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On the Behavior of Periodic Solutions of Planar Autonomous Hamiltonian Systems with Multivalued Periodic Perturbations

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Abstract. Aim of the paper is to provide a method to analyze the behavior of T-periodic solutions x_ε, $\varepsilon > 0$, of a perturbed planar Hamiltonian system near a cycle x_0, of smallest period T, of the unperturbed system. The perturbation is represented by a T-periodic multivalued map which vanishes as $\varepsilon \to 0$. In several problems from nonsmooth mechanical systems this multivalued perturbation comes from the Filippov regularization of a nonlinear discontinuous T-periodic term. Through the paper, assuming the existence of a T-periodic solution x_ε for $\varepsilon > 0$ small, under the condition that x_0 is a nondegenerate cycle of the linearized unperturbed Hamiltonian system we provide a formula for the distance between any point $x_0(t)$ and the trajectories $x_\varepsilon([0,T])$ along a transversal direction to $x_0(t)$.

Keywords. Planar Hamiltonian systems, characteristic multipliers, multivalued periodic perturbations, periodic solutions, approximation formula.

Mathematics Subject Classification (2000). 37K05, 34A60, 34C25.

1. Introduction

Let x_0 be a T-periodic cycle of the Hamiltonian system

$$\dot{x} = f(x), \tag{1}$$

where $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ is given by $f(x) = -J\nabla\phi(x), \phi \in C^2(\mathbb{R}^2, \mathbb{R})$ and J is the symplectic matrix in $\mathbb{R}^2$. Numerical simulations in the recent monographs

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[2, 3] have shown that the subharmonic Melnikov's method ([10, Chapter 4, §6], [21]) correctly predicts the existence of T-periodic solutions x_ε of the differential inclusion

$$\dot{x} \in f(x) + \varepsilon g(t, x, \varepsilon), \tag{2}$$

where $g : \mathbb{R} \times \mathbb{R}^2 \times [0,1] \to K(\mathbb{R}^2)$ is a multivalued map taking the values in the family $K(\mathbb{R}^2)$ of nonempty compact and convex sets of $\mathbb{R}^2$. Sufficient conditions for the local and global existence of at least an absolutely continuous solution of (2) starting from any initial condition can be found in ([1, Chapter 2]).

In [2, 3] the authors have observed that if θ_0 is a simple zero of the subharmonic Melnikov's bifurcation function then (2) possesses a T-periodic solution x_ε such that

$$x_\varepsilon(t) \to x_0(t + \theta_0) \quad \text{as } \varepsilon \to 0, \quad \text{uniformly in } t \in [0, T].$$

A rigorous proof of this result is provided in the papers [7, 12, 17] by means of topological degree arguments. In this paper we do not provide conditions to ensure the existence of T-periodic solutions x_ε, for $\varepsilon > 0$ small, instead we want to evaluate the distance between any point $x_0(t)$ and the curve $x_\varepsilon([0, T])$ providing in this way a tool to study the behavior of the T-periodic solutions of (2) near x_0. This tool, together with the method based on the Melnikov's bifurcation function mentioned above, permits to perform a complete analysis both for the existence and the behavior near the cycle x_0 of the T-periodic solutions x_ε to (2).

Topological degree methods to study the bifurcation from a periodic orbit for system (2), when g is a singlevalued Carathéodory function, has been also employed in [11]. The extensive references in [11] also provide an interesting overview on the methods for the the study of the bifurcation of periodic solutions in perturbed dynamical systems.

Since in this paper the existence of T-periodic solutions x_ε of (2) is assumed, we only require to the multivalued map g the minimal regularity assumptions needed for our analysis. In fact, through the paper we only assume that the map $g : \mathbb{R} \times \mathbb{R}^2 \times [0,1] \to K(\mathbb{R}^2)$ is measurable or upper semicontinuous.

The interest of considering multivalued perturbation of system (1) is mainly related to the necessity, encountered in the applications, to deal with perturbations, having jump discontinuities, of Hamiltonian autonomous systems. In fact, many physical problems are modeled by ordinary differential equations with discontinuous right hand side whose regularization produces a multivalued map (see, for instance, [1, 9, 20]). Among them we like to cite the study of the self-sustained oscillations induced by friction in one-degree of freedom mechanical systems. This problem gives rise to a planar Hamiltonian system perturbed by a periodic term of small amplitude with jump discontinuities, compare, e.g., [2, Chapter 15] where the analysis was numerically performed by means of the Melnikov method.

The paper is organized as follows. In Section 2, assuming that the linearized system

$$\dot{y} = f'(x_0(t))y \tag{3}$$

possesses a not T-periodic solution, we show the existence of a family $\{\Delta_\varepsilon\}_{\varepsilon>0}$ of real numbers with $\Delta_\varepsilon \to 0$ as $\varepsilon \to 0$ such that

$$\frac{\|x_\varepsilon(t+\Delta_\varepsilon) - x_0(t)\|}{\varepsilon} \le const \quad \text{for any } t \in [0,T] \text{ and any } \varepsilon > 0. \tag{4}$$

This property has been already established by the authors in [17, 18] in the case when x_0 is an isolated limit cycle and g in (2) is a singlevalued continuous function. In Section 3 we employ property (4) together with a suitably defined multivalued function $M^\perp \in C^0(\mathbb{R}, \mathbb{R})$ to obtain

$$x_\varepsilon(t+\Delta_\varepsilon) - x_0(t) \in \varepsilon M^\perp(t)y(t) + \alpha_\varepsilon(t)\dot{x}_0(t) + o(\varepsilon), \tag{5}$$

where y is a not T-periodic solution of the linearized system (3) and $\alpha_\varepsilon(t)$ is a scalar function infinitesimal as $\varepsilon \to 0$ of order greater or equal to 1. The function $\alpha_\varepsilon(t)$ is given in the formula (39) of the paper. The formula to represent the function $M^\perp$ is provided in Section 3, thus (5) gives an explicit formula for the distance between the trajectories x_0 and x_ε along a transversal direction to x_0. Finally, in Section 4 we specialize the formula for $M^\perp$ in the case when the Hamiltonian system (1) possesses symmetry properties, as often is the case in the applications.

2. Evaluation of the distance between the periodic solutions of the perturbed system and the cycle of the unperturbed one

In this Section we establish the validity of inequality (4) which is the starting point for (5). This result does not depend on the perturbation term g, indeed the only property we need is that the cycle x_0 is nondegenerate according to the following definition, see [23].

Definition 2.1. We say that the cycle x_0 of an autonomous system as (1) is nondegenerate if the linearized system (3) has a not T-periodic solution.

If (1) is Hamiltonian and x_0 is nondegenerate then the period T of x_0 is noncritical and viceversa (compare [5]).

Definition 2.2. ([20, Definition 2.2.1]) A function $x : [0,T] \to \mathbb{R}^2$ is said to be a solution of the differential inclusion (2) on $[0,T]$ if x is absolutely continuous and the inclusion in (2) holds for almost all (a.a.) $t \in [0,T]$.

Definition 2.3. ([13, Definition 1.3.1]) For any $\varepsilon > 0$ the multivalued map $g(\cdot, \varepsilon) : \mathbb{R} \times \mathbb{R}^2 \to K(\mathbb{R}^2)$ is said to be measurable if, for any open $V \subset \mathbb{R}^2$, the set $g^{-1}(V, \varepsilon) := \{(t, x) \in \mathbb{R} \times \mathbb{R}^2 : g(t, x, \varepsilon) \cap V \neq \emptyset\}$ is measurable.

We assume the following condition.

(H): for any bounded set $B \subset \mathbb{R}^2$ there exists $\mu_B \in L^\infty_{loc}(\mathbb{R})$ such that

$$\|g(t, x, \varepsilon)\| := \sup\{\|y\| : y \in g(t, x, \varepsilon)\} \leq \mu_B(t)$$

for all $t \in \mathbb{R}$, $x \in B$ and $\varepsilon \in [0, 1]$.

Here $L^\infty_{loc}(\mathbb{R})$ denotes the space of locally essentially bounded functions, namely the space of the functions whose restrictions to any compact set of $\mathbb{R}$ are essentially bounded. Note that the notion of nondegenerate cycles has been used in [17, 18] in a stronger sense, i.e., x_0 is called nondegenerate if the linearized system (3) has only one characteristic multiplier equal to $+1$.

In order to introduce the family $\{\Delta_\varepsilon\}_{\varepsilon > 0}$, following [18], we define a curve $S \in C(\mathbb{R}, \mathbb{R}^2)$ as follows

$$\begin{aligned} S(v) &= \Omega(T, 0, h(v)) \\ h(v) &= x_0(0) + A_1 v, \end{aligned} \tag{6}$$

where $\Omega(\cdot, t_0, \xi)$ is the solution of (1) satisfying $\Omega(t_0, t_0, \xi) = \xi$ and A_1 is an arbitrary 2×1 vector such that the 2×2 matrix $(\dot{x}_0(0), A_1)$ is nonsingular.

The following result shows that the curve S intersects x_0 transversally.

Lemma 2.4 ([18, Lemma 2.2]). *Assume $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$. Let x_0 be a nondegenerate T-periodic cycle of* (1). *Then $\dot{x}_0(0) \notin S'(0)(\mathbb{R})$.*

Using the previous Lemma we can prove the following result.

Lemma 2.5. *Assume $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and that $g : \mathbb{R} \times \mathbb{R}^2 \times [0, 1] \to K(\mathbb{R}^2)$ is measurable and satisfying* (H). *Let x_0 be a nondegenerate T-periodic cycle of* (1). *Let x_ε be a T-periodic solution to perturbed system* (2) *satisfying*

$$\|x_\varepsilon(t) - x_0(t)\| \to 0$$

as $\varepsilon \to 0$ uniformly with respect to $t \in \mathbb{R}$, then there exists $\varepsilon_0 > 0$ and $r_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the equation $x_\varepsilon(\Delta) = S(v)$ has a unique solution $(\Delta_\varepsilon, v_\varepsilon)$ in $[-r_0, r_0] \times \{v \in \mathbb{R} : |v| \leq r_0\}$. Moreover, the functions $\varepsilon \to \Delta_\varepsilon$, $\varepsilon \to v_\varepsilon$ are continuous at $\varepsilon = 0$ with $\Delta_0 = 0$ and $v_0 = 0$.

In the case when g in (2) is singlevalued and continuous Lemma 2.5 is a simple consequence of Lemma 2.4 ([18, Corollary 2.3]). In the present case of g multivalued map we should provide a proof.

Proof. Define the function $F : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ as $F((t,v),\varepsilon) = x_\varepsilon(t) - S(v)$, then $F((0,0),0) = 0$. Our assumptions and definitions (6) guarantee that F is a continuous function at the points $\mathbb{R}^2 \times \{0\}$. Since $F(\cdot,0)$ is differentiable at $(0,0)$ and $F'_{(t,v)}((0,0),0) = (\dot{x}_0(0), -S'(0))$ is nonsingular by Lemma 2.4, then there exists $r_0 > 0$ such that $d(F(\cdot,0),[-r_0,r_0] \times [-r_0,r_0],0) \neq 0$. Here $d(\Phi,V,0)$ denotes the topological degree of the map Φ in the set V with respect to 0, see, for instance, [16]. Therefore, there exists $\varepsilon_0 > 0$ such that

$$d(F(\cdot,\varepsilon),[-r_0,r_0] \times [-r_0,r_0],0) \neq 0 \quad \text{for any } \varepsilon \in [0,\varepsilon_0].$$

This implies that for any $\varepsilon \in [0,\varepsilon_0]$, by the solution property of the topological degree, there exists at least one pair $(\Delta_\varepsilon, v_\varepsilon) \in [-r_0,r_0] \times [-r_0,r_0]$ such that $x_\varepsilon(\Delta_\varepsilon) - S(v_\varepsilon) = 0$.

Let us show that this solution is unique in $[-r_0,r_0] \times [-r_0,r_0]$ provided that $r_0 > 0$ and $\varepsilon_0 > 0$ are sufficiently small. Assume the contrary, hence there exist $\varepsilon_k \to 0$ as $k \to \infty$ and $(\widetilde{\Delta}_{\varepsilon_k}, \widetilde{v}_{\varepsilon_k}) \to (0,0)$ as $k \to \infty$ such that

$$x_{\varepsilon_k}(\widetilde{\Delta}_{\varepsilon_k}) - S(\widetilde{v}_{\varepsilon_k}) = 0 \quad \text{and} \quad (\widetilde{\Delta}_{\varepsilon_k}, \widetilde{v}_{\varepsilon_k}) \neq (\Delta_{\varepsilon_k}, v_{\varepsilon_k}) \quad \text{for any } k \in \mathbb{N}.$$

Since $S : [-r_0,r_0] \to S([-r_0,r_0])$ is invertible then $(\widetilde{\Delta}_{\varepsilon_k}, \widetilde{v}_{\varepsilon_k}) \neq (\Delta_{\varepsilon_k}, v_{\varepsilon_k})$ implies $\widetilde{\Delta}_{\varepsilon_k} \neq \Delta_{\varepsilon_k}$, say $\widetilde{\Delta}_{\varepsilon_k} < \Delta_{\varepsilon_k}$. On the other hand $\dot{x}(0) \neq 0$ and so we can assume $\widetilde{v}_{\varepsilon_k} \neq v_{\varepsilon_k}$. For any $v_1, v_2 \in \mathbb{R}^2$ we define $\angle(v_1,v_2)$ as follows

$$\angle(v_1,v_2) = \arccos \frac{\langle v_1, v_2 \rangle}{\|v_1\| \cdot \|v_2\|}.$$

Then we have $\angle(x_{\varepsilon_k}(\Delta_{\varepsilon_k}) - x_{\varepsilon_k}(\widetilde{\Delta}_{\varepsilon_k}), \dot{x}_0(0)) = \angle(S(v_{\varepsilon_k}) - S(\widetilde{v}_{\varepsilon_k}), \dot{x}_0(0))$. Passing to a subsequence if necessary we have that $\left\{\frac{v_{\varepsilon_k} - \widetilde{v}_{\varepsilon_k}}{|v_{\varepsilon_k} - \widetilde{v}_{\varepsilon_k}|}\right\}_{k=1}^{\infty}$ converges. Denote by $q \in \mathbb{R}$, $|q| = 1$, the limit of this sequence. Then

$$\angle(S(v_{\varepsilon_k}) - S(\widetilde{v}_{\varepsilon_k}), \dot{x}_0(0)) \to \angle(S'(0)q, \dot{x}_0(0)) \quad \text{as } k \to \infty,$$

with $\angle(S'(0)q, \dot{x}_0(0)) \neq 0$, since, by Lemma 2.4, $\dot{x}(0) \notin S'(0)(\mathbb{R})$. Therefore, there exists $\alpha > 0$ such that

$$\left|\angle(x_{\varepsilon_k}(\Delta_{\varepsilon_k}) - x_{\varepsilon_k}(\widetilde{\Delta}_{\varepsilon_k}), \dot{x}_0(0))\right| \geq \alpha > 0 \quad \text{for any } k \in \mathbb{N}. \tag{7}$$

On the other hand $t \to x_{\varepsilon_k}(t)$ is a solution of (2) then, by Filippov's Lemma [8], (see also [4, Theorem 1.5.10]), there exists a singlevalued measurable function $h_{\varepsilon_k} : [0,T] \to \mathbb{R}^2$ such that

$$\begin{aligned} \dot{x}_{\varepsilon_k}(t) &= f(x_{\varepsilon_k}(t)) + \varepsilon_k h_{\varepsilon_k}(t) && \text{for a.a. } t \in [0,T] \\ h_{\varepsilon_k}(t) &\in g(t, x_{\varepsilon_k}(t), \varepsilon_k) && \text{for a.a. } t \in [0,T]. \end{aligned}$$

Therefore

$$x_{\varepsilon_k}(\Delta_{\varepsilon_k}) - x_{\varepsilon_k}(\widetilde{\Delta}_{\varepsilon_k}) = \int_{\widetilde{\Delta}_{\varepsilon_k}}^{\Delta_{\varepsilon_k}} f(x_{\varepsilon_k}(\tau))\, d\tau + \varepsilon_k \int_{\widetilde{\Delta}_{\varepsilon_k}}^{\Delta_{\varepsilon_k}} h_{\varepsilon_k}(\tau)\, d\tau.$$

Due to the uniform convergence of x_ε to x_0 as $\varepsilon \to 0$ we have $\sup_{k\in N}\{\|x_{\varepsilon_k}(\tau)\| : \tau \in [0,T]\} < \infty$. Thus the assumptions on f and g permit to conclude that

$$\angle(x_{\varepsilon_k}(\Delta_{\varepsilon_k}) - x_{\varepsilon_k}(\widetilde{\Delta}_{\varepsilon_k}), \dot{x}_0(0)) \to \angle(f(x_0(0)), \dot{x}_0(0)) \quad \text{as } k \to \infty,$$

hence $\angle(f(x_0(0)), \dot{x}_0(0)) = 0$ since $f(x_0(0)) = \dot{x}_0(0)$. This is a contradiction with (7) and so the proof is complete. □

We are now in the position to prove inequality (4).

Theorem 2.6. *Assume $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and $g : \mathbb{R} \times \mathbb{R}^2 \times [0,1] \to K(\mathbb{R}^2)$ is measurable and satisfying* (H). *Let x_ε be a T-periodic solution to the perturbed system* (2) *satisfying*

$$\|x_\varepsilon(t) - x_0(t)\| \to 0 \quad \text{as} \quad \varepsilon \to 0 \tag{8}$$

uniformly with respect to $t \in [0,T]$, where x_0 is a nondegenerate T-periodic cycle of the unperturbed system (1). *Let $\varepsilon_0 > 0$ and $\{\Delta_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]} \subset \mathbb{R}$ be as in Lemma* 2.5. *Then there exists $M > 0$ such that*

$$\|x_\varepsilon(t + \Delta_\varepsilon) - x_0(t)\| \le M\varepsilon \quad \text{for any } t \in [0,T] \text{ and any } \varepsilon \in (0, \varepsilon_0].$$

Proof. In the sequel $\varepsilon \in (0, \varepsilon_0]$ and $\tau \in [0,T]$. Consider the change of variables $\nu_\varepsilon(\tau) = \Omega(0, \tau, x_\varepsilon(\tau + \Delta_\varepsilon))$ in system (2). Observe that

$$x_\varepsilon(\tau + \Delta_\varepsilon) = \Omega(\tau, 0, \nu_\varepsilon(\tau)). \tag{9}$$

Taking the derivative in (9) with respect to τ we obtain

$$\dot{x}_\varepsilon(\tau + \Delta_\varepsilon) = f(\Omega(\tau, 0, \nu_\varepsilon(\tau))) + \Omega'_\xi(\tau, 0, \nu_\varepsilon(\tau))\dot{\nu}_\varepsilon(\tau). \tag{10}$$

On the other hand from (2) we have

$$\dot{x}_\varepsilon(\tau + \Delta_\varepsilon) \in f(\Omega(\tau, 0, \nu_\varepsilon(\tau))) + \varepsilon g(\tau + \Delta_\varepsilon, \Omega(\tau, 0, \nu_\varepsilon(\tau)), \varepsilon). \tag{11}$$

Since $\Omega'_\xi(\tau, 0, \nu_\varepsilon(\tau))$ is the fundamental matrix of a linear system thus it is invertible, then from (10) and (11) it follows

$$\dot{\nu}_\varepsilon(\tau) \in \varepsilon \left(\Omega'_\xi(\tau, 0, \nu_\varepsilon(\tau))\right)^{-1} g(\tau + \Delta_\varepsilon, \Omega(\tau, 0, \nu_\varepsilon(\tau)), \varepsilon),$$

and $\nu_\varepsilon(0) = x_\varepsilon(\Delta_\varepsilon) = x_\varepsilon(T + \Delta_\varepsilon) = \Omega(T, 0, \nu_\varepsilon(T))$. Since g is measurable then again by Filippov's Lemma there exists a measurable singlevalued function

$h_\varepsilon : [0,T] \to \mathbb{R}^2$ such that $h_\varepsilon(\tau) \in \left(\Omega'_\xi(\tau,0,\nu_\varepsilon(\tau))\right)^{-1} g(\tau+\Delta_\varepsilon, \Omega(\tau,0,\nu_\varepsilon(\tau)),\varepsilon)$ for a.a. $\tau \in [0,T]$, and $\dot{\nu}_\varepsilon(\tau) = \varepsilon h_\varepsilon(\tau)$, for a.a. $\tau \in [0,T]$. Therefore, $h_\varepsilon \in L^\infty([0,T],\mathbb{R}^2)$ and

$$\nu_\varepsilon(\tau) = \Omega(T,0,\nu_\varepsilon(T)) + \varepsilon \int_0^\tau h_\varepsilon(s)\,ds \quad \text{for any } \tau \in [0,T]. \tag{12}$$

Since, for any $\tau \geq 0$, $\nu_\varepsilon(\tau) \to x_0(0)$ as $\varepsilon \to 0$ we can write $\nu_\varepsilon(\tau)$ in the following form

$$\nu_\varepsilon(\tau) = x_0(0) + \varepsilon\mu_\varepsilon(\tau). \tag{13}$$

Now we prove that the functions μ_ε are bounded on $[0,T]$ uniformly with respect to $\varepsilon \in (0,\varepsilon_0]$. For this, we first subtract $x_0(0)$ from both sides of (12), with $\tau = T$, obtaining

$$\varepsilon\mu_\varepsilon(T) = \varepsilon\,\Omega'_\xi(T,0,x_0(0))\mu_\varepsilon(T) + o(\varepsilon\mu_\varepsilon(T)) + \varepsilon \int_0^T h_\varepsilon(s)\,ds, \tag{14}$$

where, from (13), $\frac{o(\varepsilon\mu_\varepsilon(T))}{\|\varepsilon\mu_\varepsilon(T)\|} \to 0$ as $\varepsilon \to 0$. Since $x_\varepsilon(\Delta_\varepsilon) \in S\left(\{v \in \mathbb{R} : |v| \leq r_0\right)$, then by Lemma 2.5 there exists $v_\varepsilon \in \mathbb{R}$, $|v_\varepsilon| \leq r_0$, such that

$$x_\varepsilon(\Delta_\varepsilon) = \Omega(T,0,x_0(0) + A_1 v_\varepsilon) \tag{15}$$

and $v_\varepsilon \to 0$ as $\varepsilon \to 0$. Now by using (15) we can represent $\varepsilon\mu_\varepsilon(T)$ as follows

$$\varepsilon\mu_\varepsilon(T) = \nu_\varepsilon(T) - x_0(0) = \Omega(0,T,\Omega(T,0,x_0(0)+A_1v_\varepsilon)) - x_0(0) = A_1 v_\varepsilon. \tag{16}$$

Therefore (14) can be rewritten as follows

$$A_1 v_\varepsilon = \Omega'_\xi(T,0,x_0(0))A_1 v_\varepsilon + o(A_1 v_\varepsilon) + \varepsilon \int_0^T h_\varepsilon(s)\,ds. \tag{17}$$

Let us show that there exists $M_1 > 0$ such that

$$|v_\varepsilon| \leq \varepsilon M_1 \quad \text{for any } \varepsilon \in (0,\varepsilon_0]. \tag{18}$$

Arguing by contradiction we assume that there exist sequences $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\varepsilon_0]$, $\varepsilon_k \to 0$ as $k \to \infty$, such that $|v_{\varepsilon_k}| = \varepsilon_k c_k$, where $c_k \to \infty$ as $k \to \infty$. Let $q_k = \frac{v_{\varepsilon_k}}{|v_{\varepsilon_k}|}$, then from (17) we have

$$A_1 q_k = \Omega'_\xi(T,0,x_0(0))A_1 q_k + \frac{o(A_1 v_{\varepsilon_k})}{|v_{\varepsilon_k}|} + \frac{1}{c_k}\int_0^T h_{\varepsilon_k}(s)\,ds, \tag{19}$$

where $\frac{o(A_1 v_{\varepsilon_k})}{|v_{\varepsilon_k}|} \to 0$ as $k \to \infty$, in fact $\frac{o(A_1 v_{\varepsilon_k})}{|v_{\varepsilon_k}|} = \frac{o(A_1 v_{\varepsilon_k})}{\|A_1 v_{\varepsilon_k}\|} \cdot \frac{\|A_1 v_{\varepsilon_k}\|}{|v_{\varepsilon_k}|}$.

Let $B = \{\nu_\varepsilon(\tau) : \tau \in [0,T], \varepsilon \in [0,1]\}$. The continuity of Ω and condition (8) imply that B is bounded. Since also $(\Omega'_\xi)^{-1}$ is continuous, we can find $\Lambda > 0$

satisfying $\left\| \left(\Omega'_\xi(T,0,v_\varepsilon(\tau))\right)^{-1} \right\| \le \Lambda$ for any $\tau \in [0,T]$ and any $\varepsilon \in [0,\varepsilon_0]$. Therefore, from assumption (H) we obtain

$$\left\| \int_0^T h_\varepsilon(s)\,ds \right\| \le \varepsilon\Lambda \int_0^T \mu_B(s+\Delta_\varepsilon)\,ds < +\infty \quad \text{for } \varepsilon \in [0,\varepsilon_0]. \tag{20}$$

Without loss of generality we may assume that the sequence $\{q_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ converges, let $q_0 = \lim_{k\to\infty} q_k$ with $|q_0| = 1$. By passing to the limit as $k \to \infty$ in (19) we have that $A_1 q_0 = \Omega'_\xi(T,0,x_0(0))A_1 q_0$. Therefore $A_1 q_0$ is the initial condition of a T-periodic solution to (3). On the other hand the cycle x_0 is nondegenerate, hence $A_1 q_0$ is linearly dependent with $\dot{x}_0(0)$ contradicting the choice of A_1. Thus (18) is true for some $M_1 > 0$. From (13) and the fact that $\nu_\varepsilon(0) = x_\varepsilon(\Delta_\varepsilon)$ we have

$$\begin{aligned} \|x_\varepsilon(\Delta_\varepsilon) - x_0(0)\| &= \varepsilon\|\mu_\varepsilon(0)\| \\ &\le \varepsilon\|\mu_\varepsilon(T)\| + \|\varepsilon\mu_\varepsilon(T) - \varepsilon\mu_\varepsilon(0)\| \\ &= \varepsilon\|\mu_\varepsilon(T)\| + \|\nu_\varepsilon(T) - \nu_\varepsilon(0)\|. \end{aligned} \tag{21}$$

From (12) and (20) we have that there exists $M_2 > 0$ such that

$$\|\nu_\varepsilon(T) - \nu_\varepsilon(0)\| \le \varepsilon M_2 \quad \text{for any } \varepsilon \in (0,\varepsilon_0]. \tag{22}$$

Therefore, combining (16) with (18) and taking into account (22), from (21) we have that $\|x_\varepsilon(\Delta_\varepsilon) - x_0(0)\| \le \varepsilon\|A_1\|M_1 + \varepsilon M_2$ for any $\varepsilon \in (0,\varepsilon_0]$. Since $\dot{x}_\varepsilon(t+\Delta_\varepsilon) \in f(x_\varepsilon(t+\Delta_\varepsilon)) + \varepsilon g(t+\Delta_\varepsilon, x_\varepsilon(t+\Delta_\varepsilon),\varepsilon)$ and g is measurable then Filippov's Lemma ensures the existence of a measurable singlevalued function $m_\varepsilon : [0,T] \to \mathbb{R}^2$ such that

$$\dot{x}_\varepsilon(t+\Delta_\varepsilon) = f(x_\varepsilon(t+\Delta_\varepsilon)) + \varepsilon m_\varepsilon(t) \quad \text{for a.a. } t \in [0,T]$$

and $m_\varepsilon(t) \in g(t+\Delta_\varepsilon, x_\varepsilon(t+\Delta_\varepsilon),\varepsilon)$ for a.a. $t \in [0,T]$. This allows to conclude that

$$x_\varepsilon(t+\Delta_\varepsilon) - x_0(t) = x_\varepsilon(\Delta_\varepsilon) - x_0(0) + \int_0^t \left(f(x_\varepsilon(s+\Delta_\varepsilon)) - f(x_0(s))\right)ds + \varepsilon\int_0^t m_\varepsilon(s)ds.$$

Therefore, there exists a constant $M_3 \ge 0$ such that, for any $\varepsilon \in (0,\varepsilon_0]$, we have

$$\|x_\varepsilon(t+\Delta_\varepsilon) - x_0(t)\| \le \varepsilon(\|A_1\|M_1 + M_2) + M_3\int_0^t \|x_\varepsilon(s+\Delta_\varepsilon) - x_0(s)\|ds + \varepsilon M_3. \tag{23}$$

By means of the Gronwall-Bellman Lemma (compare, e.g., [6, Chapter II, § 11]), inequality (23) implies

$$\|x_\varepsilon(t+\Delta_\varepsilon) - x_0(t)\| \le \varepsilon\left(\|A_1\|M_1 + M_2 + M_3\right) e^{M_3 T} \quad \text{for any } \varepsilon \in (0,\varepsilon_0].$$

and thus the proof is complete. □

Remark 2.7. Observe that Theorem 2.6 does not require that (1) is a Hamiltonian system, indeed the crucial assumption is that the linearized system (3) has a not T-periodic solution.

3. First approximation formula for periodic solutions of the perturbed system

Denote by $\widetilde{z}$ a non-trivial T-periodic solution of the adjoint system

$$\dot{z} = -(f'(x_0(t)))^* z. \tag{24}$$

Observe that, since $+1$ is a characteristic multiplier of (3) then $+1$ is also a characteristic multiplier of (24), see [6, Chapter III, §23]), and so $\widetilde{z}$ exists.

Let $t_* \in [0,T]$ such that $\widetilde{z}_1(t_*) = 0$, hence $\widetilde{z}_2(t_*) \neq 0$. We begin the section by studying the behavior, as $\varepsilon \to 0$, of the scalar product $\left\langle \widetilde{z}(t), \frac{x_\varepsilon(t+\Delta_\varepsilon)-x_0(t)}{\varepsilon} \right\rangle$ which is the starting point for deriving the first approximation formula (5). To this end we denote by $\widehat{z} = (\widehat{z}_1, \widehat{z}_2)$ any solution of (24) defined in $[0,T]$ linearly independent with $\widetilde{z}$ and introduce the multivalued map $M^\perp : [0,T] \to K(\mathbb{R})$ as follows

$$\begin{aligned} M^\perp(t) = \Bigg\{ & \gamma(t_*) \int_{t-T}^{t} \langle -\widehat{z}(\tau), h(\tau)\rangle \, d\tau : \\ & h \in L^\infty([-T,T],\mathbb{R}^2), h(t) \in g(t, x_0(t), 0) \text{ for a.a } t \in [-T,T] \Bigg\}, \end{aligned} \tag{25}$$

where $\gamma(t_*) = \frac{\widetilde{z}_2(t_*)}{\widehat{z}_2(T+t_*)-\widehat{z}_2(t_*)}$. We can prove the following result.

Theorem 3.1. *Assume $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and $g : \mathbb{R} \times \mathbb{R}^2 \times [0,1] \to K(\mathbb{R}^2)$ upper semicontinuous and satisfying* (H). *Let x_ε be a T-periodic solution to the perturbed system* (2) *such that*

$$\|x_\varepsilon(t+\Delta_\varepsilon) - x_0(t)\| \leq M\varepsilon \quad \textit{for any } t \in [0,T] \textit{ and any } \varepsilon \in (0, \varepsilon_0],$$

where $\Delta_\varepsilon \to 0$ as $\varepsilon \to 0$, M and ε_0 are positive constants and x_0 is a nondegenerate cycle of the Hamiltonian system (1). *Then*

$$\lim_{\varepsilon \to 0} \rho \left(\frac{1}{\varepsilon} \langle \widetilde{z}(t), x_\varepsilon(t+\Delta_\varepsilon) - x_0(t) \rangle, \frac{\widetilde{z}_2(t_*)}{\widehat{z}_2(T+t_*) - \widehat{z}_2(t_*)} M^\perp(t) \right) = 0$$

uniformly with respect to $t \in [0,T]$, where for any $v \in \mathbb{R}^n$ and $S \subset \mathbb{R}^n$ the distance $\rho(v,S)$ is defined as $\rho(v,S) = \inf_{s \in S} \|v - s\|$.

To prove Theorem 3.1 we need the following Lemma.

Lemma 3.2. *Assume that the T-periodic system*

$$\dot{u} = A(t)u, \quad u \in \mathbb{R}^2 \tag{26}$$

has the characteristic multiplier $+1$ of algebraic multiplicity 2. Let us denote by $\widetilde{u} = (\widetilde{u}_1, \widetilde{u}_2)$ a T-periodic solution of (26) *such that $\widetilde{u}_1(0) = 0$, $\widetilde{u}_2(0) \neq 0$. Denote by $\widehat{u} = (\widehat{u}_1, \widehat{u}_2)$ any solution of* (26) *satisfying $\widehat{u}_1(0) \neq 0$. Then*

$$\widehat{u}(t+T) = \widehat{u}(t) + \frac{\widehat{u}_2(T) - \widehat{u}_2(0)}{\widetilde{u}_2(0)} \widetilde{u}(t) \quad \textit{for any } t \in \mathbb{R}.$$

This result has been proved in [19, Lemma 4.2] under the additional assumption $\widehat{u}_2(0) = 0$. Though it is immediate to see that avoiding this assumption does not affect the proof of [19, Lemma 4.2] at all we provide here a proof of Lemma 3.2 for a sake of completeness.

Proof. Denote the fundamental matrix of system (26) by X such that $X(0) = I$. Since $X(T)\binom{0}{1} = \binom{0}{1}$, then $X(T) = \left(\begin{smallmatrix} a & 0 \\ b & 1 \end{smallmatrix}\right)$ with $a, b \in \mathbb{R}$. By our assumption $X(T)$ has two eigenvalues equal to $+1$, therefore $X(T) = \left(\begin{smallmatrix} 1 & 0 \\ b & 1 \end{smallmatrix}\right)$, $b \in \mathbb{R}$. We have

$$X(t+T)\widehat{u}(0) = X(t)X(T)\widehat{u}(0) = X(t)\widehat{u}(0) + X(t)\begin{pmatrix} 0 \\ b\widehat{u}_1(0) \end{pmatrix} = X(t)\widehat{u}(0) + \frac{b\widehat{u}_1(0)}{\widetilde{u}_2(0)}\widetilde{u}(t).$$

On the other hand

$$X(T)\widehat{u}(0) = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\widehat{u}(0) = \widehat{u}(0) + \begin{pmatrix} 0 \\ b\widehat{u}_1(0) \end{pmatrix},$$

which implies $b\widehat{u}_1(0) = \widehat{u}_2(T) - \widehat{u}_2(0)$. This completes the proof. □

Proof of Theorem 3.1. In what follows $\varepsilon \in (0, \varepsilon_0]$; $t, \tau \in [-T, T]$ and $\widetilde{z}$, $\widehat{z}$ are the functions introduced at the beginning of this section. Let A be a nonsingular 2×2 matrix such that

$$\widehat{z}(0)^* A = (0, 1). \tag{27}$$

Let $Y(t)$ be the fundamental matrix of the linearized system (3) with initial condition $Y(0) = A$. Let

$$Z(t) = (Y(t)^*)^{-1} \tag{28}$$

and define $a_\varepsilon \in C([-T, T], \mathbb{R}^2)$ as $a_\varepsilon(t) = Z(t)^* \frac{x_\varepsilon(t+\Delta_\varepsilon) - x_0(t)}{\varepsilon}$. Then we have

$$x_\varepsilon(t + \Delta_\varepsilon) - x_0(t) = \varepsilon Y(t) a_\varepsilon(t). \tag{29}$$

In what follows by $o(\varepsilon)$, $\varepsilon > 0$, we will denote a function, which may depend also on other variables, having the property that $\frac{o(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0$ uniformly with respect to these variables when they belong to any bounded set. Since

$$\dot{x}_\varepsilon(t + \Delta_\varepsilon) \in f(x_\varepsilon(t + \Delta_\varepsilon)) + \varepsilon g(t + \Delta_\varepsilon, x_\varepsilon(t + \Delta_\varepsilon), \varepsilon) \quad \text{for a.a. } t \in \mathbb{R}$$

then, again by Filippov's Lemma there exists a measurable singlevalued function $h_\varepsilon : \mathbb{R} \to \mathbb{R}^2$ such that

$$\dot{x}_\varepsilon(t + \Delta_\varepsilon) = f(x_\varepsilon(t + \Delta_\varepsilon)) + \varepsilon h_\varepsilon(t) \quad \text{for a.a. } t \in \mathbb{R} \tag{30}$$

and $h_\varepsilon(t) \in g(t + \Delta_\varepsilon, x_\varepsilon(t + \Delta_\varepsilon), \varepsilon)$ for a.a. $t \in \mathbb{R}$. By subtracting (1) where $x(t)$ is replaced by $x_0(t)$ from (30) we obtain

$$\dot{x}_\varepsilon(t + \Delta_\varepsilon) - \dot{x}_0(t) = f'(x_0(t))(x_\varepsilon(t + \Delta_\varepsilon) - x_0(t)) + \varepsilon h_\varepsilon(t) + o_t(\varepsilon) \tag{31}$$

for a.a. $t \in [-T, T]$. Here $\varepsilon \to o_t(\varepsilon)$ is such that $o_{t+T}(\cdot) = o_t(\cdot)$ for any $t \in \mathbb{R}$. By substituting (29) into (31) we have

$$\varepsilon \dot{Y}(t) a_\varepsilon(t) + \varepsilon Y(t) \dot{a}_\varepsilon(t) = \varepsilon f'(x_0(t)) Y(t) a_\varepsilon(t) + \varepsilon h_\varepsilon(t) + o_t(\varepsilon)$$

for a.a. $t \in [-T, T]$. Since $f'(x_0(t))Y(t) = \dot{Y}(t)$ the last formula can be rewritten as follows

$$\varepsilon Y(t) \dot{a}_\varepsilon(t) = \varepsilon h_\varepsilon(t) + o_t(\varepsilon) \quad \text{for a.a. } t \in [-T, T]. \tag{32}$$

By means of Perron's Lemma [22] (see also Demidovich [6, Sec. III, §12]), formula (27) implies that $\widehat{z}(t)^* Y(t) = (0, 1)$ for any $t \in \mathbb{R}$. Therefore, applying $\widehat{z}(t)^*$ to both sides of (32) we have

$$\varepsilon (\dot{a}_{\varepsilon,2})(t) = \varepsilon \widehat{z}(t)^* h_\varepsilon(t) + \widehat{z}(t)^* o_t(\varepsilon) \quad \text{for a.a. } t \in [-T, T],$$

where $a_{\varepsilon,2}(t)$ is the second component of the vector $a_\varepsilon(t)$, and so

$$a_{\varepsilon,2}(t) = a_{\varepsilon,2}(t_0) + \int_{t_0}^{t} \langle \widehat{z}(\tau), h_\varepsilon(\tau) \rangle \, d\tau + \int_{t_0}^{t} \left\langle \widehat{z}(\tau), \frac{o_\tau(\varepsilon)}{\varepsilon} \right\rangle d\tau \tag{33}$$

for all $t, t_0 \in [-T, T]$. From (28) we have that $Z(0)^* Y(0) = I$. Therefore $([Z(0)]_2)^* A = (0, 1)$, where $[Z(0)]_2$ denotes the second column of $Z(0)$. Thus $[Z(0)]_2 = \widehat{z}(0)$. Therefore $a_{\varepsilon,2}(t) = \left\langle \widehat{z}(t), \frac{x_\varepsilon(t+\Delta_\varepsilon) - x_0(t)}{\varepsilon} \right\rangle$. Since $\widehat{z}$ is linearly independent with $\widetilde{z}$ then $\widehat{z}_1(t_*) \neq 0$. Since system (1) is Hamiltonian then the algebraic multiplicity of the characteristic multiplier $+1$ of linearized system (3) is equal to 2. By Lemma 3.2 we have

$$\widehat{z}(t) = \widehat{z}(t - T) + \frac{\widehat{z}_2(T + t_*) - \widehat{z}_2(t_*)}{\widetilde{z}_2(t_*)} \widetilde{z}(t) = \widehat{z}(t - T) + \frac{1}{\gamma(t_*)} \widetilde{z}(t),$$

that implies $a_{\varepsilon,2}(t_0) = a_{\varepsilon,2}(t_0 - T) + \frac{1}{\gamma(t_*)} \left\langle \widetilde{z}(t_0), \frac{x_\varepsilon(t_0+\Delta_\varepsilon) - x_0(t_0)}{\varepsilon} \right\rangle$. Substituting the last formula into (33) we obtain

$$\int_{t_0}^{t_0 - T} \langle \widehat{z}(\tau), h_\varepsilon(\tau) \rangle \, d\tau = -\frac{1}{\gamma(t_*)} \left\langle \widetilde{z}(t_0), \frac{x_\varepsilon(t_0+\Delta_\varepsilon) - x_0(t_0)}{\varepsilon} \right\rangle - \int_{t_0}^{t_0 - T} \left\langle \widehat{z}(\tau), \frac{o_\tau(\varepsilon)}{\varepsilon} \right\rangle d\tau. \tag{34}$$

We claim that

$$\lim_{\varepsilon \to 0} \rho \left(\gamma(t_*) \int_{t_0}^{t_0 - T} \langle \widehat{z}(\tau), h_\varepsilon(\tau) \rangle \, d\tau, M^\perp(t) \right) = 0, \tag{35}$$

uniformly with respect to $t_0 \in [0, T]$, with h_ε defined as in (30). To prove this we observe that the subset of $\mathbb{R}$ given by

$$M := \Big\{ \gamma(t_*) \int_{-T}^{T} \langle \widehat{z}(\tau), h(\tau) \rangle \, d\tau \; : \\ h \in L^\infty([-T, T], \mathbb{R}^2) \text{ and } h(t) \in g(t, x_0(t), 0) \text{ for a.a. } t \in [-T, T] \Big\}$$

is nonempty and compact; hence, for each $\varepsilon \in (0, \varepsilon_0]$, there exists $k_\varepsilon : \mathbb{R} \to \mathbb{R}$ such that $k_\varepsilon(t) \in g(t, x_0(t), 0)$ for a.a. t and

$$\rho\left(\gamma(t_*)\int_{-T}^{T}\langle\widehat{z}(\tau), h_\varepsilon(\tau)\rangle\, d\tau, M\right) = |\gamma(t_*)|\left|\int_{-T}^{T}\langle\widehat{z}(\tau), h_\varepsilon(\tau) - k_\varepsilon(\tau)\rangle\, d\tau\right|.$$

The upper semicontinuity of g in the bounded set

$$[-T, T] \times \{x_\varepsilon(t) : t \in [0, T],\ \varepsilon \in [0, \varepsilon_0]\} \times [0, \varepsilon_0]$$

implies that, given $\delta > 0$, there exists $\varepsilon_1 \in (0, \varepsilon_0]$ such that, for all $\varepsilon \in [0, \varepsilon_1]$, we have that

$$h_\varepsilon(t) \in g(t + \Delta_\varepsilon, x_\varepsilon(t + \Delta_\varepsilon), \varepsilon) \subset B_\delta(g(t, x_0(t), 0)) \quad \text{for a.a. } t \in [-T, T]. \tag{36}$$

Fix an arbitrary $\delta > 0$ and let $\varepsilon_1 \in (0, \varepsilon_0]$ satisfying (36). Let $\varepsilon \in [0, \varepsilon_1]$ and $t_0 \in [0, T]$. We obtain

$$\begin{aligned}\rho\left(\gamma(t_*)\int_{t_0-T}^{t_0}\langle\widehat{z}(\tau), h_\varepsilon(\tau)\rangle\, d\tau, M^\perp(t)\right) &\le |\gamma(t_*)|\left|\int_{t_0-T}^{t_0}\langle\widehat{z}(\tau), h_\varepsilon(\tau) - k_\varepsilon(\tau)\rangle\, d\tau\right| \\ &\le |\gamma(t_*)|\int_{-T}^{T}|\langle\widehat{z}(\tau), h_\varepsilon(\tau) - k_\varepsilon(\tau)\rangle|\, d\tau \\ &\le 2T|\gamma(t_*)|\delta\|\widehat{z}\|_C.\end{aligned}$$

which implies our assertion (35). According to (34), the proof is complete. □

Remark 3.3. The assumption that (1) is Hamiltonian ensures that the linearized system (3) has a characteristic multiplier +1 of algebraic multiplicity 2 and so the assumption of Lemma 3.2. Alternatively, we could directly assume that the algebraic multiplicity of the characteristic multiplier +1 of (3) is equal to 2. The latter is a bit more general. The same consideration applies to Theorems 3.6 and 4.2 below.

We have the following result.

Lemma 3.4. *Let x_0 be a nondegenerate T-periodic cycle of the Hamiltonian system* (1). *Let $\widetilde{z}$ be any T-periodic solution of the adjoint system* (24). *Then*

$$\langle\dot{x}_0(t), \widetilde{z}(t)\rangle = 0 \quad \textit{for any } t \in \mathbb{R}. \tag{37}$$

Proof. Let $t_* \in [0, T]$ be such that $\widetilde{z}_1(t_*) = 0$. Let $\widehat{z}$ be any solution of (24) linearly independent with $\widetilde{z}$. Then from Lemma 3.2 we have

$$\langle\dot{x}_0(t), \widehat{z}(t + T)\rangle = \langle\dot{x}_0(t), \widehat{z}(t)\rangle + \frac{\widehat{z}_2(T + t_*)}{\widetilde{z}_2(t_*)}\langle\dot{x}_0(t), \widetilde{z}(t)\rangle \quad \text{for any } t \in \mathbb{R}.$$

Perron's Lemma [22] implies that $\langle\dot{x}_0(t), \widehat{z}(t + T)\rangle = \langle\dot{x}_0(t), \widehat{z}(t)\rangle$ for any $t \in \mathbb{R}$ and thus (37). □

Lemma 3.4 allows the reader to better understand the substantial difference between the situation when the cycle x_0 is isolated, which is studied in [17, 18] and the present situation when the cycle is non-isolated. In fact, in [17, 18] it is shown that $\langle \dot{x}_0(t), \widetilde{z}(t)\rangle \neq 0$, for any $t \in \mathbb{R}$, which is the contrary of (37).

Remark 3.5. Let $\widetilde{z}$ be any T-periodic solution of the adjoint system (24) and $\widehat{z}$ any solution of (24) linearly independent with $\widetilde{z}$. Lemma 3.4 ensures that $\langle \dot{x}_0(t), \widetilde{z}(t)\rangle = 0$ for any $t \in \mathbb{R}$, moreover from the Perron's Lemma $\langle \dot{x}_0(t), \widehat{z}(t)\rangle = \langle \dot{x}_0(0), \widehat{z}(0)\rangle \neq 0$ for any $t \in \mathbb{R}$. Without loss of generality we can assume that $\langle \dot{x}_0(0), \widehat{z}(0)\rangle = 1$. Let y be the function defined by

$$y(t)^* = \left(\frac{-\widehat{z}_2(t)}{\det(\widehat{z}(t), \widetilde{z}(t))}, \frac{\widehat{z}_1(t)}{\det(\widehat{z}(t), \widetilde{z}(t))}\right)$$

then

$$(\dot{x}_0(t), y(t)) = \begin{pmatrix} \widehat{z}(t)^* \\ \widetilde{z}(t)^* \end{pmatrix}^{-1} \tag{38}$$

is a matrix solution of the linearized system (3) ([6, Chapter III, §12]).

We can now formulate the following result.

Theorem 3.6. *Assume $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and $g : \mathbb{R} \times \mathbb{R}^2 \times [0,1] \to K(\mathbb{R}^2)$ upper semicontinuous and satisfying* (H). *Let x_ε be a T-periodic solution to perturbed system* (2) *such that*

$$\|x_\varepsilon(t) - x_0(t)\| \to 0 \quad as\ \varepsilon \to 0$$

uniformly with respect to $t \in [0,T]$, where x_0 is a nondegenerate T-periodic cycle of the Hamiltonian system (1). *Let $\widetilde{z}$, $\widehat{z}$ be as in Remark* 3.5 *and $\dot{x}_0$, y as in* (38). *Then there exists a family $\{\Delta_\varepsilon\}_{\varepsilon>0}$ such that $\Delta_\varepsilon \to 0$ as $\varepsilon \to 0$ and*

$$\lim_{\varepsilon\to 0} \rho\Big(x_\varepsilon(t+\Delta_\varepsilon) - x_0(t), \varepsilon M^\perp(t)y(t) + \langle \widehat{z}(t), x_\varepsilon(t-\Delta_\varepsilon) - x_0(t)\rangle \dot{x}_0(t)\Big) = 0, \tag{39}$$

uniformly with respect to $t \in [0,T]$.

Proof. The proof of Theorem 3.6 follows from the following representation

$$x_\varepsilon(t+\Delta_\varepsilon) - x_0(t) = \langle \widetilde{z}(t), x_\varepsilon(t+\Delta_\varepsilon) - x_0(t)\rangle y(t) + \langle \widehat{z}(t), x_\varepsilon(t+\Delta_\varepsilon) - x_0(t)\rangle \dot{x}_0(t),$$

which is a consequence of (38), and from Theorem 3.1. □

4. A symmetric case

In this section we consider the situation when the unperturbed Hamiltonian system (1) possesses the following symmetry properties:

$$f_1(\xi_1, \xi_2) = f_1(-\xi_1, \xi_2), \tag{40}$$
$$f_2(\xi_1, \xi_2) = -f_2(-\xi_1, \xi_2), \tag{41}$$
$$(f_1)'_{(1)}(\xi_1, \xi_2) = -(f_2)'_{(2)}(\xi). \tag{42}$$

where $(h)'_{(i)}, i = 1, 2$ denotes the derivative of h with respect to the $i-$variable. The main consequence of this symmetry assumption is given by the following Lemma whose prove is immediate.

Lemma 4.1 ([19, Lemma 4.4]). *Assume $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and that properties (40)–(42) hold true. Let x_0 be a nondegenerate cycle of the Hamiltonian system (1) and denote by y the solution of the linearized system (3) satisfying*

$$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} -\dot{x}_{0,2}(0) \\ \dot{x}_{0,1}(0) \end{pmatrix} \tag{43}$$

Then the functions $\widehat{z}(\theta) = \begin{pmatrix} y_2(\theta) \\ -y_1(\theta) \end{pmatrix}$, $\widetilde{z}(\theta) = \begin{pmatrix} -\dot{x}_{0,2}(\theta) \\ \dot{x}_{0,1}(\theta) \end{pmatrix}$, $\theta \in \mathbb{R}$, where $\dot{x}_0(\theta) = (\dot{x}_{0,1}(\theta), \dot{x}_{0,2}(\theta))$, are linearly independent solutions of the adjoint system (24).

Lemma 4.1 allows us to rewrite the multivalued map $M^\perp : [0, T] \to K(\mathbb{R})$ defined in (25) as follows

$$M^\perp(t) = \left\{ \frac{\dot{x}_{0,1}(t_*)}{y_1(T + t_*)} \int_{t-T}^{t} \det\left(-y(\tau), h(\tau)\right) d\tau : \right.$$
$$\left. h \in L^\infty([-T, T], \mathbb{R}^2) : h(t) \in g(t, x_0(t), 0) \text{ for a.a } t \in [-T, T] \right\}.$$

where $t_* \in [0, T]$ is such that $\dot{x}_{0,2}(t_*) = 0$. Therefore Theorem 3.6 takes the form of the following Theorem 4.2 when the symmetry assumptions (40)–(42) are satisfied. In particular, observe that the statement of Theorem 4.2 refers only to the linearized system (3) and not to the adjoint system (24).

Theorem 4.2. *Assume $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and $g : \mathbb{R} \times \mathbb{R}^2 \times [0, 1] \to K(\mathbb{R}^2)$ upper semicontinuous and satisfying* (H). *Let x_ε be a T-periodic solution to perturbed system (2) satisfying*

$$\|x_\varepsilon(t) - x_0(t)\| \to 0 \quad as\ \varepsilon \to 0$$

uniformly with respect to $t \in [0, T]$, where x_0 is a nondegenerate T-periodic cycle of the Hamiltonian system (1). Let y be the solution of the linearized system (3) *with the initial condition* (43). *Then there exists a family* $\{\Delta_\varepsilon\}_{\varepsilon>0}$ *such that* $\Delta_\varepsilon \to 0$ *as* $\varepsilon \to 0$ *and*

$$x_\varepsilon(t+\Delta_\varepsilon)-x_0(t) \in \varepsilon M^\perp(t)y(t)+\left\langle \begin{pmatrix} y_2(t) \\ -y_1(t) \end{pmatrix}, x_\varepsilon(t+\Delta_\varepsilon) - x_0(t) \right\rangle \dot{x}_0(t)+o(\varepsilon),$$

uniformly with respect to $t \in [0, T]$.

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