if} \quad K^X(x)\downarrow, \\ a & \text{otherwise.} \end{cases}$$

Theorem 3. Let $\mathcal{A}$ be a family of Σ^0_{n+2}-sets, let $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$, and let A, B be any sets of $\mathcal{A}$. Then

1) $\alpha^K_A \equiv_{\mathbf{0}'} \alpha$;

2) $\alpha \leqslant \alpha^K_A$;

3) $\alpha < \alpha^K_A$ iff α is not complete w.r.t. A;

4) if $A \neq B$ then

$$\mathbf{inf}(\mathrm{deg}(\alpha^K_A), \mathrm{deg}(\alpha^K_B)) = \mathrm{deg}(\alpha).$$

Question. *Is it true that*

$$((\alpha_a^{\mathbf{0^{(n)}}})_b^{\mathbf{0^{(n)}}})_a^{\mathbf{0^{(n)}}} \equiv (\alpha_a^{\mathbf{0^{(n)}}})_b^{\mathbf{0^{(n)}}}?$$

In particular, is it true that

$$((\alpha_a^K)_b^K)_a^K \equiv (\alpha_a^K)_b^K)?$$

Answer [Zarif Khisamiev] Both possibilities are valid.

6. Distributive substructures of Rogers semilattices

Let $\mathcal{A}$ be a non-trivial countable set. Denote by $\mathcal{L}(A)$ the set of degrees of all numberings of $\mathcal{A}$ with partial order induced by reducibility of numberings. $\mathcal{L}(A)$ is upper semilattice. For any numbering α of $\mathcal{A}$, let $\mathcal{C}(\mathcal{A})_\alpha$ stands for the subsemilattice of $\mathcal{L}(A)$ which contains deg(α) and is closed under completion operator.

Theorem 12[Z.Khisamiev]. If α is non-complete w.r.t. some element of $\mathcal{A}$ then $\mathcal{C}(\mathcal{A})_\alpha$ is infinite distributive lattice with finite principal ideals.

$S(\mathcal{A}) \leftrightharpoons$ the set of the special objects of $\mathcal{A}$.

Theorem 13[Z.Khisamiev]. Let α, β be any numberings of the sets $\mathcal{A}, \mathcal{B}$. Then $\mathcal{C}(\mathcal{A})_\alpha \approx \mathcal{C}(\mathcal{B})_\beta$ iff $|S(\mathcal{A})| = |S(\mathcal{B})|$ and $|\mathcal{A} \setminus S(\mathcal{A})| = |\mathcal{B} \setminus S(\mathcal{B})|$.

Theorem 14[Z.Khisamiev]. For every non-empty countable sets $\mathcal{B} \subseteq \mathcal{A}$, there exists a complete numbering α whose special objects are exactly elements of $\mathcal{B}$.

Theorem 15[Z.Khisamiev]. Let $\mathcal{B} \subseteq \mathcal{A} \subseteq \mathbf{\Sigma}^0_{n+2}$ with the common $\perp$. For every $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ and $\beta \in \mathrm{Com}^0_{n+2}(\mathcal{B})$ there exists $\gamma \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ s.t.
(1) $\alpha \oplus \beta \leqslant \gamma$;
(2) γ is complete w.r.t. every element of $\mathcal{B}$;
(3) γ is not complete w.r.t. every element of $\mathcal{A} \setminus \mathcal{B}$.

Open questions on complete numberings

Question 1. *Is any Σ^0_{n+2}-computable minimal numbering of non-trivial family $\mathcal{A}$ always incomplete?*

Question 2. *Let $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ be a numbering of a non-trivial family $\mathcal{A}$ and suppose that α is not complete w.r.t. $A \in \mathcal{A}$. Does there exist a numbering β s.t. $\alpha < \beta < \alpha^K_A$?*

Question 3. *Let $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ be a numbering of a non-trivial family $\mathcal{A}$ and suppose that α is not complete w.r.t. $A \in \mathcal{A}$. Does there exist a non-splittable numbering β s.t. $\alpha < \beta < \alpha^K_A$?*

Question 4. *Does every Σ^0_{n+2}-computable family $\mathcal{A}$ have a Σ^0_{n+2}-computable uniformly complete numbering?*

6. Elementary properties and isomorphic types of Rogers semilattices

The purpose is to show differences in the elementary theories of Rogers semilattices of arithmetical numberings, depending on structural invariants of the given families of arithmetical sets.

Everyone who has ever dealt with the classical theory of computable numberings is well aware that general facts about Rogers semilattices of families of c.e. sets are very rare, and at the same time it is very difficult to establish elementary properties that distinguish given structures. Opposite to the classical case, the elementary theories of Rogers semilattices of arithmetical numberings for the level two and higher seem more exciting. Now, we briefly examine some algebraic and elementary properties of the Rogers semilattices $\mathcal{R}^0_{n+2}(\mathcal{A})$ for various $\mathcal{A}$.

Cardinality, Lattice Properties, Undecidability

Theorem 16 [A.Khutoretsky]. For every family $\mathcal{A}$ of c.e. sets, if the Rogers semilattice $\mathcal{R}^0_1(\mathcal{A})$ contains at least two elements then it is infinite.

Theorem 17 [V.Selivanov]. For every family $\mathcal{A}$ of c.e. sets, if the Rogers semilattice $\mathcal{R}^0_1(\mathcal{A})$ contains at least two elements then it is not a lattice.

Theorem 18 [S.Goncharov, A.Sorbi]. If a Σ^0_{n+2}-computable family $\mathcal{A}$ is not trivial then the Rogers semilattice $\mathcal{R}^0_{n+2}(\mathcal{A})$ is infinite and is not a lattice.

Question 5. Under what conditions the Rogers semilattice $\mathcal{R}^0_1(\mathcal{A})$ of a family $\mathcal{A}$ of c.e. sets is non-trivial?

Let ε^* denote the bounded distributive lattice obtained by dividing the lattice ε of all c.e. subsets of ω modulo the ideal of all finite sets. We will denote by $\widehat{\beta}$ the principal ideal of $\mathcal{R}^0_{n+1}(\mathcal{A})$,

$$\widehat{\beta} \risingdotseq \{\deg(\gamma) \mid \deg(\gamma) \leqslant \deg(\beta)\}.$$

Theorem 19[S.Podzorov]. Let $\mathcal{A}$ be any Σ^0_{n+2}-computable family. There exists a numbering $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ such that
(1) $\widehat{\alpha}$ is isomorphic to $\varepsilon^* \backslash \{\perp\}$ if the family $\mathcal{A}$ is infinite;
(2) $\widehat{\alpha}$ is isomorphic to ε^* if the family $\mathcal{A}$ is finite.

Corollary. The elementary theory of every non-trivial Rogers semilattice $\mathcal{R}^0_{n+2}(\mathcal{A})$ is hereditarily undecidable.

Question 6. Is the elementary theory of any non-trivial Rogers semilattice of a Σ^0_1-computable family hereditarily undecidable, or at least undecidable?

Extremal Elements

Theorem 11[BGS]. Let $\mathcal{A} \subseteq \Sigma^0_{n+2}$ be a finite family. Then the following statements are equivalent:
(1) $\mathcal{R}^0_{n+2}(\mathcal{A})$ has a greatest element;
(3) $\mathcal{A}$ contains a least element $\perp$ under inclusion.

Theorem 20[BG]. For every n, if $\mathcal{A}$ is an infinite Σ^0_{n+2}-computable family, then $\mathcal{R}^0_{n+2}(\mathcal{A})$ has infinitely many minimal elements.

Remark. Theorem 20 does not hold for some infinite families of c.e. sets and does hold for other ones.

Question 7 [Yu.L.Ershov]. What is the possible number of minimal elements in the Rogers semilattice $\mathcal{R}^0_1(\mathcal{A})$ of a family of c.e. sets?

The Weak Distributivity Property

Definition 6. An upper semilattice $\langle L, \vee, \leqslant \rangle$ is called *distributive* if for every $a_1, a_2, b \in L$, if $b \leq a_1 \vee a_2$ then there exist $b_1, b_2 \in L$ such that $b_1 \leq a_1, b_2 \leq a_2$ and $b = b_1 \vee b_2$.

Theorem 21[BGS]. For every n and for every finite family $\mathcal{A} \subseteq \Sigma^0_{n+1}$, $\mathcal{R}^0_{n+1}(\mathcal{A})$ is a distributive upper semilattice.

Theorem 21 does not hold for the infinite families if $n > 1$.

Definition 7. An upper semilattice $\mathfrak{L} = \langle L, \leqslant \rangle$ is *weakly distributive* if $\mathfrak{L}_\perp = \langle L \cup \{\perp\}, \leqslant_\perp \rangle$ is distributive, where $\perp \notin \mathfrak{L}$ and

$$\leqslant_\perp \leftrightharpoons \leqslant \cup \{(\perp, a) \mid a \in L \cup \{\perp\}\}.$$

Proposition[BGS]. An upper semilattice L is weakly distributive iff for every $a_1, a_2, b \in L$, if $b \leq a_1 \vee a_2$ and $b \not\leq a_1, b \not\leq a_2$ then there exist $b_1, b_2 \in L$ such that $b_1 \leq a_1, b_2 \leq a_2$ and $b = b_1 \vee b_2$.

Theorem 22[BGS]. For every n, the Rogers semilattice of any infinite Σ^0_{n+2}-computable family is not weakly distributive.

Question 8. Does there exist a computable infinite family $\mathcal{A}$ of c.e. sets such that $\mathcal{R}^0_1(\mathcal{A})$ is distributive? Does there exist a computable infinite family $\mathcal{A}$ of c.e. sets such that $\mathcal{R}^0_1(\mathcal{A})$ is weakly distributive?

Theorem 23[BGS]. For every n, there exist infinitely many Σ^0_{n+1}-computable families with elementary pairwise different Rogers semilattices.

Theorem 24[BGS]. For every n there exist $m \geq n$ and a Σ^0_{m+2}-computable family $\mathcal{B}$ such that no Rogers semilattice $\mathcal{R}^0_{n+1}(\mathcal{A})$ of any Σ^0_{n+1}-computable family $\mathcal{A}$ is isomorphic to $\mathcal{R}^0_{m+2}(\mathcal{B})$.

Theorem 25 [to be checked]. For every n and every non-trivial Σ^0_{n+6}-computable family $\mathcal{A}$, $\mathcal{R}^0_{n+6}(\mathcal{A})$ is not isomorphic to $\mathcal{R}^0_{n+1}(\mathcal{B})$ of any Σ^0_{n+1}-computable family $\mathcal{B}$.

Question 9. Is it true that Rogers semilatticies of any two non-trivial computable families of the different levels of the arithmetical hierarchy are not isomorphic?

Question 10. Do the elementary theories of the classes of the Rogers semilattices of different levels coincide?

Question 11. Is it true that for every $m \neq n$ there exist non-trivial Σ^0_{m+1}-computable family $\mathcal{A}$ and Σ^0_{n+1}-computable family $\mathcal{B}$ s.t.

$$\mathrm{Th}(\mathcal{R}^0_{m+1}(\mathcal{A})) = \mathrm{Th}(\mathcal{R}^0_{n+1}(\mathcal{B}))?$$

Question 12. Is it true that for every $m \neq n$ and for every non-trivial Σ^0_{m+1}-computable family $\mathcal{A}$ there exists Σ^0_{n+1}-computable family $\mathcal{B}$ s.t.

$$\mathrm{Th}(\mathcal{R}^0_{m+1}(\mathcal{A})) = \mathrm{Th}(\mathcal{R}^0_{n+1}(\mathcal{B}))?$$