Lyapunov Method and Convergence of the Full-Range Model of CNNs

Mauro Di Marco, Mauro Forti, Massimo Grazzini, Paolo Nistri, and Luca Pancioni

***Abstract*—This paper develops a Lyapunov approach for studying convergence and stability of a class of differential inclusions termed differential variational inequalities (DVIs). The DVIs describe the dynamics of a general system evolving in a compact convex subset of the state space. In particular, they include the dynamics of the full-range (FR) model of cellular neural networks (CNNs), which is characterized by hard-limiter nonlinearities with vertical segments in the $i-v$ characteristic. The approach is based on the following two main tools: 1) a set-valued derivative, which enables to compute the evolution of a Lyapunov function along the solutions of the DVIs without involving integrations, and 2) an extended version of LaSalle's invariance principle, which permits to study the limiting behavior of the solutions with respect to the invariant sets of the DVIs. Then, this paper establishes conditions for convergence (complete stability) of DVIs in the presence of multiple equilibrium points (EPs), global asymptotic stability (GAS), and global exponential stability (GES) of the unique EP. These conditions are applied to investigate convergence, GAS, and GES for FR-CNNs and some extended classes of FR-CNNs. It is shown that, by means of the techniques developed in this paper, the analysis of convergence and stability of FR-CNNs is no more difficult than that of the standard (S)-CNNs. In addition, there are significant cases, such as the symmetric FR-CNNs and the nonsymmetric FR-CNNs with a Lyapunov diagonally stable matrix, where the proof of convergence or global stability is much simpler than that of the S-CNNs.**

***Index Terms*—Cellular neural networks (CNNs), convergence, differential variational inequalities (DVIs), full-range (FR) model, LaSalle's invariance principle, set-valued derivative.**

I. Introduction

SINCE THEIR introduction in 1988 [1], the standard (S) cellular neural networks (CNNs) have been among the most investigated paradigms for real-time neural information processing. The S-CNNs have found important applications, particularly in the fields of image processing, robotics, and pattern recognition [2]–[5]. The CNNs, in their more general form of cellular nonlinear networks, have also been really useful for the modeling and understanding of the spatial–temporal dynamics observable in active media [6], [7].

The full-range (FR) model of CNNs has been introduced in [8] in order to obtain an advantageous VLSI implementation of CNNs with a large number of neurons. A large part of the manufactured CNN chips, such as CACE1k, ACE4k, and ACE16k chips, are based on the FR model of CNNs [8]–[11]. The FR-CNNs exploit ideal hard-limiter devices with two vertical segments in the $i-v$ characteristic, which constrain the state variables to evolve within a closed hypercube of the space. Such an improved range of the state variables has enabled one to obtain smaller power consumption, higher cell densities, and increased processing speed compared to S-CNNs. Due to the nonlinearities with vertical segments in the $i-v$ characteristic, there are values of the state variables where the velocity vector of FR-CNNs is not single valued, but instead, it belongs to a whole set of feasible velocities. From a mathematical viewpoint, the dynamics of FR-CNNs is thus described by a differential inclusion [12], which is defined by a set-valued vector field representing for each state the corresponding set of feasible velocities [8], [13].

In a recent paper [14], it has been shown by Corinto and Gilli that the S-CNN model and the FR-CNN model might display a qualitatively different dynamical behavior for the same set of parameters (interconnections and inputs). In particular, a class of completely stable (convergent) second-order S-CNNs has been considered in [14], such that the corresponding FR-CNN displays a stable limit cycle for the same set of parameters, and then, the FR-CNN is not completely stable. One main consequence is that, in the general case, stability of FR-CNNs cannot be deduced from existing results on stability for S-CNNs. Hence, it is needed to develop suitable tools, which are based on the theory of differential inclusions, for studying in a rigorous way stability and convergence of FR-CNNs.

In this paper, we consider a class of differential inclusions termed differential variational inequalities (DVIs), which describe the dynamics of a system evolving in a compact convex subset of the state space. This class includes as a special case the FR-CNNs, when the set is a closed hypercube and the vector field within the hypercube is an affine vector field. The main goal of this paper is to develop a Lyapunov approach for addressing stability and convergence of the considered DVIs. As in the standard Lyapunov approach for differential equations, it is shown that it is possible to evaluate the time evolution of an energy function along the solutions of the DVIs, without needing integrations. This is obtained via a suitable notion of set-valued derivative, which permits one to evaluate, directly from the vector field, the time derivative of the energy along each solution of the DVI. Furthermore, an extended version of LaSalle's invariance principle is proved, which enables one to study the limiting behavior of the solutions with respect to the invariant sets of the DVIs. This paper then establishes conditions ensuring convergence (complete stability) of DVIs in the presence of multiple equilibrium points (EPs), global asymptotic stability (GAS), and global exponential stability (GES) of the unique EP. The results are applied to investigate complete

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stability, GAS, and GES for some classes of symmetric and nonsymmetric FR-CNNs and some extended classes of FR-CNNs.

The notion of set-valued derivative in this paper has been inspired by an analogous notion of derivative that has been introduced by Shevitz and Paden [15] and subsequently improved by Bacciotti and Ceragioli [16]. The derivative in those papers holds for *general systems of differential inclusions*, and as such, it has a broader applicability with respect to the derivative in this paper, which holds for the subclass of differential inclusions given by the DVIs. However, it is proved that, by exploiting the *peculiar geometric structure of the solutions of DVIs*, we are able to obtain advantages for the class of DVIs with respect to the derivative in [15] and [16]. Indeed, we will see that there are important cases, such as the symmetric FR-CNNs, for which the derivative introduced here can be successfully applied to prove convergence via a Lyapunov function, while the application of the derivative in [15] and [16] fails to prove convergence.

The structure of this paper is outlined as follows. Section II collects the needed mathematical preliminaries, while Section III presents the DVI model studied in this paper. Then, in Sections IV and V, we develop an extended Lyapunov method for DVIs. Sections VI and VII give the main results on convergence, GAS, and GES of DVIs and FR-CNNs, respectively. After some specific examples in Section VIII, Section IX draws the main conclusions of this paper.

Notation

Let $\mathbb{R}^n$ be the real n-space. Given matrix $A \in \mathbb{R}^{n\times n}$, by A', we mean the transpose of A, while $[A]_S = (1/2)(A' + A)$ is the symmetric part of A. In particular, by E_n, we denote the $n \times n$ identity matrix. Given the column vectors $x, y \in \mathbb{R}^n$, we denote by $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ the scalar product of x and y, while $\|x\| = \sqrt{\langle x, x\rangle}$ is the Euclidean norm of x. Given a set $D \subset \mathbb{R}^n$, by $\mathrm{cl}(D)$ and $\mathrm{bd}(D)$, we denote the closure of D and the boundary of D, respectively, while $\mathrm{dist}(x, D) = \inf_{y\in D} \|x-y\|$ is the distance of vector $x \in \mathbb{R}^n$ from D. Moreover, $\overline{\mathrm{co}}(D)$ is the closure of the convex hull of D. By $B(z, r) = \{y \in \mathbb{R}^n : \|y-z\| < r\}$, we mean an n-dimensional open ball with center $z \in \mathbb{R}^n$ and radius r.

II. Preliminaries

In this section, we recall some definitions and properties that are needed in this paper. The reader is referred to [12], [17], and [18] for a more thorough treatment.

A. Set-Valued Maps and Generalized Gradient

By $F : D \multimap \mathbb{R}^n$, $D \subset \mathbb{R}^n$, we denote a set-valued map, i.e., a map that associates to any $x \in D$ a nonempty subset $F(x) \subset \mathbb{R}^n$. We say that F is upper semicontinuous at $x \in D$ if and only if, for any neighborhood N containing $F(x)$, there exists a neighborhood M of x such that $F(M \cap D) \subset N$. Moreover, F is said to be upper semicontinuous if and only if it is upper semicontinuous at every $x \in D$. A function $\phi : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$ open, is said to be locally Lipschitz if, for each $x \in U$, there exist $\rho_x > 0$ and $\ell_x > 0$ such that $|\phi(y) - \phi(z)| \le \ell_x\|y-z\|, \forall y, z \in B(x, \rho_x) \subset U$. If ϕ is locally Lipschitz, then ϕ is differentiable for almost all (a.a.) $x \in U$ (in the sense of Lebesgue measure). Moreover, for any $x \in U$, the generalized gradient of ϕ at x can be defined as follows [18]:

$$\partial\phi(x) = \overline{\mathrm{co}}\left\{\lim_{n\to\infty} \nabla\phi(x_n) : x_n \to x, x_n \notin N, x_n \notin \Omega\right\}$$

where $\Omega \subset \mathbb{R}^n$ is the set of points where ϕ is not differentiable, $N \subset \mathbb{R}^n$ is an arbitrary set of zero measure, and $\nabla\phi(x_n)$ is the gradient of ϕ evaluated at x_n. It can be shown that $\partial\phi : U \multimap \mathbb{R}^n$ is a set-valued map with nonempty compact convex values $\partial\phi(x) \subset \mathbb{R}^n$ [19, Prop. 2.1.2, p. 27]. Moreover, the restriction of $\partial\phi$ to any compact set $Q \subset U$ is an upper semicontinuous set-valued map [19, Prop. 2.1.5, p. 29].

B. Nonpathological Functions

Definition 1 ([20], [21]): Let J be a compact interval of $\mathbb{R}$. A function $\phi : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$ open, is said to be *nonpathological* if it is locally Lipschitz and, for every absolutely continuous function $x : J \to \mathbb{R}^n$ and for a.a. $t \in J$, the set $\partial\phi(x(t))$ is a subset of an affine subspace orthogonal to $\dot{x}(t)$. ∎

Nonpathological functions are a wide class, including Clarke's regular functions [19] and convex, semiconvex, concave, and semiconcave functions [21]. Let U be an open subset of $\mathbb{R}^n$. We recall that a function $\phi : U \to \mathbb{R}$ is said to be *semiconvex* if, for any open bounded convex set $V \subset U$, there exists $\zeta > 0$ such that the function $x \to \phi(x) + \zeta\|x\|^2$ is convex in V. Moreover, $\phi : U \to \mathbb{R}$ is said to be *semiconcave* if $-\phi$ is semiconvex.

The following fundamental property permits one to evaluate the derivative of the composition of a nonpathological function and an absolutely continuous function [21, Prop. 1], [20].

1) Property 1 (Chain Rule): If $\phi : \mathbb{R}^n \to \mathbb{R}$ is nonpathological and $x : J \to \mathbb{R}^n$ is absolutely continuous, then, for a.a. $t \in J$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(x(t)) = \langle \dot{x}(t), p\rangle \qquad \forall p \in \partial\phi(x(t)).$$

∎

C. Tangent and Normal Cones

Let $Q \subset \mathbb{R}^n$ be a nonempty closed convex set. The tangent cone to Q at $x \in Q$ is defined as

$$T_Q(x) = \left\{v \in \mathbb{R}^n : \liminf_{\rho\to 0^+} \frac{\mathrm{dist}(x + \rho v, Q)}{\rho} = 0\right\}$$

while the normal cone to Q at $x \in Q$ is given by

$$N_Q(x) = \{p \in \mathbb{R}^n : \langle p, v\rangle \le 0, \forall v \in T_Q(x)\}.$$

Both $T_Q(x)$ and $N_Q(x)$ are nonempty closed convex cones in $\mathbb{R}^n$ (possibly reduced to $\{0\}$). Furthermore, N_Q is a *monotone* operator, i.e., for any $x, y \in Q$ and any $n_x \in N_Q(x)$, $n_y \in N_Q(y)$, we have $\langle x-y, n_x-n_y\rangle \ge 0$ [12, Prop. 1, p. 159]. The orthogonal set to $N_Q(x)$ is defined as

$$N_Q^{\perp}(x) = \{v \in \mathbb{R}^n : \langle p, v\rangle = 0,\ \forall p \in N_Q(x)\}.$$

It is known that $N_Q^{\perp}(x)$ is a vector subspace of $\mathbb{R}^n$ such that $N_Q^{\perp}(x) \subseteq T_Q(x)$. The aforementioned cones, evaluated at some points of a closed convex set $Q \subset \mathbb{R}^2$, are reported in Fig. 1.

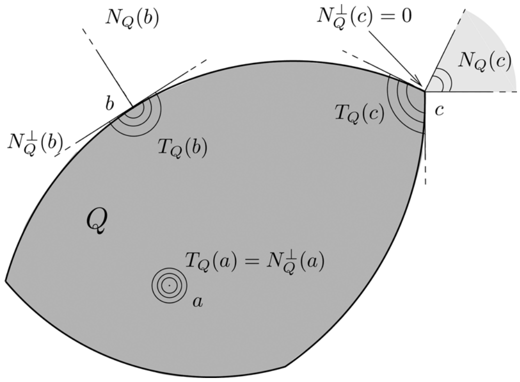

Fig. 1. Closed and convex set $Q \subset \mathbb{R}^2$ with cones T_Q, N_Q, and $N_Q^\perp$ at some points of Q (the cones are translated into the corresponding points). Note that $N_Q(a) = 0$, $N_Q^\perp(a) = T_Q(a) = \mathbb{R}^2$, while $N_Q^\perp(c) = 0$.

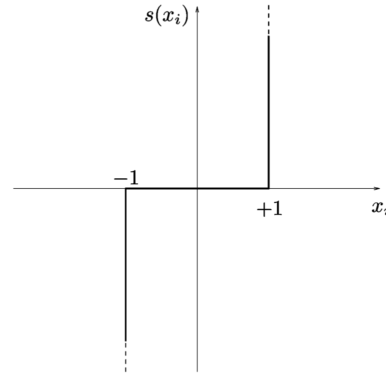

Fig. 2. Hard-limiter nonlinearity used in the FR-CNN model.

Finally, for any $x \in \mathbb{R}^n$, there exists a unique point $\mathcal{P}_Q(x) \in Q$ satisfying

$$\|x - \mathcal{P}_Q(x)\| = \operatorname{dist}(x, Q) = \min_{y \in Q} \|y - x\|.$$

The operator $\mathcal{P}_Q$ is called the projector of best approximation on Q.

III. A Class of DVIs

Let $Q \subset \mathbb{R}^n$ be a nonempty compact convex set. Furthermore, let $f : Q \multimap \mathbb{R}^n$ be an upper semicontinuous map with nonempty compact convex values. As in [12, Ch. 5], we consider the class of DVIs

$$\dot{x} \in f(x) - N_Q(x) \tag{1}$$

where $x \in Q$ and $N_Q(x)$ denotes the normal cone to Q at x.

Our interest for the DVI (1) derives mainly from the fact that it includes as a special case the differential inclusion that models the dynamics of FR-CNNs. In fact, an FR-CNN can be described by [8], [13]

$$\dot{x} \in Ax + I - S(x) \tag{2}$$

where $x \in K = [-1, 1]^n$, $A \in \mathbb{R}^{n \times n}$ is the interconnection matrix, $I \in \mathbb{R}^n$ is the constant input, and $S(x) = (s(x_1), s(x_2), \ldots, s(x_n))' : K \multimap \mathbb{R}^n$, with

$$s(\rho) = \begin{cases} (-\infty, 0], & \rho = -1 \\ 0, & \rho \in (-1, 1) \\ [0, +\infty), & \rho = 1 \end{cases} \tag{3}$$

being the ideal hard-limiter nonlinearity with two vertical segments shown in Fig. 2. It has been observed in [13] that we have $S(x) = N_K(x)$ for any $x \in K$. Then, the FR-CNN (2) can be equivalently described by the DVI

$$\dot{x} \in Ax + I - N_K(x)$$

which is included in (1).

Another interesting subclass of (1) is given by gradient (G) DVIs of the type

$$\dot{x} \in -\partial\phi(x) - N_Q(x) \tag{4}$$

where $\phi : U \to \mathbb{R}$ is a nonpathological function and U is an open set containing Q (see Section II). Note that if $A = A' \in \mathbb{R}^{n \times n}$, $I \in \mathbb{R}^n$, and $\phi(x) = -(1/2)x'Ax - x'I : \mathbb{R}^n \to \mathbb{R}$, then (4) reduces to an FR-CNN with a symmetric interconnection matrix A.

In a recent work [22], the concept of DVIs in [12] has been extended in order to provide a general mathematical framework for modeling and analyzing problems that simultaneously involve dynamics, inequalities, and discontinuities, as it happens in hybrid systems or variable structure systems.

We say that $x : J \to \mathbb{R}^n$ is a solution of (1) in the compact interval $J \subset \mathbb{R}$ if x is absolutely continuous on J, $x(t) \in Q$ for any $t \in J$ and $\dot{x}(t) \in f(x(t)) - N_Q(x(t))$ for a.a. $t \in J$ [12]. Furthermore, we say that $x(t)$, $t \geq 0$, is a solution of (1) for $t \geq 0$ if, for any compact interval $J \subset [0, +\infty)$, x is a solution of (1) in J. By an EP $\xi \in Q$ of (1), we mean a constant solution $x(t) = \xi$, $t \geq 0$, of (1). We remark that $\xi \in Q$ is an EP of (1) if and only if ξ satisfies the algebraic inclusion $0 \in f(\xi) - N_Q(\xi)$.

Property 2: Suppose that $f : Q \multimap \mathbb{R}^n$ is an upper semicontinuous map with nonempty compact convex values. Then, the following hold: 1) For any $x_0 \in Q$, there exists at least a solution $x : [0, +\infty) \to Q$ of (1) with initial condition $x(0) = x_0$, and 2) the set E of the EPs of (1) is a nonempty compact subset of Q. ■

Proof: The result in point 1) derives from [12, Th. 1, p. 267]. The same theorem also ensures that $E \neq \emptyset$. Since $E \subset Q$, to prove point 2), it suffices to show that E is closed. To this end, note that $\sup \|f(Q)\| < +\infty$ [12, Prop. 3, p. 42], and consider the differential inclusion

$$\dot{x} \in F_r(x) = f(x) - N_Q(x) \cap \operatorname{cl}(B(0, r))$$

where $+\infty > r > \sup \|f(Q)\|$. It can be easily verified that $F_r : Q \multimap \mathbb{R}^n$ is an upper semicontinuous map with nonempty compact convex values. Let $\{\xi_k\}$ be a sequence of points in E that converges to z as $k \to \infty$. We want to prove that $z \in E$. Clearly, $z \in Q$ and $0 \in F_r(\xi_k)$ for any $k = 1, 2, \ldots$. Suppose, for contradiction, that $0 \notin F_r(z)$. Since $F_r(z)$ is compact, there exists $\epsilon > 0$ such that $0 \notin F_r(z) + B(0, \epsilon)$. Furthermore, since F_r is upper semicontinuous at z, there exists $\delta > 0$ such that $F_r(B(z, \delta) \cap Q) \subseteq F_r(z) + B(0, \epsilon)$. However, for sufficiently large k, we have $\xi_k \in B(z, \delta)$ and $0 \in F_r(\xi_k)$, which is a contradiction. ■

In the general case, there exist multiple solutions of (1) starting at a given initial condition. However, as shown in the next result, there are significant cases where the uniqueness of the solution is guaranteed.

Property 3: Let $U \subset \mathbb{R}^n$ be an open set containing Q. Suppose that $f : U \to \mathbb{R}^n$ is a single-valued map, which is locally Lipschitz in U, or that $f = -\partial\phi$, where $\phi : U \to \mathbb{R}$ is semiconvex. Then, for any $x_0 \in Q$, there exists a unique solution $x : [0, +\infty) \to Q$ of (1) with initial condition $x(0) = x_0$. ■

Proof: Let $x(t)$, $t \geq 0$, be a solution of (1) with initial condition $x(0) = x_0$ (Property 2). Suppose that there exists a second solution $z(t)$, $t \geq 0$, of (1) with initial condition $z(0) = x_0$, and let $\Delta(t) = \|x(t) - z(t)\|^2/2$. We have $\Delta(0) = 0$ and for a.a. $t \geq 0$

$$\begin{aligned}\dot{\Delta}(t) &= \langle x(t) - z(t), \dot{x}(t) - \dot{z}(t)\rangle \\ &= \langle x(t) - z(t), v_x(t) - \gamma_x(t) - v_z(t) + \gamma_z(t)\rangle \\ &= \langle x(t) - z(t), v_x(t) - v_z(t)\rangle \\ &\quad - \langle x(t) - z(t), \gamma_x(t) - \gamma_z(t)\rangle\end{aligned}$$

where $\gamma_x(t) \in N_Q(x(t))$, $\gamma_z(t) \in N_Q(z(t))$, $v_x(t) \in f(x(t))$, and $v_z(t) \in f(z(t))$. Since N_Q is a monotone operator (Section II), we have $\langle x(t) - z(t), \gamma_x(t) - \gamma_z(t)\rangle \geq 0$, hence

$$\dot{\Delta}(t) \leq \langle x(t) - z(t), v_x(t) - v_z(t)\rangle .$$

Suppose that $f : U \to \mathbb{R}^n$ is locally Lipschitz. Then, f is Lipschitz on Q, i.e., there exists $L > 0$ such that $\|v_x(t) - v_z(t)\| = \|f(x(t)) - f(z(t))\| \leq L\,\|x(t) - z(t)\|$. It follows that for a.a. $t \geq 0$, we have

$$\begin{aligned}\dot{\Delta}(t) &\leq \langle x(t) - z(t), f(x(t)) - f(z(t))\rangle \\ &\leq \|x(t) - z(t)\|\,\|f(x(t)) - f(z(t))\| \\ &\leq L\,\|x(t) - z(t)\|^2 = 2L\Delta(t).\end{aligned}$$

Suppose now that $f = -\partial\phi$, where $\phi : U \to \mathbb{R}$ is a semiconvex function in U. For sufficiently small $\epsilon > 0$, consider the open bounded convex set $Q + B(0, \epsilon) \subset U$. Since ϕ is semiconvex, there exists $\zeta > 0$ such that $\phi_\zeta(x) = \phi(x) + \zeta\|x\|^2$ is convex in V. Furthermore, by [23, Th. 10.4, p. 86], ϕ_ζ is Lipschitz on the compact set $Q + \mathrm{cl}(B(0, \epsilon/2)) \subset Q + B(0, \epsilon)$. Then, ϕ_ζ is regular on $Q + B(0, \epsilon/2)$[19, Prop. 2.3.6, p. 40], and $\partial\phi_\zeta$ is a monotone map on $Q + B(0, \epsilon/2)$[19, Prop. 2.2.9 p. 37]. Finally, since ϕ is the sum of two regular functions (ϕ_ζ, which is convex, and $-\zeta\|\cdot\|^2$, which is continuously differentiable), then it is regular [19, Prop. 2.2.9, p. 37]. In particular, we have $v_x(t) = -w_x(t) + 2\zeta x(t)$ for some $w_x(t) \in \partial\phi_\zeta(x(t))$ and $v_z(t) = -w_z(t) + 2\zeta z(t)$ for some $w_z(t) \in \partial\phi_\zeta(z(t))$. By letting $L = 2\zeta$, we obtain for a.a. $t \geq 0$

$$\begin{aligned}\dot{\Delta}(t) &\leq \langle x(t) - z(t), v_x(t) - v_z(t)\rangle \\ &= \langle x(t) - z(t), -w_x(t) + 2\zeta x(t) + w_z(t) - 2\zeta z(t)\rangle \\ &= 2\zeta\,\|x(t) - z(t)\|^2 - \langle x(t) - z(t), w_x(t) - w_z(t)\rangle \\ &\leq 2L\Delta(t)\end{aligned}$$

where we have taken into account that since $\partial\phi_\zeta$ is a monotone operator, we have $\langle x(t) - z(t), w_x(t) - w_z(t)\rangle \geq 0$.

In both cases, we can apply the Gronwall lemma to the nonnegative function Δ and obtain that for $t \geq 0$, we have $0 \leq \Delta(t) \leq \Delta(0)\exp\{2Lt\} = 0$. Hence, $z(t) = x(t)$ for $t \geq 0$. ■

Definition 2: Let $x(t), t \geq 0$, be a solution of (1). The ω-limit set of x, which we denote by ω_x, is the set of points $y \in \mathbb{R}^n$ such that there exists a sequence $\{t_k\}$, with $t_k \to +\infty$ as $k \to +\infty$, such that $\lim_{k\to+\infty} x(t_k) = y$. ■

Property 4: Suppose that $f : Q \multimap \mathbb{R}^n$ is an upper semicontinuous map with nonempty compact convex values and that $x(t)$, $t \geq 0$, is a solution of (1). Then, ω_x is a nonempty compact connected subset of Q, and $\lim_{t\to+\infty} \mathrm{dist}(x(t), \omega_x) = 0$. Furthermore, ω_x is weakly invariant, i.e., for any $y \in \omega_x$, there exists at least a solution $z : [0, +\infty) \to Q$ of (1) such that $z(0) = y$ and $z(t) \subset \omega_x$ for any $t \in [0, +\infty)$. ■

Proof: As in the proof of Property 2, consider

$$\dot{x} \in F_r(x) = f(x) - N_Q(x) \cap \mathrm{cl}\,(B(0, r)) \tag{5}$$

where $+\infty > r > \sup\|f(Q)\|$ and $F_r : Q \multimap \mathbb{R}^n$ is an upper semicontinuous map with nonempty compact convex values. By an argument, as in the proof of [13, Prop. 5], it can be shown that if $x : [0, +\infty) \to Q$ is a solution of (1), then x is also a solution of (5) for $t \geq 0$. Since x is bounded for $t \geq 0$, the result follows from the results in [24, pp. 129–130]. ■

Consider a solution $x(t)$, $t \geq 0$, of (1). By definition, x evolves within the set Q for all $t \geq 0$, i.e., it is a *viable* solution to (1). This fact enables to prove the next result, giving some geometrical properties of the time derivative $\dot{x}$, which will be useful in the definition of the set-valued derivative of a candidate Lyapunov function for (1) (see Section IV).

Property 5: Suppose that $f : Q \multimap \mathbb{R}^n$ is an upper semicontinuous map with nonempty compact convex values and that $x(t), t \geq 0$, is a solution to (1). Then, x is differentiable for a.a. $t \geq 0$, and we have

$$\dot{x}(t) \in \mathcal{P}_{T_Q(x(t))} f(x(t)) \cap N_Q^{\perp}(x(t)) .$$

■

Proof: Since x is absolutely continuous on any compact interval in $[0, +\infty)$, then x is differentiable for a.a. $t \in [0, +\infty)$. Let $t > 0$ be such that x is differentiable at t. From [12, Prop. 2, p. 266], we have that

$$\dot{x}(t) \in \mathcal{P}_{T_Q(x(t))} f(x(t)) .$$

To prove that $\dot{x}(t) \in N_Q^{\perp}(x(t))$, let $h > 0$, and note that since $x(t)$ and $x(t+h)$ belong to Q, we have $\mathrm{dist}(x(t) + h\dot{x}(t), Q) \leq \|x(t) + h\dot{x}(t) - x(t+h)\|$. Dividing by h, and accounting for the differentiability of x at time t, we obtain

$$\lim_{h\to 0^+} \frac{\mathrm{dist}(x(t) + h\dot{x}(t), Q)}{h} = 0$$

hence we have $\dot{x}(t) \in T_Q(x(t))$.

Now, suppose that $h \in (-t, 0)$. Since, once more, $x(t)$ and $x(t+h)$ belong to Q, we have $0 \leq \mathrm{dist}(x(t) + (-h)(-\dot{x}(t)), Q)/(-h) \leq \|(x(t) + h\dot{x}(t) - x(t+h))\|/(-h)$. Let $\rho = -h$. Then

$$\lim_{\rho\to 0^+} \frac{\mathrm{dist}(x(t) + \rho(-\dot{x}(t)), Q)}{\rho} = 0$$

hence, by definition, $-\dot{x}(t) \in T_Q(x(t))$.

To complete the proof, it suffices to observe that $T_Q(x) \cap (-T_Q(x)) = N_Q^{\perp}(x)$ for any $x \in Q$. In fact, if $v \in T_Q(x) \cap (-T_Q(x))$ and $p \in N_Q(x)$, then $\langle v, p\rangle \le 0$ and $\langle -v, p\rangle \le 0$, respectively. This means that $\langle v, p\rangle = 0$, i.e., $v \in N_Q^{\perp}(x)$. Conversely, if $v \in N_Q^{\perp}(x)$ and $p \in N_Q(x)$, then we have $\langle v, p\rangle = 0$ and $\langle -v, p\rangle = 0$, respectively. Hence, $v \in T_Q(x) \cap (-T_Q(x))$. ∎

IV. Set-Valued Derivative and Lyapunov Functions

Consider a nonpathological function $\phi : U \to \mathbb{R}$, where $U \subset \mathbb{R}^n$ is an open set containing Q. Our goal is to evaluate the time evolution of ϕ along the solutions of (1) without involving the integration of (1). To this end, the following notion of the set-valued derivative of ϕ will play a crucial role.

Definition 3: Let $\phi : U \to \mathbb{R}$ be a nonpathological function. The (set-valued) derivative of ϕ with respect to (1) at a point $x \in Q$ is given by

$$\mathcal{D}\phi(x) = \left\{ a \in \mathbb{R} : \exists v \in \mathcal{P}_{T_Q(x)} f(x) \cap N_Q^{\perp}(x) : a = \langle v, p\rangle, \forall p \in \partial\phi(x) \right\}. \quad (6)$$

∎

The next basic properties of $\mathcal{D}\phi$ can be proved.

Property 6: Let $\phi : U \to \mathbb{R}$ be a nonpathological function. Then, the following hold.

1) For any $x \in Q$, $\mathcal{D}\phi(x)$ is a (possibly empty) compact interval of $\mathbb{R}$.
2) If $\xi \in Q$ is an EP of (1), then $0 \in \mathcal{D}\phi(\xi)$.
3) If $\phi \in C^1(U)$, then, for any $x \in Q$, we have

$$\mathcal{D}\phi(x) = \emptyset$$

if $\mathcal{P}_{T_Q(x)} f(x) \cap N_Q^{\perp}(x) = \emptyset$, whereas

$$\mathcal{D}\phi(x) = \left\{ \langle v, \nabla\phi(x)\rangle, v \in \mathcal{P}_{T_Q(x)} f(x) \cap N_Q^{\perp}(x) \right\}$$

if $\mathcal{P}_{T_Q(x)} f(x) \cap N_Q^{\perp}(x) \neq \emptyset$.
4) If $\phi \in C^1(U)$ and $f : U \to \mathbb{R}^n$ is single valued, then, for any $x \in Q$, the derivative $\mathcal{D}\phi(x)$ is either the empty set or a singleton and is given by

$$\mathcal{D}\phi(x) = \emptyset$$

if $\mathcal{P}_{T_Q(x)} f(x) \notin N_Q^{\perp}(x)$, whereas

$$\mathcal{D}\phi(x) = \left\langle \mathcal{P}_{T_Q(x)} f(x), \nabla\phi(x)\right\rangle$$

if $\mathcal{P}_{T_Q(x)} f(x) \in N_Q^{\perp}(x)$.

∎

Proof:

1) The proof is given in Appendix I. The same Appendix I also provides a geometric interpretation of some basic aspects concerning the definition of $\mathcal{D}\phi$.
2) If ξ is an EP of (1), then there exist $v_\xi \in f(\xi)$ and $\gamma_\xi \in N_Q(\xi)$ such that $v_\xi - \gamma_\xi = 0$. We obtain $\mathcal{P}_{T_Q(\xi)}(v_\xi) = \mathcal{P}_{T_Q(\xi)}(\gamma_\xi) = 0 \in N_Q^{\perp}(\xi)$. Moreover, $\left\langle \mathcal{P}_{T_Q(\xi)}(v_\xi), p\right\rangle = 0$ for any $p \in \partial\phi(\xi)$, and so, $0 \in \mathcal{D}\phi(\xi)$.

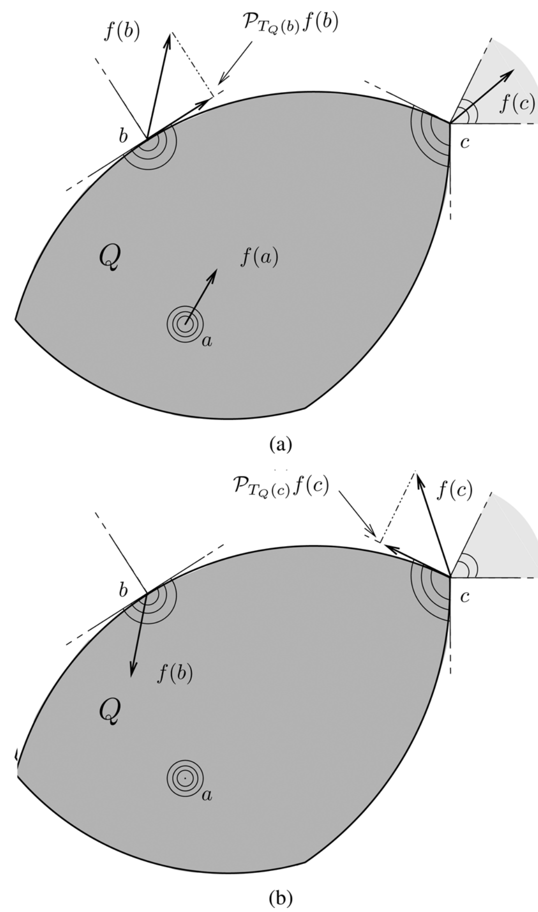

Fig. 3. Vector fields involved in the computation of the derivative $\mathcal{D}\phi$ for $f : U \to \mathbb{R}^2$ that is single-valued and $\phi \in C^1(U)$. (a) Three cases (points a, b, and c) where $\mathcal{D}\phi$ is a singleton. (b) Two cases (points b and c) where $\mathcal{D}\phi$ is empty.

3) and 4) The results directly follow from (6) and the fact that ϕ is continuously differentiable at x, hence we have $\partial\phi(x) = \{\nabla\phi(x)\}$.

∎

Fig. 3 shows the vector fields involved in the computation of the derivative $\mathcal{D}\phi$ in a simple 2-D case where $f : U \to \mathbb{R}^2$ is single valued and $\phi \in C^1(U)$ (cf. point 4) in Property 6). Fig. 3(a) considers three points a, b, $c \in Q$, as in Fig. 1. For point a, we have $f(a) \in T_Q(a)$, hence $\mathcal{D}\phi(a) = \langle \mathcal{P}_{T_Q(a)} f(a), \nabla\phi(a)\rangle$ is single valued. Similarly, for point b, we have $\mathcal{P}_{T_Q(b)} f(b) \in N_Q^{\perp}(b)$ and then $\mathcal{D}\phi(b) = \langle \mathcal{P}_{T_Q(b)} f(b), \nabla\phi(b)\rangle$. For c, we have $\mathcal{P}_{T_Q(c)} f(c) = 0 \in N_Q^{\perp}(c)$ and $\mathcal{D}\phi(c) = \langle \mathcal{P}_{T_Q(c)} f(c), \nabla\phi(c)\rangle = 0$ (note that c is an EP of the DVI). Fig. 3(b) considers instead two situations where $\mathcal{D}\phi$ is empty. In particular, at point b, we have $f(b) \in T_Q(b)$, but $f(b) \notin N_Q^{\perp}(b)$, hence $\mathcal{D}\phi(b) = \emptyset$. Similarly, for point c, we have $\mathcal{P}_{T_Q(c)} f(c) \notin N_Q^{\perp}(c)$, hence $\mathcal{D}\phi(c) = \emptyset$.

Property 7: Let $\phi : U \to \mathbb{R}$ be a nonpathological function. The set-valued derivative of ϕ with respect to the G-DVI (4) has the following properties.

1) We have $0 \in \mathcal{D}\phi(\xi)$ if and only if $\xi \in Q$ is an EP of (4).
2) If $\phi \in C^1(U)$, then, for any $x \in Q$, we have

$$\mathcal{D}\phi(x) = \emptyset$$

if $\mathcal{P}_{T_Q(x)}(-\nabla\phi(x)) \notin N_Q^{\perp}(x)$, whereas

$$\mathcal{D}\phi(x) = -\left\|\mathcal{P}_{T_Q(x)}\left(-\nabla\phi(x)\right)\right\|^2 \leq 0$$

if $\mathcal{P}_{T_Q(x)}\left(-\nabla\phi(x)\right) \in N_Q^{\perp}(x)$. ■

Proof:

1) If ξ is an EP of (1), then we have $0 \in \mathcal{D}\phi(\xi)$ from point 2) in Property 6.
Suppose now that $0 \in \mathcal{D}\phi(y)$ for some $y \in Q$. Then, there exists $u \in f(y) = -\partial\phi(y)$, such that $\langle \mathcal{P}_{T_Q(y)}(u), p\rangle = 0$ for any $p \in \partial\phi(y)$. Since $\mathcal{P}_{T_Q(y)}(u) \in N_Q^{\perp}(y)$, we have that $u = \mathcal{P}_{T_Q(y)}(u) + \mathcal{P}_{N_Q(y)}(u)$, with $\langle \mathcal{P}_{T_Q(y)}(u), \mathcal{P}_{N_Q(y)}(u)\rangle = 0$, hence

$$\begin{aligned} 0 &= \left\langle \mathcal{P}_{T_Q(y)}(u), -u\right\rangle \\ &= -\left\langle \mathcal{P}_{T_Q(y)}(u), \mathcal{P}_{T_Q(y)}(u) + \mathcal{P}_{N_Q(y)}(u)\right\rangle \\ &= -\left\|\mathcal{P}_{T_Q(y)}(u)\right\|^2 \end{aligned}$$

and so $\mathcal{P}_{T_Q(y)}(u) = 0$. Noting that $0 = \mathcal{P}_{T_Q(y)}(u) = u - \mathcal{P}_{N_Q(y)}(u)$, where $u \in f(y)$ and $\mathcal{P}_{N_Q(y)}(u) \in N_Q(y)$, it follows that y is an EP of (4).

2) The result follows from point 4) in Property 6, considering that $f(x) = -\nabla\phi(x)$ and noting that $-\nabla\phi(x) = \mathcal{P}_{T_Q(x)}(-\nabla\phi(x)) + \mathcal{P}_{N_Q(x)}(-\nabla\phi(x))$, where $\langle \mathcal{P}_{T_Q(x)}(-\nabla\phi(x)), \mathcal{P}_{N_Q(x)}(-\nabla\phi(x))\rangle = 0$. ■

The result that follows gives a fundamental link between the time derivative of ϕ along the solutions of (1) and the set-valued derivative $\mathcal{D}\phi$.

Lemma 1: Let $\phi : U \to \mathbb{R}$ be a nonpathological function, and let $x(t)$, $t \geq 0$, be a solution of (1). Then, $\phi(x(\cdot))$ is differentiable for a.a. $t \geq 0$, and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi\left(x(t)\right) \in \mathcal{D}\phi\left(x(t)\right) \neq \emptyset$$

for a.a. $t \geq 0$. ■

Proof: The function $\phi(x(\cdot))$ is absolutely continuous on any compact interval in $[0, +\infty)$, since it is the composition of a locally Lipschitz function and an absolutely continuous function. Then, for a.a. $t \geq 0$, we have that $x(\cdot)$ and $\phi(x(\cdot))$ are differentiable at t. Furthermore, by the chain rule (Property 1), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi\left(x(t)\right) = \langle \dot{x}(t), p\rangle \qquad \forall p \in \partial\phi\left(x(t)\right)$$

and from Property 5

$$\dot{x}(t) \in \mathcal{P}_{T_Q(x(t))} f\left(x(t)\right) \cap N_Q^{\perp}\left(x(t)\right).$$

Then, from the definition (6) of set-valued derivative, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi\left(x(t)\right) \in \mathcal{D}\phi\left(x(t)\right)$$

for a.a. $t \geq 0$. ■

Definition 4: Let $\phi : U \to \mathbb{R}$ be a nonpathological function. We say that ϕ is a Lyapunov function for (1) if, for any $x \in Q$, we have $\max \mathcal{D}\phi(x) \leq 0$ or $\mathcal{D}\phi(x) = \emptyset$. If, in addition, we have $\max \mathcal{D}\phi(x) < 0$ or $\mathcal{D}\phi(x) = \emptyset$ when x is not an EP of (1), then ϕ is said to be a strict Lyapunov function for (1). ■

The next theorem is the main result in this section.

Theorem 1: Suppose that $\phi : U \to \mathbb{R}$ is a nonpathological Lyapunov function for (1). If $x(t)$, $t \geq 0$, is a solution of (1), then $\phi(x(\cdot))$ is a nonincreasing function for $t \geq 0$, and there exists the $\lim_{t\to+\infty} \phi(x(t)) = \phi(\infty) > -\infty$. ■

Proof: By Lemma 1, we have that for a.a. $t \geq 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi\left(x(t)\right) \in \mathcal{D}\phi\left(x(t)\right) \leq \max \mathcal{D}\phi\left(x(t)\right) \leq 0.$$

Since the function $\phi(x(\cdot))$ is absolutely continuous on any compact interval in $[0, +\infty)$, it follows that it is nondecreasing for $t \geq 0$. Moreover, ϕ is bounded from below in the compact set Q, hence there exists the $\lim_{t\to+\infty} \phi(x(t)) = \phi(\infty) > -\infty$. ■

V. LaSalle's Invariance Principle for DVIs

In the standard framework of differential equations, LaSalle's invariance principle is one of the most powerful tools for studying convergence toward equilibria of dynamical systems that possess a Lyapunov function. In this section, we provide an extended version of this principle, which is applicable to the class of DVIs (1). The extension is based on the notion of the set-valued derivative of a nonpathological function developed in Section IV.

Theorem 2: Suppose that $f : Q \multimap \mathbb{R}^n$ is an upper semicontinuous map with nonempty compact convex values. Let $U \subset \mathbb{R}^n$ be an open set containing Q, and let $\phi : U \to \mathbb{R}$ be a nonpathological Lyapunov function for (1). Let $Z = \{x \in Q : 0 \in \mathcal{D}\phi(x)\}$, and let M be the largest weakly invariant subset of (1) contained in $\mathrm{cl}(Z)$. If $x(t)$, $t \geq 0$, is a solution of (1), then we have

$$\lim_{t\to+\infty} \mathrm{dist}(x(t), M) = 0.$$

■

Proof: It suffices to show that $\omega_x \subseteq M$ (Property 4). It is known from Theorem 1 that $\phi(x)$ is a nonincreasing function on $[0, +\infty)$ and $\phi(x(t)) \to \phi(\infty) > -\infty$ as $t \to +\infty$. For any $z \in \omega_x$, there exists a sequence $\{t_k\}$, with $t_k \to +\infty$ as $k \to +\infty$, such that $x(t_k) \to z$ as $k \to +\infty$. From the continuity of ϕ, we have $\phi(z) = \lim_{t_k\to+\infty} \phi(x(t_k)) = \phi(\infty)$, hence ϕ is constant on ω_x.

Let $z_0 \in \omega_x$, and let $z(t)$, $t \geq 0$, be a solution of (1) such that $z(0) = z_0$ and $z(t) \subseteq \omega_x$ for $t \geq 0$ (see Property 4), hence $\phi(z(t)) = \phi(\infty)$ for $t \geq 0$. From Theorem 1, it follows that for a.a. $t \geq 0$, we have $0 = d\phi(z(t))/dt \in \mathcal{D}\phi(z(t))$, and so, $z(t) \in Z$ for a.a. $t \geq 0$. It follows that we have $z_0 = z(0) \in \mathrm{cl}(Z)$. The choice of z_0 in ω_x is arbitrary, hence we obtain $\omega_x \subseteq \mathrm{cl}(Z)$. Since ω_x is weakly invariant, we conclude that $\omega_x \subseteq M$. ■

Remark: The notion of set-valued derivative $\mathcal{D}\phi$ of a nonpathological function ϕ in Section IV and the extended version of LaSalle's invariance principle in Theorem 2 have been inspired by previous relevant works on analogous issues [15], [16]. The reader is also referred to [25] for another extended version of LaSalle's invariance principle, which has been applied to estimate the location of complex attractors of a class of discontinuous dynamical systems.

Consider the general class of differential inclusions

$$\dot{x} \in F(x) \tag{7}$$

where $x \in \mathbb{R}^n$ and $F : \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper semicontinuous map with nonempty compact convex values. In [16], the following set-valued derivative of a nonpathological function $\phi : \mathbb{R}^n \to \mathbb{R}$ has been introduced for (7):

$$\widetilde{\mathcal{D}}\phi(x) = \{a \in \mathbb{R} : \exists v \in F(x) : a = \langle v, \eta \rangle,\ \forall \eta \in \partial\phi(x)\} \tag{8}$$

where $x \in \mathbb{R}^n$. If we apply this definition to the class of DVIs (1), and consider that $\mathcal{P}_{T_Q(x)} f(x) \cap N_Q^{\perp}(x) \subseteq \mathcal{P}_{T_Q(x)} f(x) \subseteq F(x) = f(x) - N_Q(x)$, we have

$$\mathcal{D}\phi(x) \subseteq \widetilde{\mathcal{D}}\phi(x)$$

for any $x \in Q$.[1] We will see (Remark 3 in Section VII-A) that there are important cases, such as the symmetric FR-CNNs, where a given function ϕ satisfies $\mathcal{D}\phi(x) \leq 0$ or $\mathcal{D}\phi(x) = \emptyset$ for any $x \in Q$, but $\widetilde{\mathcal{D}}\phi(x)$ also assumes positive values for some $x \in Q$. The advantages in the use of $\mathcal{D}\phi$ are due to the fact that, in its definition, we exploit the particular structure of the velocity $\dot{x}$ for the solutions of the DVI (1) evolving within the compact convex set Q (see Property 5). Such a structure of $\dot{x}$ cannot be exploited in the definition of $\widetilde{\mathcal{D}}\phi$ for general systems, as in (7). ■

VI. Convergence and Global Stability of DVIs

Definition 5: The DVI (1) is said to be quasi-convergent if, for any solution $x(t)$, $t \geq 0$, of (1), we have

$$\lim_{t \to +\infty} \operatorname{dist}(x(t), E) = 0$$

where E is the set of EPs of (1). Furthermore, the DVI (1) is said to be convergent if, for any solution $x(t)$, $t \geq 0$, of (1), there exists $\xi \in E$ such that

$$\lim_{t \to +\infty} x(t) = \xi.$$

■

LaSalle's invariance principle in Theorem 2 yields the next result on quasi-convergence and convergence of (1).

Theorem 3: Suppose that $f : Q \multimap \mathbb{R}^n$ is an upper semicontinuous map with nonempty compact convex values. Let $U \subset \mathbb{R}^n$ be an open set containing Q, and let $\phi : U \to \mathbb{R}$ be a nonpathological strict Lyapunov function for (1). Then, (1) is quasi-convergent. If, in addition, the EPs of (1) are isolated, then (1) is convergent. ■

Proof: Let $x(t)$, $t \geq 0$, be a solution of (1). Theorem 2 implies that $\lim_{t \to +\infty} \operatorname{dist}(x(t), M) = 0$, where M is the largest weakly invariant subset of (1) in $\operatorname{cl}(Z)$. It suffices to show that we have $E = M$. Of course, $E \subseteq M \subseteq \operatorname{cl}(Z)$. On the other hand, since ϕ is a strict Lyapunov function for (1), we also have $E = Z$. Since E is a closed subset of Q (Property 2), we conclude that $E = \operatorname{cl}(E) = \operatorname{cl}(Z)$, hence $E = M$. This proves that (1) is quasi-convergent.

Suppose now that the EPs of (1) are isolated. Convergence of (1) follows from the connectedness of the omega-limit set of each solution of (1) (Property 4). ■

For the G-DVIs (4), it is straightforward to derive the following result.

Theorem 4: Function $\phi : U \to \mathbb{R}$ is a nonpathological strict Lyapunov function for the G-DVI (4). Then, (4) is quasi-convergent. If the EPs of (4) are isolated, then (4) is convergent. ■

Proof: Suppose that for $x \in Q$, we have $\mathcal{D}\phi(x) \neq \emptyset$, and let $a \in \mathcal{D}\phi(x)$. Then, there exists $u \in \partial\phi(x)$ such that $\mathcal{P}_{T_Q(x)}(-u) \in N_Q^{\perp}(x)$ and $\langle \mathcal{P}_{T_Q(x)}(-u), p \rangle = a$ for any $p \in \partial\phi(x)$. We obtain $a = -\langle \mathcal{P}_{T_Q(x)}(-u), -u \rangle = -\langle \mathcal{P}_{T_Q(x)}(-u), \mathcal{P}_{T_Q(x)}(-u) + \mathcal{P}_{N_Q(x)}(-u) \rangle$. Then, $a = -\|\mathcal{P}_{T_Q(x)}(-u)\|^2 \leq 0$, i.e., ϕ is a Lyapunov function for (4). From point 1) in Property 7, we also conclude that $\mathcal{D}\phi$ is a strict Lyapunov function for (4). ■

Theorems 3 and 4 give conditions for convergence of the solutions of the DVI (1) in the general case where there are multiple EPs for (1). Some specific applications to FR-CNNs will be described in Section VII-A. The property of convergence in the presence of multiple EPs is of fundamental importance for the neural network applications to image processing, the implementation of associative memories, and several other signal processing tasks [1], [2], [26], [27]. There are other interesting applications where it is needed that a neural network has a unique globally stable EP [28]–[33]. For example, in the solution of global optimization problems in real time, the property of global stability prevents the network from getting stuck at some local minimum of the cost function that is desired to be minimized. In the remaining part of this section, we give conditions for global stability of (1), which involve the set-valued derivative of a nonpathological Lyapunov function, as introduced in Section IV. Some applications to FR-CNNs will be presented in Section VII-B.

Definition 6: The EP $\xi \in Q$ of (1) is said to be locally stable if, for any $\epsilon > 0$, there exists $\rho > 0$ such that we have $\|x(t) - \xi\| < \epsilon$, $t \geq 0$, for any solution $x(t)$, $t \geq 0$, of (1) satisfying $x(0) \in Q$ and $\|x(0) - \xi\| < \rho$. ■

Definition 7: The EP $\xi \in Q$ of (1) is said to be globally asymptotically stable (GAS) if ξ is locally stable and globally attractive, i.e., for any solution $x(t)$, $t \geq 0$, of (1), we have $\lim_{t \to +\infty} x(t) = \xi$. ■

Definition 8: The EP $\xi \in Q$ of (1) is said to be globally exponentially stable (GES) if there exists $a > 0$ such that, for any solution $x(t)$, $t \geq 0$, of (1), we have

$$\|x(t) - \xi\| \leq \psi(\|x(0) - \xi\|) \exp(-at)$$

for $t \geq 0$, where $\psi : [0, +\infty) \to [0, +\infty)$ is a continuous function on $[0, +\infty)$ such that $\psi(0) = 0$ and $\psi(\rho) > 0$ for $\rho > 0$. ■

The next results on GAS and GES of (1) can be proved.

[1] An analogous inclusion holds when comparing $\mathcal{D}\phi$ with the derivative in [15].

Theorem 5: Suppose that $f : Q \multimap \mathbb{R}^n$ is an upper semicontinuous map with nonempty compact convex values. Let $U \subset \mathbb{R}^n$ be an open set containing Q, and let $\xi \in Q$ be an EP of (1). Suppose that $\phi : U \to \mathbb{R}$ is a nonpathological Lyapunov function for (1), such that the following hold: 1) $\phi(x) - \phi(\xi) > 0$ for any $x \in Q \setminus \{\xi\}$, and 2) for any $x \in Q \setminus \{\xi\}$, we have either $\mathcal{D}\phi(x) = \emptyset$ or $\max \mathcal{D}\phi(x) < 0$. Then, ξ is a GAS EP of (1). ■

Proof: The hypothesis in point 2) in Property 6 implies that ξ is the unique EP of (1); moreover, ϕ is a strict Lyapunov function for (1). Then, global convergence to the EP ξ follows from Theorem 3.

The proof of local stability of the EP ξ is basically the same as that for standard differential equations, and it is omitted here for brevity. ■

Theorem 6: Suppose that $f : Q \multimap \mathbb{R}^n$ is an upper semicontinuous map with nonempty compact convex values. Let $U \subset \mathbb{R}^n$ be an open set containing Q, and let $\xi \in Q$ be an EP of (1). Let $\phi : U \to \mathbb{R}$ be a nonpathological Lyapunov function for (1), such that the following hold: 1) There exist $\alpha, \beta, \eta, \lambda > 0$ such that, for any $x \in Q$, we have

$$\alpha \|x - \xi\|^{\eta} \leq \phi(x) - \phi(\xi) \leq \beta \|x - \xi\|^{\lambda} \tag{9}$$

and 2) there exists $\gamma > 0$ such that, for any $x \in Q$, we have either $\mathcal{D}\phi(x) = \emptyset$ or

$$\max \mathcal{D}\phi(x) \leq -\gamma \left(\phi(x) - \phi(\xi)\right).$$

Then, ξ is a GES EP of (1). In particular, for any solution $x(t)$, $t \geq 0$, of (1), we have

$$\|x(t) - \xi\| \leq \mu(t) = \left(\frac{\beta}{\alpha}\right)^{1/\eta} \|x(0) - \xi\|^{\lambda/\eta} \exp\left(\frac{-\gamma t}{\eta}\right) \tag{10}$$

for $t \geq 0$. ■

Proof: Let $x(t)$, $t \geq 0$, be a solution of (1), and consider the function $\Delta(t) = \phi(x(t)) - \phi(\xi) \geq 0$, $t \geq 0$, which is absolutely continuous on any compact interval in $[0, +\infty)$. Accounting for Lemma 1, for a.a. $t \in [0, +\infty)$, we have $\mathcal{D}\phi(x(t)) \neq \emptyset$ and $\dot{\Delta}(t) = \mathrm{d}\phi(x(t))/\mathrm{d}t \in \mathcal{D}\phi(x(t)) \leq \max \mathcal{D}\phi(x(t)) \leq -\gamma\left(\phi(x(t)) - \phi(\xi)\right) = -\gamma\Delta(t)$. Thus, $\Delta(t) \leq \Delta(0)\exp(-\gamma t)$ for $t \geq 0$. By recalling (9), we have

$$\begin{aligned}\alpha \|x(t) - \xi\|^{\eta} &\leq \Delta(t) \leq \Delta(0)\exp(-\gamma t)\\ &\leq \beta \|x(0) - \xi\|^{\lambda} \exp(-\gamma t)\end{aligned}$$

from which we obtain (10). ■

VII. Convergence and Global Stability of FR-CNNs

Let us consider the class of FR-CNNs

$$\dot{x} \in Ax + I - N_K(x) \tag{11}$$

where $x \in K = [-1, 1]^n$, $A \in \mathbb{R}^{n \times n}$ is the neuron interconnection matrix and $I \in \mathbb{R}^n$ is the constant input. Note that, as a consequence of Property 3, for any $x_0 \in K$, there is a unique solution $x : [0, +\infty) \to K$ of (11) with initial condition $x(0) = x_0$. In this section, we will exploit the results obtained in Section V in order to obtain basic results on convergence and global stability of (11). Then, we briefly address convergence of some extended classes of FR-CNNs.

A. Convergence of Symmetric FR-CNNs

Suppose that the neuron interconnection matrix $A = A'$ is symmetric, and consider for (11) the (candidate) Lyapunov function

$$\phi(x) = -\frac{1}{2}x' A x - x' I \tag{12}$$

where $x \in \mathbb{R}^n$. Note that the FR-CNN (11) is equivalent to the G-DVI

$$\dot{x} \in -\nabla\phi(x) - N_K(x)$$

for $x \in K$. Then, Theorem 4 yields the following convergence result.

Theorem 7: If $A = A'$, then ϕ, as in (12), is a strict Lyapunov function for the FR-CNN (11). Hence, the FR-CNN (11) is quasi-convergent, and it is convergent if the EPs of (11) are isolated. ■

Remarks:

1) The result on convergence in Theorem 7 coincides with that obtained in a recent paper ([13, Th. 1]). It is noted that Theorem 7 is a direct consequence of the extended version of LaSalle's invariance principle in Theorem 2 and the fact that a symmetric FR-CNN, being a G-DVI, admits a strict Lyapunov function.[13, Theorem 1] has been proved by using a different method, not based on an invariance principle, which requires the analysis of the behavior of the solutions close to the set of the EPs of the FR-CNNs (see the comparatively more complex proof of [13, Th. 1]). ■

2) Consider the S-CNN model [1]

$$\dot{x} = -x + AG(x) + I \tag{13}$$

where $x \in \mathbb{R}^n$, $G(x) = (g(x_1), g(x_2), \ldots, g(x_n))' : \mathbb{R}^n \to \mathbb{R}^n$, and $g(\rho) = (1/2)(|\rho + 1| - |\rho - 1|) : \mathbb{R} \to \mathbb{R}$ is the piecewise-linear neuron activation. When the neuron interconnection matrix $A = A'$ is symmetric, (13) admits the Lyapunov function [1]

$$\psi(x) = -\frac{1}{2}G'(x)(A - E_n)G(x) - G'(x)I \tag{14}$$

where $x \in \mathbb{R}^n$. One main problem is that ψ *is not a strict Lyapunov function* for the symmetric S-CNN (13).[2] Then, in order to prove quasi-convergence or convergence of (13) by means of LaSalle's invariance principle, it is needed to investigate the geometry of the largest invariant sets of (13) where the time derivative of ψ along solutions of (13) vanishes [34]. Such an analysis turns out to be quite complex

[2]Indeed, in partial and total saturation regions of (13), the time derivative of ψ along solutions of (13) vanishes in sets of points that are larger than the sets of EPs of (13).

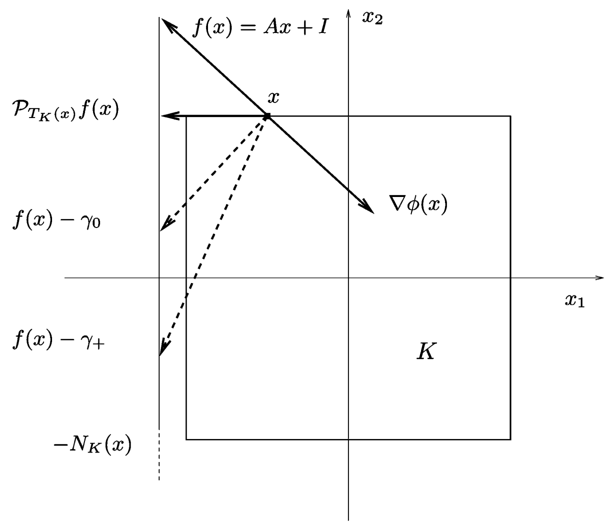

Fig. 4. Comparison between the set-valued derivatives $\mathcal{D}\phi$ and $\widetilde{\mathcal{D}}\phi$ for a second-order FR-CNN. Consider a point x on an edge of K where $T_K(x) = \{(x_1, x_2) \in \mathbb{R}^2 : -\infty < x_1 < +\infty, x_2 \leq 0\}$, $N_K(x) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, x_2 \geq 0\}$, and $N_K^\perp(x) = \{(x_1, x_2) \in \mathbb{R}^2 : -\infty < x_1 < +\infty, x_2 = 0\}$. We have $\mathcal{P}_{T_K(x)}f(x) \in N_K^\perp(x)$ and $\mathcal{D}\phi(x) = \langle \mathcal{P}_{T_K(x)}f(x), \nabla\phi(x)\rangle = -\|\mathcal{P}_{T_K(x)}f(x)\|^2 < 0$. The derivative $\widetilde{\mathcal{D}}\phi(x)$, which is given by $\widetilde{\mathcal{D}}\phi(x) = \{\langle u, \nabla\phi(x)\rangle, u \in f(x) - N_K(x)\}$, instead assumes both positive and negative values. For example, the figure considers a vector $\gamma_0 \in N_K(x)$ such that we have $\widetilde{\mathcal{D}}\phi(x) \ni 0 = \langle f(x) - \gamma_0, \nabla\phi(x)\rangle$, and a vector $\gamma_+ \in N_K(x)$ such that $\widetilde{\mathcal{D}}\phi(x) \ni \langle f(x) - \gamma_+, \nabla\phi(x)\rangle > 0$.

(see [35] for details).[3] It is worth to stress that, according to Theorem 7, ϕ, as in (12), *is a strict Lyapunov function* for the symmetric FR-CNN (11), hence the proof of quasi-convergence or convergence of (11) is a direct consequence of the generalized version of LaSalle's invariance principle in Theorem 2. ∎

3) Let us compare the definition of set-valued derivative $\mathcal{D}\phi$ in this paper, with the derivative $\widetilde{\mathcal{D}}\phi$ in (8), which has been introduced in [16]. Consider a second-order symmetric FR-CNN, whose solutions evolve in the square $K = [-1, 1]^2$, and the candidate Lyapunov function ϕ in (12). Suppose that there is a point x belonging to an edge of K, such that $\mathcal{P}_{T_K(x)}f(x) = \mathcal{P}_{T_K(x)}(Ax + I) \in N_K^\perp(x)$ (see Fig. 4). Due to the continuity of f, there is a neighborhood of x along the edge, such that $\mathcal{P}_{T_K(y)}(Ay + I) \in N_K^\perp(y)$ for y in the neighborhood. If a solution of the FR-CNN passes through x, then it is seen to slide along the edge during some time interval.

We have $\nabla\phi(x) = -f(x)$ and

$$\mathcal{D}\phi(x) = \left\langle \mathcal{P}_{T_K(x)}f(x), \nabla\phi(x)\right\rangle = -\left\|\mathcal{P}_{T_K(x)}f(x)\right\|^2 < 0.$$

Instead, from (8), we obtain

$$\begin{aligned}\widetilde{\mathcal{D}}\phi(x) &= \{\langle u, \nabla\phi(x)\rangle, u \in f(x) - N_K(x)\}\\ &= \left[-\|f(x)\|^2, +\infty\right)\end{aligned}$$

hence $\widetilde{\mathcal{D}}\phi(x)$ is a half-line containing both negative and positive values (see Fig. 4 for a geometric interpretation). Therefore, by means of the derivative $\mathcal{D}\phi$, we can conclude that ϕ, as in (12), is a Lyapunov function for (11) (see also the proof of Theorem 7), while it cannot be proved that ϕ is a Lyapunov function for (11) using $\widetilde{\mathcal{D}}\phi$. ∎

[3]We also refer the reader to another approach for convergence of symmetric S-CNNs [36], which exploits a Lojasiewicz inequality to circumvent difficulties encountered in the use of the nonstrict Lyapunov function (14) for the symmetric S-CNNs.

B. Global Stability of Nonsymmetric FR-CNNs

In this section, we give a basic result on GES of the unique EP of the FR-CNN (11), which is applicable to a large class of nonsymmetric interconnection matrices A.

Suppose that $-A$ is Lyapunov diagonally stable (LDS) [37], i.e., there exists a positive-definite diagonal matrix $\alpha = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^{n\times n}$, such that $[\alpha(-A)]_S = (1/2)(\alpha(-A) + (-A)'\alpha)$ is positive definite.

From Property 2, there exists at least an EP $\xi \in K$ of (11). Consider for (11) the candidate Lyapunov function

$$\phi(x) = \frac{1}{2}(x - \xi)'\alpha(x - \xi)$$

for $x \in \mathbb{R}^n$. We have

$$\frac{\alpha_m}{2}\|x - \xi\|^2 \leq \phi(x) - \phi(\xi) \leq \frac{\alpha_M}{2}\|x - \xi\|^2$$

for $x \in \mathbb{R}^n$, where $\alpha_m = \min_{i=1,2,\ldots,n}\{\alpha_i\} > 0$ and $\alpha_M = \max_{i=1,2,\ldots,n}\{\alpha_i\} > 0$.

Let

$$\lambda_m = \min_{i=1,2,\ldots,n} \lambda_i > 0$$

where $\lambda_i > 0, i = 1, 2, \ldots, n$, are the eigenvalues of $[\alpha(-A)]_S$. Then, the following holds.

Theorem 8: If $-A$ is LDS, then the FR-CNN (11) has a unique GES EP $\xi \in K$. Furthermore, for any solution $x(t)$, $t \geq 0$, of (1), we have

$$\|x(t) - \xi\| \leq \sqrt{\frac{\alpha_M}{\alpha_m}}\|x(0) - \xi\|\exp\left(\frac{-\lambda_m t}{2\alpha_M}\right)$$

for $t \geq 0$. ∎

Proof: It suffices to show that we have

$$\max \mathcal{D}\phi(x) \leq -\frac{\lambda_m}{\alpha_M}\phi(x)$$

for any $x \in K$, such that $\mathcal{D}\phi(x) \neq \emptyset$. Then, the result follows from Theorem 6.

Let $x \in K$ be such that $\mathcal{D}\phi(x) \neq \emptyset$. Also, let $\gamma_x = \mathcal{P}_{N_K(x)}(Ax + I) \in N_K(x)$, hence $\mathcal{P}_{T_K(x)}(Ax + I) = Ax + I - \mathcal{P}_{N_K(x)}(Ax + I) = Ax + I - \gamma_x \in N_K^\perp(x)$. Since $\xi \in K$ is an EP of (11), there exists $\gamma_\xi \in N_K(\xi)$ such that $A\xi + I - \gamma_\xi = 0$.

We have

$$\begin{aligned}\mathcal{D}\phi(x) &= \left\langle \mathcal{P}_{T_K(x)}(Ax + I), \nabla\phi(x)\right\rangle\\ &= \left\langle \mathcal{P}_{T_K(x)}(Ax + I) + A\xi + I - \gamma_\xi, \nabla\phi(x)\right\rangle\\ &= -(x - \xi)'\alpha(-A)(x - \xi) - (x - \xi)'\alpha(\gamma_x - \gamma_\xi).\end{aligned}$$

Exploiting Lemma 2 in Appendix II, we obtain $(x-\xi)'\alpha(\gamma_x-\gamma_\xi)\geq 0$, and hence

$$\begin{aligned}\mathcal{D}\phi(x) &= -(x-\xi)'\alpha(-A)(x-\xi)-(x-\xi)'\alpha(\gamma_x-\gamma_\xi)\\ &\leq -(x-\xi)'\alpha(-A)(x-\xi)\\ &= -(x-\xi)'\left[\alpha(-A)\right]_S(x-\xi)\\ &\leq -\frac{\lambda_m}{\alpha_M}\alpha_M\|x-\xi\|^2\leq -\frac{\lambda_m}{\alpha_M}\phi(x).\end{aligned}$$

■

Remarks:

1) The hypothesis of LDS interconnection matrices in Theorem 8 is on the basis of results on GAS or GES of several other neural network models, including the Hopfield neural networks, the Cohen–Grossberg neural networks, and the S-CNNs (see, e.g., [28]–[33]). We refer the reader to those papers for interesting applications of GAS or GES of neural networks in the solution of optimization problems in real time. ■
2) A result on GES of FR-CNNs, which is analogous to that in Theorem 8, has been established in a recent paper [38, Th. 5] by means of a different method that is not based on the use of the generalized derivative $\mathcal{D}\phi$ and an invariance principle. We refer the reader to [38] for a general comparison of the conditions ensuring GES of FR-CNNs and S-CNNs also in the presence of delay. Again, it is worth to stress the quite simple proof of GES in Theorem 8, and the comparatively more complex proof of GAS or GES, under analogous assumptions, for S-CNNs [29], [39], [40]. ■

C. Extended FR-CNNs

By means of the results in Section VI, we can conceive some classes of extended FR-CNNs, which enjoy the property of convergence and, as such, are potentially useful for solving signal processing tasks in real time or implementing content addressable memories.

As a first example, consider the DVI

$$\dot{x}\in Ax+I-N_Q(x) \tag{15}$$

where $Q\in\mathbb{R}^n$ is a nonempty compact convex set. An interesting case would be that where Q is defined by a finite set of affine constraints (Q is a polytope). The DVI (15) can be considered as an extended FR-CNN; indeed, it reduces to (11) in the case where $Q=K=[-1,1]^n$.

As a second example, consider the DVI

$$\dot{x}\in -\partial\phi(x)-N_K(x) \tag{16}$$

where $\phi:\mathbb{R}^n\to\mathbb{R}$ is a nonpathological function. An interesting case is that where ϕ is a square function plus a finite combination of absolute-value functions of the state variables. Again, (16) reduces to (11) when $A=A'$ is symmetric and $\phi(x)=-(1/2)x'Ax-x'I$.

The following result is an immediate consequence of Theorem 4.

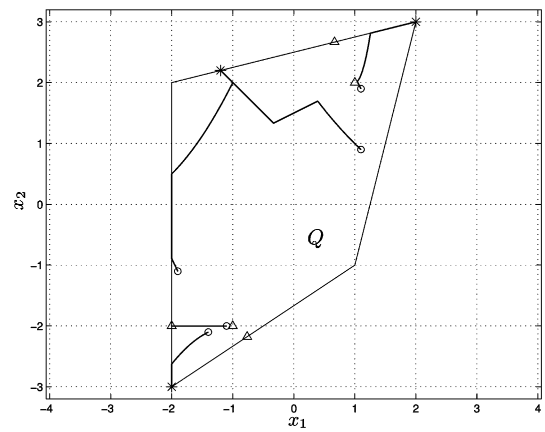

Fig. 5. Five trajectories of (17) with different initial conditions (○). The stable EPs of (17) are denoted by $*$, and the unstable EPs are represented by $\triangle$.

Proposition 1: The DVI (16) is quasi-convergent, and it is convergent when the EPs of (16) are isolated. Similarly, the DVI (15) is quasi-convergent if $A=A'$, and it is convergent if the EPs of (15) are isolated. ■

The study of the possible application of the extended FR-CNNs (15) and (16) to specific processing tasks goes beyond the scope of this paper, and it will be the subject of future investigations. It should be remarked that, in order to maintain the advantages in the VLSI implementation of the FR paradigm, the selection of the set Q in (15), and the function ϕ in (16), should preserve as far as possible the sparse and local interconnection structure characterizing the CNNs.

VIII. Examples

1) Example 1: Consider the function

$$\phi(x_1,x_2)=|-x_1+2x_2-3|+2|x_1+x_2-1|-\frac{1}{2}x_1^2-x_2^2$$

where $x=(x_1,x_2)'\in\mathbb{R}^2$, and the second-order G-DVI

$$\dot{x}\in -\partial\phi(x)-N_Q(x) \tag{17}$$

where Q is the polytope shown in Fig. 5. Note that ϕ is not convex in Q, but it is semiconvex. By evaluating $\partial\phi$, we obtain

$$\begin{aligned}\partial\phi(x)=&\begin{bmatrix}-1&2\\2&2\end{bmatrix}\begin{bmatrix}\operatorname{sign}(-x_1+2x_2-3)\\ \operatorname{sign}(x_1+x_2-1)\end{bmatrix}\\ &+\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}\end{aligned}$$

where

$$\operatorname{sign}(\rho)=\begin{cases}-1, & \rho<0\\ [-1,1], & \rho=0\\ 1, & \rho>0\end{cases}$$

is the set-valued map representing the characteristic of a hard comparator.

The G-DVI (17) has three stable EPs at $\xi_1=(2,3)'$ and $\xi_2=(-2,-3)'$, which are vertices of Q, and $\xi_3=(-6/5,11/5)'$ on an edge of Q. Furthermore, (17) has three unstable EPs on

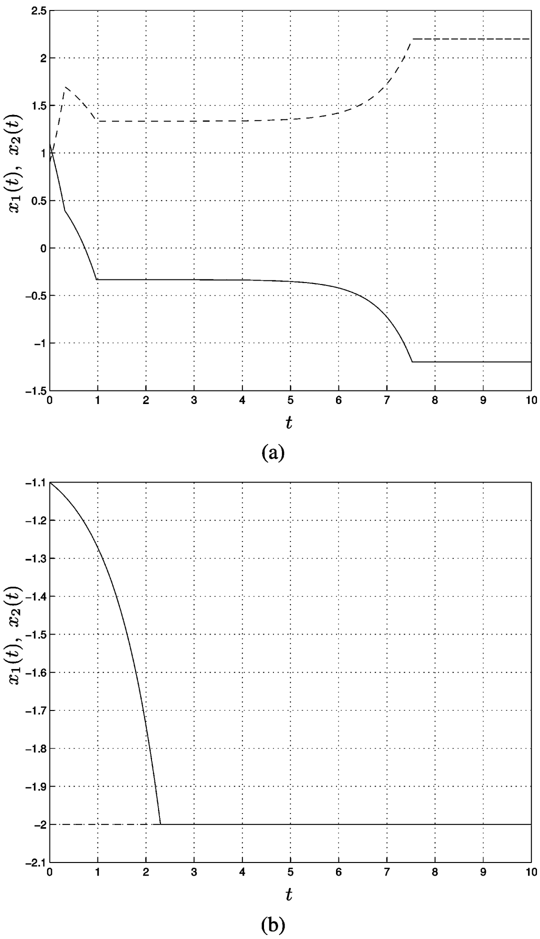

Fig. 6. Time-domain evolution of the state variables (solid) $x_1(t)$ and (dashed) $x_2(t)$ for the solution of (17) (a) with initial condition $(x_1(0), x_2(0)) = (1.1, 0.9)$ and (b) with initial condition $(x_1(0), x_2(0)) = (-1.1, -2)$.

edges of Q at $\xi_4 = (-2,-2)'$, $\xi_5 = (2/3, 8/3)'$, and $\xi_6 = (-13/17, -37/17)'$, and two additional unstable EPs at $\xi_7 = (-1,-2)'$ and $\xi_8 = (1,2)'$, respectively, in the interior of Q.

We have simulated the dynamics of (17) by using MATLAB. Fig. 5 shows five trajectories of (17) starting at different initial conditions. Fig. 6 shows the time evolution of the two trajectories starting at $x(0) = (1.1, 0.9)'$ and $x(0) = (-1.1, -2)'$, respectively. It is seen that each solution converges toward an EP of (17), in accordance with Theorem 4. Note, in particular, that the solution starting at $x(0) = (-1.1, -2)'$ converges toward the unstable saddle-type EP $\xi_4 = (-2,-2)'$ along the stable manifold of the EP, while all other solutions converge toward one of the stable EPs.

2) Example 2: As a second example, consider the function

$$\phi(x_1, x_2) = |-x_1 + 2x_2 - 3| + 2|x_1 + x_2 - 1| + \frac{1}{2}x_1^2 + x_2^2 - \frac{1}{3}x_1 + \frac{23}{3}x_2$$

where $x = (x_1, x_2)' \in \mathbb{R}^2$, and the G-DVI

$$\dot{x} \in -\partial\phi(x) - N_Q(x) \tag{18}$$

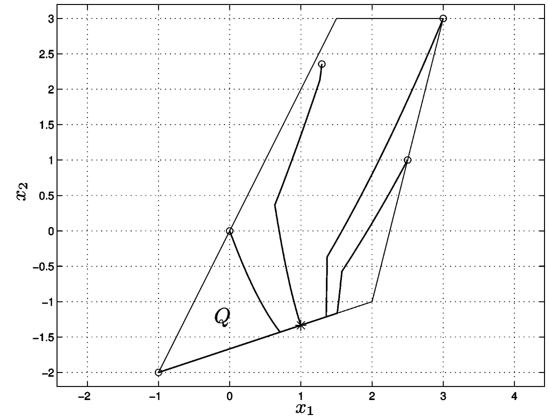

Fig. 7. Five trajectories of (18) with different initial conditions. The unique GES EP of (18) is denoted by $*$.

where Q is the polytope in Fig. 7. Now, ϕ is convex in Q, and we have

$$\partial\phi(x) = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \text{sign}(-x_1 + 2x_2 - 3) \\ \text{sign}(x_1 + x_2 - 1) \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} \\ \frac{-23}{3} \end{bmatrix}.$$

The G-DVI (18) has a unique EP at $\xi = (1, -4/3)'$ on an edge of Q. It can be verified that all the assumptions of Theorem 6 are satisfied with the following parameter values: $\alpha = 0.54$, $\beta = 9.33$, $\eta = 2$, $\lambda = 1$, and $\gamma = 2$. Hence, ξ is a GES EP of (18). Fig. 7 shows five trajectories of (18) with different initial conditions, which are seen to converge toward the EP ξ as $t \to +\infty$. In particular, Fig. 8 shows the time evolution of the trajectory starting at $x(0) = (2.5, 1)'$, and compares for this solution the norm $\|x(t) - \xi\|$ with the upper bound $\mu(t)$ in (10) of Theorem 6. It is seen that (10) is satisfied, although, as it usually happens, the bound $\mu(t)$ turns out to be quite conservative.

IX. Conclusion

This paper has developed a Lyapunov approach for studying stability and convergence of a class of DVIs. The approach is based on the notion of set-valued derivative of a nonpathological Lyapunov function, and an extended version of LaSalle's invariance principle for the DVIs.

General conditions have been established, which guarantee convergence of DVIs in the presence of multiple EPs, and GAS and GES of the unique EP. The conditions have been used to derive results on convergence, GAS, and GES for the FR model of CNNs and some extended classes of FR-CNNs.

The results obtained show that, by means of the developed Lyapunov approach, the analysis of stability and convergence of FR-CNNs is no more difficult than that of the S-CNNs. In addition, in some relevant cases, such as the symmetric FR-CNNs and the FR-CNNs with an LDS interconnection matrix, the analysis of convergence with the approach developed here is much simpler than that of the S-CNNs. In fact, one basic result obtained here is that a symmetric FR-CNN admits a strict Lyapunov function, and thus, it is convergent as a direct

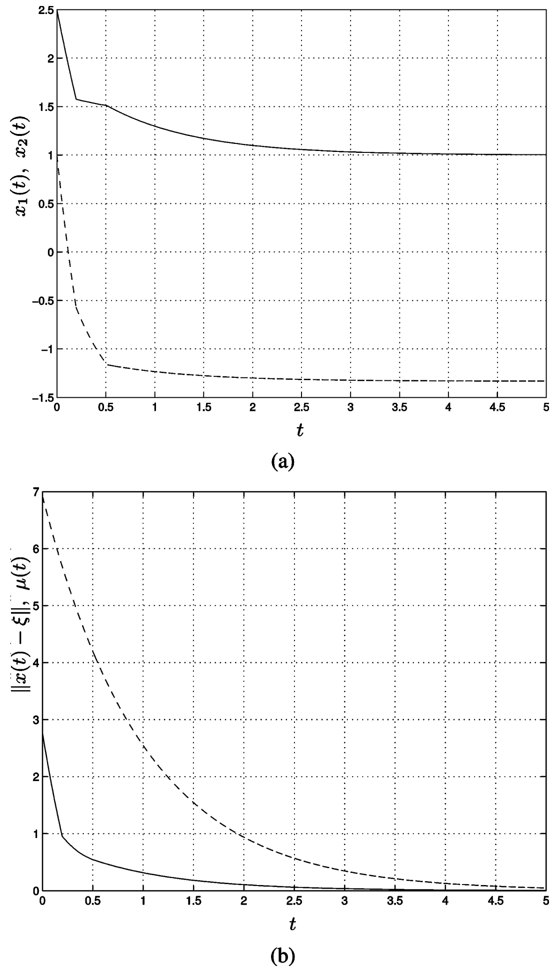

Fig. 8. Time-domain evolution of the solution of (17) (a) with initial conditions $(x_1(0), x_2(0)) = (2.5, 1)$ and (b) comparison of (solid) $\|x(t) - \xi\|$ with (dashed) the bound $\mu(t)$ in (10) for the same solution.

consequence of the extended version of LaSalle's invariance principle. On the contrary, as is well known, the Lyapunov function discovered by Chua and Yang for symmetric S-CNNs is not strict, and its use leads to a more complex and elaborate convergence proof. An analogous conclusion has been reached for LDS interconnection matrices.

Appendix I

Some Geometrical Properties of $\mathcal{D}\phi$: In this section, we discuss some geometrical aspects relative to the set-valued derivative $\mathcal{D}\phi$ in Definition 3. Then, we give the proof of point 1) in Property 6.

Let us introduce the map $P : Q \multimap \mathbb{R}^n$ given by $P(x) = N_Q(x) + N_Q^{\perp}(x)$, $x \in Q$, and the related vector field $f_P : Q \multimap \mathbb{R}^n$ as $f_P(x) = f(x) \cap P(x)$, $x \in Q$. Clearly, $P(x)$ is a nonempty closed convex cone and $f_P(x)$ is a (possibly empty) compact convex set for any $x \in Q$.

Proposition 2: For any $x \in Q$, we have that

$$\mathcal{P}_{N_Q^{\perp}(x)} f_P(x) = \mathcal{P}_{T_Q(x)} f(x) \cap N_Q^{\perp}(x).$$

■

Proof: Let $v \in P_{T_Q(x)} f(x) \cap N_Q^{\perp}(x)$, hence there exists $w \in f(x)$ such that $v = \mathcal{P}_{T_Q(x)} w = w - \mathcal{P}_{N_Q(x)} w$. Therefore, $w = u + v$, $u = \mathcal{P}_{N_Q(x)} w$, and so, $w \in f_P(x)$, since $u \in N_Q(x) \subset \left(N_Q^{\perp}(x)\right)^{\perp}$ and $v \in N_Q^{\perp}(x)$. On the other hand, the decomposition of w along the subspaces $\left(N_Q^{\perp}(x)\right)^{\perp}$ and $N_Q^{\perp}(x)$ is unique, thus $v = \mathcal{P}_{N_Q^{\perp}(x)} w \in \mathcal{P}_{N_Q^{\perp}(x)} f_P(x)$.

Now, let $v \in \mathcal{P}_{N_Q^{\perp}(x)} f_P(x)$, hence there exists $w \in f_P(x) \subseteq f(x)$ such that $v = \mathcal{P}_{N_Q^{\perp}(x)} w$. To conclude the proof, it suffices to observe that $v = \mathcal{P}_{T_Q(x)} w$, because $v \in N_Q^{\perp}(x) \subset \mathrm{bd}(T_Q(x))$, and the projection of w onto $T_Q(x)$ is unique. ■

Observe that if $w \in f(x) \setminus f_P(x)$, $x \in Q$, then $\mathcal{P}_{T_Q(x)} f(x) \cap N_Q^{\perp}(x) = \emptyset$. Therefore, if $\mathcal{D}\phi(x) \neq \emptyset$, then $f_P(x) \neq \emptyset$. The next result characterizes the subset of $f_P(x)$, and the corresponding subset of $\mathcal{P}_{N_Q^{\perp}(x)} f_P(x)$, for which $\mathcal{D}\phi(x) \neq \emptyset$. To this end, fix any $p_0 \in \partial\phi(x)$, $x \in Q$, and introduce the map $f_\phi : Q \multimap \mathbb{R}^n$ as follows:

$$f_\phi(x) = \mathcal{P}_{N_Q^{\perp}(x)} f_P(x) \cap (-p_0 + \partial\phi(x))^{\perp}$$

for $x \in Q$, where $(-p_0 + \partial\phi(x))^{\perp}$ is the orthogonal set to $-p_0 + \partial\phi(x)$.

Proposition 3: Assume that $\mathcal{D}\phi(x) \neq \emptyset$ at $x \in Q$. Fix any $p_0 \in \partial\phi(x)$. Then, we have

$$\mathcal{D}\phi(x) = \{\langle v, p_0 \rangle, v \in f_\phi(x)\}.$$

■

Proof: Let $a \in \mathcal{D}\phi(x)$; hence, by Proposition 2, there exists $w \in f_P(x)$ such that $v = \mathcal{P}_{N_Q^{\perp}(x)} w$ satisfies $\langle v, p \rangle = a$ for any $p \in \partial\phi(x)$. Consider now $\langle v, -p_0 + p \rangle = \langle v, -p_0 \rangle + \langle v, p \rangle = -a + a = 0$, hence $v \in (-p_0 + \partial\phi(x))^{\perp}$. In conclusion, there exists $v \in f_\phi(x)$ such that $a = \langle v, p_0 \rangle$.

Conversely, let $a = \langle v, p_0 \rangle$ for some $v \in f_\phi(x) \neq \emptyset$. By Proposition 2, we have that $v \in \mathcal{P}_{T_Q(x)} f(x) \cap N_Q^{\perp}(x)$. Let $p \in \partial\phi(x)$, then $\langle v, -p_0 + p \rangle = 0$, since $v \in (-p_0 + \partial\phi(x))^{\perp}$. Therefore, $\langle v, p \rangle = \langle v, p_0 \rangle = a$ whenever $p \in \partial\phi(x)$, thus $a \in \mathcal{D}\phi(x)$. This concludes the proof. ■

We are now in a position to give the proof of point 1) in Property 6. By Proposition 2, we have that $\mathcal{P}_{T_Q(x)} f(x) \cap N_Q^{\perp}(x) = \mathcal{P}_{N_Q^{\perp}(x)} f_P(x)$, $x \in Q$. The set $\mathcal{P}_{N_Q^{\perp}(x)} f_P(x)$ is compact and convex, since it is the projection of the compact and convex set $f_P(x)$ onto the subspace $N_Q^{\perp}(x)$. Furthermore, the set $f_\phi(x) = \mathcal{P}_{N_Q^{\perp}(x)} f_P(x) \cap (-p_0 + \partial\phi(x))^{\perp}$, where p_0 is a fixed vector in $\partial\phi(x)$, is also compact and convex, since it is the intersection of the compact and convex set $\mathcal{P}_{N_Q^{\perp}(x)} f_P(x)$ with the subspace $(-p_0 + \partial\phi(x))^{\perp}$. By Proposition 3, if $\mathcal{D}\phi(x) \neq \emptyset$, we have $\mathcal{D}\phi(x) = \{\langle v, p_0 \rangle, v \in f_\phi(x)\}$. Hence, $\mathcal{D}\phi(x)$ is a compact convex set of $\mathbb{R}$, i.e., $\mathcal{D}\phi(x)$ is a bounded closed interval of $\mathbb{R}$.

Appendix II

A Result on Normal Cones to a Closed Hypercube:

Lemma 2: Let $a, b \in K = [-1, 1]^n$, $u \in N_K(a)$, and $v \in N_K(b)$. If $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ is such that $d_i \geq 0$, $i = 1, 2, \ldots, n$, then $\langle b - a, D(v - u) \rangle \geq 0$. ■

Proof: We start to prove that if $c = (c_1, c_2, \ldots, c_n)' \in K$ and $w = (w_1, w_2, \ldots, w_n)' \in N_K(c)$, then $Dc \in N_K(c)$. Let $i \in \{1, 2, \ldots, n\}$, and observe that $N_{[-1,1]}(c_i) = s(c_i)$ [see (3)]. Since $N_K(c) = N_{[-1,1]}(c_1) \times N_{[-1,1]}(c_2) \times \cdots \times N_{[-1,1]}(c_n)$ ([12, Prop. 9, p. 224]), we have $w_i = 0$ if $|c_i| < 1$, so $0 = D_i w_i \in N_{[-1,1]}(c_i)$. Moreover, if $c_i = 1$, then $w_i \geq 0$ and $0 \leq D_i w_i \in N_{[-1,1]}(c_i) = [0, +\infty)$. Finally, if $c_i = -1$, we have $w_i \leq 0$, so $0 \geq D_i w_i \in N_{[-1,1]}(c_i) = (-\infty, 0]$. As a consequence, $Dw = (D_1 w_1, D_2 w_2, \ldots, D_n w_n)' \in N_{[-1,1]}(c_1) \times N_{[-1,1]}(c_2) \times \cdots \times N_{[-1,1]}(c_n) = N_K(c)$.

Since $a, b \in K = [-1,1]^n$, $u \in N_K(a)$, and $v \in N_K(b)$, we have $Du \in N_K(u)$ and $Dv \in N_K(v)$. The result follows from the fact that the normal cone N_K is a monotone operator (Section II). ■

REFERENCES

[1] L. O. Chua and L. Yang, "Cellular neural networks: Theory," *IEEE Trans. Circuits Syst.*, vol. 35, no. 10, pp. 1257–1272, Oct. 1988.

[2] L. O. Chua, *CNN: A Paradigm for Complexity*. Singapore: World Scientific, 1998.

[3] B. E. Shi, P. Arena, and A. Zarandi, "Special issue on CNN technology and active wave computing," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 51, no. 5, pp. 939–947, May 2004.

[4] G. Cserey, A. Falus, and T. Roska, "Immune response inspired spatial–temporal target detection algorithms with CNN-UM," *Int. J. Circuit Theory Appl.*, vol. 34, no. 1, pp. 21–47, Jan./Feb. 2006.

[5] T. Roska, P. Arena, and C. Rekeczky, "Special issue on CNN technology (Part 1)," *Int. J. Circuit Theory Appl.*, vol. 34, no. 1, pp. 127–140, Jan. 2006.

[6] L. O. Chua, "Special issue on nonlinear waves, patterns and spatio-temporal chaos in dynamic arrays," *IEEE Trans. Circuits Syst. I*, vol. 42, no. 10, pp. 557–823, Oct. 1995.

[7] I. Petrás, T. Roska, and L. O. Chua, "New spatial–temporal patterns and the first programmable on-chip bifurcation test bed," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 50, no. 5, pp. 619–633, May 2003.

[8] A. Rodríguez-Vázquez, S. Espejo, R. Domínguez-Castro, J. L. Huertas, and E. Sánchez-Sinencio, "Current-mode techniques for the implementation of continuous- and discrete-time cellular neural networks," *IEEE Trans. Circuits Syst. II, Analog Digit. Signal Process.*, vol. 40, no. 3, pp. 132–146, Mar. 1993.

[9] S. Espejo, R. Carmona, R. Domínguez-Castro, and A. Rodríguez-Vázquez, "A VLSI-oriented continuous-time CNN model," *Int. J. Circuit Theory Appl.*, vol. 24, no. 3, pp. 341–356, 1996.

[10] G. L. Cembrano, A. Rodríguez-Vázquez, S. E. Meana, and R. Domínguez-Castro, "ACE16k: A 128X128 focal plane analog processor with digital I/O," *Int. J. Neural Syst.*, vol. 13, no. 6, pp. 427–434, Dec. 2003.

[11] A. Rodríguez-Vázquez, G. Liñán-Cembrano, L. Carranza, E. Roca-Moreno, R. Carmona-Galán, F. Jiménez-Garrido, R. Domínguez-Castro, and S. E. Meana, "ACE16k: The third generation of mixed-signal SIMD-CNN ACE chips towards VSoCs," *IEEE Trans. Circuits Syst. I, reg. Papers*, vol. 51, no. 5, pp. 851–863, May 2004.

[12] J. P. Aubin and A. Cellina, *Differential Inclusions*. Berlin, Germany: Springer-Verlag, 1984.

[13] G. D. Sandre, M. Forti, P. Nistri, and A. Premoli, "Dynamical analysis of full-range cellular neural networks by exploiting differential variational inequalities," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 54, no. 8, pp. 1736–1749, Aug. 2007.

[14] F. Corinto and M. Gilli, "Comparison between the dynamic behaviour of Chua-Yang and full-range cellular neural networks," *Int. J. Circuit Theory Appl.*, vol. 31, no. 5, pp. 423–441, Sep./Oct. 2003.

[15] D. Shevitz and B. Paden, "Lyapunov stability theory of nonsmooth systems," *IEEE Trans. Autom. Control*, vol. 39, no. 9, pp. 1910–1914, Sep. 1994.

[16] A. Bacciotti and F. Ceragioli, "Stability and stabilization of discontinuous systems and nonsmooth Lyapunov functions," *Esaim, COCV*, vol. 4, pp. 361–376, 1999.

[17] J. P. Aubin and H. Frankowska, *Set-Valued Analysis*. Boston, MA: Birkauser, 1990.

[18] T. Rockafellar and R. Wets, *Variational Analysis*. Berlin, Germany: Springer-Verlag, 1997.

[19] F. H. Clarke, *Optimization and Non-Smooth Analysis*. New York: Wiley, 1983.

[20] M. Valadier, "Entrainement unilatéral, lignes de descente, fonctions Lipschitziennes non pathologiques," *Comptes Rendu Academie des Sciences. Series I, Mathematique. Paris*, vol. 8, pp. 241–244, 1989.

[21] A. Bacciotti and F. Ceragioli, "Nonpathological Lyapunov functions and discontinuous Carathéodory systems," *Automatica*, vol. 42, no. 3, pp. 453–458, Mar. 2006.

[22] J. Pang and D. Stewart, "Differential variational inequalities," *Math. Program., Ser. A*, vol. 113, no. 2, pp. 345–424, Jan. 2008.

[23] R. T. Rockafellar, *Convex Analysis*. Princeton, NJ: Princeton Univ. Press, 1970.

[24] A. F. Filippov, *Differential Equations With Discontinuous Right-Hand Side. Mathematics and Its Applications (Soviet Series)*. Boston, MA: Kluwer, 1988.

[25] S. Deriviere and M. Aziz-Alaoui, "A method to analyze complete stability of analytic neural networks," Université de Sherbrooke, Sherbrooke, QC, Canada, 2006, Tech. Rep. 31.

[26] L. O. Chua and T. Roska, "Stability of a class of nonreciprocal cellular neural networks," *IEEE Trans. Circuits Syst.*, vol. 37, no. 12, pp. 1520–1527, Dec. 1990.

[27] P. Thiran, G. Setti, and M. Hasler, "An approach to information propagation in 1-D cellular neural networks. Part I: Local diffusion," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 45, no. 8, pp. 777–789, Aug. 1998.

[28] E. Kaszkurewicz and A. Bhaya, "On a class of globally stable neural circuits," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 41, no. 2, pp. 171–174, Feb. 1994.

[29] M. Forti and A. Tesi, "New conditions for global stability of neural networks with application to linear and quadratic programming problems," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 42, no. 7, pp. 354–366, Jul. 1995.

[30] S. Hu and J. Wang, "Absolute exponential stability of a class of continuous-time recurrent neural networks," *IEEE Trans. Neural Netw.*, vol. 14, no. 1, pp. 35–45, Jan. 2003.

[31] W. Lu and T. Chen, "New conditions on global stability of Cohen–Grossberg neural networks," *Neural Comput.*, vol. 15, no. 5, pp. 1173–1189, May 2003.

[32] H. Qi and L. Qi, "Deriving sufficient conditions for global asymptotic stability of delayed neural networks via nonsmooth analysis," *IEEE Trans. Neural Netw.*, vol. 15, no. 1, pp. 99–109, Jan. 2004.

[33] J. Cao and J. Wang, "Global asymptotic and robust stability of recurrent neural networks with time delays," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 52, no. 2, pp. 417–426, Feb. 2005.

[34] M. W. Hirsch, "Systems of differential equations which are competitive or cooperative. I: Limit sets," *SIAM J. Math. Anal.*, vol. 13, pp. 167–179, 1982.

[35] S. S. Lin and C. W. Shih, "Complete stability for standard cellular neural networks," *Int. J. Bifurc. Chaos*, vol. 9, pp. 909–918, 1999.

[36] M. Forti and A. Tesi, "The Lojasiewicz exponent at an equilibrium point of a standard CNN is 1/2," *Int. J. Bifurc. Chaos*, vol. 16, no. 8, pp. 2191–2205, Aug. 2006.

[37] D. Hershkowitz, "Recent directions in matrix stability," *Linear Algebra Appl.*, vol. 171, pp. 161–186, 1992.

[38] M. Di Marco, M. Forti, M. Grazzini, and L. Pancioni, "On global exponential stability of standard and full-range CNNs," *Int. J. Circuit Theory Appl.*, vol. 36, no. 5/6, pp. 653–680, Jul. 2008.

[39] Y. Zhang, P. Heng, and A. Fu, "Estimate of exponential convergence rate and exponential stability for neural networks," *IEEE Trans. Neural Netw.*, vol. 10, no. 6, pp. 1487–1493, Nov. 1999.

[40] X. Liang and J. Si, "Global exponential stability of neural networks with globally Lipschitz continuous activations and its application to linear variational inequality problem," *IEEE Trans. Neural Netw.*, vol. 12, no. 2, pp. 349–359, Mar. 2001.

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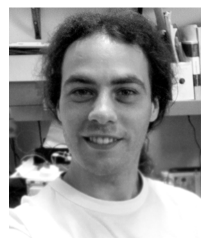

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