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On Bifurcation of Periodic Solutions for Functional Differential Equations of the Neutral Type with Small Delay[1]

M. I. Kamenskii,* Yu. V. Lysakova,* and P. Nistri**

* *Voronezh State University, Voronezh, Russia*
** *Siena University, Siena, Italy*

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Abstract—The class is singled out of systems described by ordinary differential equations unsolved relative to a derivative, in which a small delay leads to bifurcation of periodic solutions from the equilibrium state. The direct application of the classical results of M.A. Krasnosel'skii to these systems is made difficult in view of the complex character of the dependence on a bifurcation parameter, which is a small delay. The problem on bifurcation of periodic solutions for the stated systems is solved by methods of the theory of rotation of condensing vector fields.

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1. INTRODUCTION

A large class of nonlinear control systems is described by ordinary differential equations unsolved relative to a derivative. This will take place, for example, in feedback systems, when the control is set up on the basis of analysis of not only the system state, but also of the speed of its change. In this case, a small delay related to the time of passage of a signal is commonly not taken into account, etc. It is considered that in this case the qualitative pattern of the process does not change. However, in other problems, examples are well known of the presence of critical values of the parameter that lead to a qualitative change of the pattern of the process, for example, to the birth of periodic modes from the equilibrium state that then loses the stability. This phenomenon bears the name of bifurcation of forced oscillations from the equilibrium position. Its mathematical model was built up by Krasnosel'skii (see, for example [1]). We note that Krasnosel'skii considered the problem for the birth of periodic solutions from the state of equilibrium of the n-dimensional system of ordinary differential equations, the right side of which continuously depends on the parameter. Thus, in view of finite dimensionality of the system, the right side is uniformly continuous in the parameter. But the simplest system with a delay, which was dealt with above, has the form

$$x'(t) = \Phi(t, x(t - \varepsilon h), x'(t - \varepsilon h)), \tag{1}$$

where $\Phi : R^1 \times R^m \times R^m \to R^m$, $h > 0$, while the ε is a small positive parameter. Following the terminology from [2], these equations are called equations of the neutral type. In this case, the continuity by ε will not be uniform and the Krasnosel'skii method cannot be directly used for analysis of the problem on bifurcation of the periodic solutions of the Eq. (1). We will also note that the operators, whose fixed points define periodic solutions of the Eq. (1), will not be entirely

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continuous in appropriate function spaces. This requires the application of the theory of condensing operators, which is developed in [3].

We note that at $\varepsilon = 0$, the Eq. (1) converts to the ordinary differential equation that is not solved relative to a derivative:

$$x'(t) = \Phi(t, x(t), x'(t)). \tag{2}$$

The primary purpose of this article is the proof of the following fact: *if the Eq.* (2) *in the neighborhood of the equalibrium state is equivalent to the ordinary differential equation*

$$x'(t) = \varphi(t, x(t)),$$

for which the Krasnosel'skii bifurcation conditions for ordinary differential equations are fulfilled, then a small delay leads to bifurcation of periodic solutions for the Eq. (1). The results are stated for the functional differential equations of the neutral type of common form, which are described below.

2. STATEMENT OF THE PROBLEM

We will consider the equation

$$x'(t) = f(t, \varepsilon, W(\varepsilon)x_t, W(\varepsilon)x'_t), \tag{3}$$

where

$$f : R^1 \times [0,1] \times C([-h,0], R^m) \times C([-h,0], R^m) \to R^m,$$

while the expressions $x_t, x'_t \in C([-h,0], R^m)$ denote

$$x_t(s) = x(t+s),\ x'_t(s) = x'(t+s) \quad (s \in [-h,0]),$$

and the operator $W(\varepsilon), \varepsilon \in [0,1]$, acts in $C([-h,0], R^m)$ by the rule

$$[W(\varepsilon)u](s) = u(\varepsilon s) \quad (s \in [-h,0]).$$

Thus, the ε specifies a value of delay in the Eq. (3). For the given $\varepsilon \in [0,1]$, a value of the derivative of the unknown function x at the instant of time t depends on the behavior of the function x and its derivative in the section $[t - \varepsilon h, t]$.

We will assume that the following condition is fulfilled:

(A_1) *The function* $x(t) \equiv 0$ *is the solution of the Eq.* (3), *i.e.,*

$$f(t, \varepsilon, 0, 0) \equiv 0. \tag{4}$$

If $\varepsilon = 0$, then the Eq. (3) will have the form (2), where $\Phi : R^1 \times R^m \times R^m \longrightarrow R^m$ is the operator defined by the equality

$$\Phi(t, u, v) = f(t, 0, I_h u, I_h v).$$

Here I_h is the canonical embedding of R^m in $C([-h,0], R^m)$, i.e., the I_h is the operator acting from the R^m to the $C([-h,0], R^m)$ by the rule

$$(I_h u)(s) \equiv u \quad \text{for all} \quad u \in R^m.$$

We note that the Eq. (2) is the ordinary differential equation unsolved relative to the derivative. We assume that

(A_2) *The operator f is T-periodic by the first variable, continuous by the aggregate of variables, and satisfies the Lipschitz condition with the constant $l < 1$ by the fourth variable.*

We will further consider that the operator $f(t, \varepsilon, \cdot, \cdot)$ can be expanded in a Taylor series at the point $(t, \varepsilon, 0, 0)$, in which case all differentials of the $2, 3, \ldots, (k-1)$th order are equal to zero, i.e.,

$$f(t, \varepsilon, u, v) = a(t, \varepsilon)u + b(t, \varepsilon)v + C(t, \varepsilon, u, v) + D(t, \varepsilon, u, v), \tag{5}$$

where

$$a(t, \varepsilon) = f'_u(t, \varepsilon, 0, 0), \tag{6}$$

$$b(t, \varepsilon) = f'_v(t, \varepsilon, 0, 0), \tag{7}$$

and $C : R^1 \times [0, 1] \times C([-h, 0], R^m) \times C([-h, 0], R^m) \to R^m$ is the homogeneous operator of the order of k by the aggregate of the third and the fourth variable, i.e.,

$$C(t, \varepsilon, \lambda u, \lambda v) = \lambda^k C(t, \varepsilon, u, v).$$

The operator $D(t, \varepsilon, u, v)$ has the higher order of smallness than the k by the aggregate of the third and the fourth variable, i.e.,

$$\frac{\|D(t, \varepsilon, u, v)\|}{(\|u\| + \|v\|)^k} \to 0 \quad \text{at} \quad u, v \to 0$$

is uniform relative to the t, ε.

By the theorem on the implicit function (see, for example [4]), the Eq. (2) is equivalent to the ordinary differential equation in the normal form

$$x'(t) = \varphi(t, x(t)), \tag{8}$$

where $\varphi : R^1 \times R^m \to R^m$ admits in the neighborhood of the point $(t, 0)$ the presentation that is similar to the (5):

$$\varphi(t, x) = [I - b(t)]^{-1} a(t)x + C_\varphi(t, x) + D_\varphi(t, x), \tag{9}$$

where $C_\varphi : R^1 \times R^m \to R^m$ is the homogeneous operator by the second variable of the order of k and $\|D_\varphi(t, x)\| = o(\|x\|^k)$. Here $a(t) = a(t, 0)I_h$, $b(t) = b(t, 0)I_h$.

It follows from the equality (4) that

$$\varphi(t, 0) = 0. \tag{10}$$

Thus, $x(t) \equiv 0$ is the solution of the Eq. (8).

We will further assume that

(A_3) *1 is the simple multiplicator of the system*

$$x'(t) = [I - b(t)]^{-1} a(t)x(t). \tag{11}$$

We will denote by $x_0(t)$ the nonzero T-periodic solution of the system (11), such that the $\|x_0(0)\| = 1$. We will further consider that the x_0 is twice continuously differentiable and fulfilled:

(A_4) *System*

$$x'(t) = a(t, \varepsilon)W(\varepsilon)x_t + b(t, \varepsilon)W(\varepsilon)x'_t \tag{12}$$

has no nonzero T-periodic solutions for any $\varepsilon \neq 0$.

To fulfill (A_4) under the conditions of differentiability by ε of the operators a and b defined by the equalities (6) and (7), it is sufficient that the following expression be different from zero:

$$\int_0^T < (I - b(t,0))^{-1}(a(t,0)Ax_0'(t) + b(t,0)Ax_0''(t) + a_2'(t,0)I_h x_0(t) + b_2'(t,0)I_h x_0'(t)),\ z(t) > dt, \quad (13)$$

where

$$a_2'(t,0) = \left.\frac{\partial a(t,\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}, \quad b_2'(t,0) = \left.\frac{\partial b(t,\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0},$$

and the operator A acting from R^m in $C([-h,0],R^m)$ is defined by the formula

$$(Au)(s) = s(I_h u)(s) = su, \quad u \in R^m,$$

while $z(t)$ is the nonzero T-periodic solution conjugated to (11) of the system

$$z'(t) = -a^*(t)(I - b^*(t))^{-1}z(t).$$

If $\int_0^T < z(t),\ x_0(t) > dt = 1$, then the expression (13) is the coefficient μ_1 in the expansion

$$\mu^\varepsilon = 1 + \varepsilon\mu_1 + o(\varepsilon) \quad (14)$$

of the simple eigenvalue of $\mu^\varepsilon \to 1$ at $\varepsilon \to 0$ of the shift operator $U(T,\varepsilon)$ by trajectories of the Eq. (12) in time T. This expansion is valid in view of [5] and the fact (see [3]) that the U condenses by the aggregate of variables ε and the initial value. Details see in [5] and [3].

Let e_0 be the normalized eigenvector of the shift operator $V(T)$ by trajectories of the system (11). The eigenvector corresponds to the eigenvalue equal to 1, which is the initial condition of the T-periodic solution $x_0(t)$ of the system (11), i.e., $x_0(0) = e_0$.

Let g_0 be the eigenvector of the operator $V^*(T)$ conjugated to the operator $V(T)$, which corresponds to the eigenvalue 1 and is normalized by the condition:

$$(e_0, g_0) = 1. \quad (15)$$

It can be said without loss of generality that the $g_0 = z(0)$.

We introduce the following designation:

$$\xi_0 = \int_0^T \Big(U^{-1}(\tau,0)C_\varphi(\tau, x_0(\tau)), g_0\Big)\, d\tau.$$

3. FORMULATION OF BASIC RESULTS

We will denote by C_T^1 the set of continuously differentiable T-periodic functions with values in R^m. The norm in C_T^1 is defined in the following way:

$$\|x\|_{C_T^1} = \|x\|_{C_T} + \|\dot{x}\|_{C_T},$$

where C_T is the set of continuous T-periodic functions.

Definition 1. We suppose that $x(t) \equiv 0$ is the solution of the Eq. (3) at any ε, in which case at $\varepsilon = 0$, this is the isolated T-periodic solution. Then, the $\varepsilon = 0$ is said to be the point of bifurcation of solutions of the Eq. (3) if there exists a sequence of values of the parameter $\varepsilon_n \to 0$ and an appropriate sequence of nonzero T-periodic solutions $x^{\varepsilon_n} \to 0$ of the equation

$$(x^{\varepsilon_n})'(t) = f(t, \varepsilon_n, W(\varepsilon_n)x_t^{\varepsilon_n}, W(\varepsilon_n)(x^{\varepsilon_n})'_t).$$

Theorem 1. *Let conditions* (A_1)–(A_4) *be fulfilled.*

Then for the $\varepsilon = 0$ *to be the point of bifurcation of solutions of the Eq.* (3), *it is necessary and sufficient that there should exist a sequence* $\varepsilon_n \to 0$, *for the elements of which one of the equalities is fulfilled:*

$$\operatorname{sgn}(1 - \mu^{\varepsilon_n}) = \operatorname{sgn}\xi_0 \tag{16}$$

or

$$\operatorname{sgn}(1 - \mu^{\varepsilon_n}) = (-1)^{(k+1)}\operatorname{sgn}\xi_0. \tag{17}$$

If $\varepsilon = 0$ *is the point of bifurcation, then the bifurcation solution* x_{ε_n} *of the Eq.* (3) *satisfies the relation*

$$1 - \mu^{\varepsilon_n} \sim \|x_{\varepsilon_n}\|_{C_T^1}^{k-1}.$$

Remark 1. Equalities (16) and (17) denote the following: at the even k, bifurcation always takes place, while at the odd k, there is a need for the fulfillment of the equality (16).

Remark 2. If x_0'', $\dfrac{\partial a}{\partial \varepsilon}$, $\dfrac{\partial b}{\partial \varepsilon}$ are continuous, then the equalities (16) and (17) simply mean that either

$$-\mu_1 = \operatorname{sgn}\xi_0,$$

or

$$\mu_1 = (-1)^k\operatorname{sgn}\xi_0,$$

where the μ_1 is defined by the expression (13). In this case, the bifurcation solution x_ε exists at all comparatively small values of the parameter ε.

4. CONCLUSIONS

In this article, we solved an important problem on bifurcation of forced oscillations from the equilibrium position for systems described by differential equations unsolved relative to a derivative in the case when the bifurcation parameter is a small delay. The necessary and sufficient conditions of this bifurcation are set out. The method is developed of the investigation of the above systems with the aid of topological methods of nonlinear analysis, which are related to the notion of rotation of condensing vector fields.

APPENDIX

We will present the pattern of the proof of Theorem 1. By replacing the variables $x'(t) = \omega(t)$, $x(0) = \lambda$, $u = (\lambda, \omega)$, the Eq. (3) reduces to the equivalent operator equation

$$u = H(\varepsilon, u), \quad \text{where} \quad H(\varepsilon, u) = G(\varepsilon, u) + C(\varepsilon, u) + D(\varepsilon, u). \tag{18}$$

Here $G : [0,1] \times E \to E$ is the operator strongly continuous in ε and linear in u, while $C : [0,1] \times E \to E$ is the uniform operator of the order of k and $\|D(\varepsilon, u)\| = o(\|u\|^k)$. For the obtained operator equation, the fulfillment of the following conditions is verified:

(1) operator H is the condensing one in the aggregate of variables ε, u;

(2) operator $G(0,\cdot)$ has a simple eigenvalue equal to 1, and 1 is not the eigenvalue of the operator $G(\varepsilon,\cdot)$ for all $\varepsilon \neq 0$;

(3) eigenvalues close to unity of the operator $U(T,\varepsilon)$ of shift by trajectories of the system (12) and similar eigenvalues of the operator $G(\varepsilon,\cdot)$ lie on one side from 1;

(4) $\operatorname{sgn}\xi_0 = \operatorname{sgn}\widetilde{\xi}_0$, where $\widetilde{\xi}_0 = (\widetilde{g}_0, C(0,\widetilde{e}_0))$, $\widetilde{e}_0$ is the eigenvector of the operator $G(0,\cdot)$, which corresponds to the eigenvalue 1, while $\widetilde{g}_0$ is the eigenvalue of the operator $G^*(0,\cdot)$, which corresponds to the eigenvalue 1 and is normalized by the condition $(\widetilde{e}_0, \widetilde{g}_0) = 1$.

Further, we build up cones of the following types:

$$K_\lambda = \{u : \|P(0)u\| \le \lambda \|(I - P(0))u\|, (u, \widetilde{g}_0) > 0\},$$
$$K_{-\lambda} = \{u : \|P(0)u\| \le \lambda \|(I - P(0))u\|, (u, \widetilde{g}_0) < 0\},$$

where $P(\varepsilon)u = u - \dfrac{(u,\widetilde{g}_0)}{(\widetilde{e}_\varepsilon, \widetilde{g}_0)}\widetilde{e}_\varepsilon$. Here $\widetilde{e}_\varepsilon$ is the eigenvector of the operator $G(\varepsilon,\cdot)$, which corresponds to the eigenvalue of $\widetilde{\mu}^\varepsilon \to 1$ at $\varepsilon \to 0$. After this, on cones K_λ and $K_{-\lambda}$, we consider the intersection with the spherical layer $L(\varepsilon) = \{u : c_\varepsilon < \|u\| < r_0\}$, where $c_\varepsilon > 0$. Then, on this intersection, we build up homotopy of the field $I - H(\varepsilon, u)$ to the one-dimensional field $I - (I - P(\varepsilon))H(\varepsilon, u)$. The conditions (16), (17) enable us to establish the difference from zero of rotation of the obtained field on $L(\varepsilon) \bigcap K_\lambda$ in the case of the even k and on $L(\varepsilon) \bigcap K_{-\lambda}$ and $L(\varepsilon) \bigcap K_\lambda$ in the case of the odd k and, hence, to prove the availability of nonzero solutions in Eq. (18).

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