

An Averaging Method for Singularly Perturbed Systems with Hysteresis in Infinite Dimensional Spaces

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Submitted by V. Sree Hari Rao

Received on May 07, 2007, and in final form on July 15, 2007

Abstract

For a singular perturbation problem in infinite dimensional spaces with an hysteresis operator of Krasnosel'skii-Pokrovskii type we provide conditions ensuring the existence of periodic solutions for small values of the perturbation parameter $\epsilon > 0$ and we study their behavior as $\epsilon \rightarrow 0$. The problem is modelled by a system of semilinear equations whose linear parts are infinitesimal generators of C_0 -semigroups and the nonlinear operators are of high frequency with respect to time. The approach is based on the topological degree theory for condensing operators together with the averaging method.

⁰AMS (MOS) 2001 Mathematics Subject Classification: Primary: 34G20, 34D15, 34C29. Secondary: 47D03, 47J40, 47B99.

1. Introduction

Some of the authors have treated in [10] the problem of the existence and of the dependence on a small parameter $\epsilon > 0$ of the periodic solutions of a singularly perturbed system of semilinear parabolic equation with high frequency nonlinear inputs containing the output $w(\cdot)$ of an hysteresis operator Γ of Krasnosel'skii-Pokrovskii type, see e.g. M.A. Krasnosel'skii et al. [19] and Macki et al. [25], acting on the slow variable x . The linear parts were represented by infinitesimal generators of analytic semigroups whose inverse were assumed to be completely continuous. The aim of this paper is to deal with the same problem in the case when the linear operators are the infinitesimal generators of C_0 -semigroups. Specifically, the system under consideration has the form

$$\begin{cases} x'(\tau) + A_1x(\tau) = f_1(\tau/\epsilon, x(\tau), y(\tau), w(\tau)), \\ w(\tau) = \Gamma[0, Px_0, w_0]Px(\tau), \\ \epsilon y'(\tau) + A_2y(\tau) = f_2(\tau/\epsilon, x(\tau), y(\tau), w(\tau)), \end{cases} \tag{S}$$

where ϵ is a small positive parameter, $-A_1$ and $-A_2$ are infinitesimal generators of C_0 -semigroups of linear operators e^{-A_1t} and e^{-A_2t} , acting in the Banach spaces E_1 and E_2 , Γ is an hysteresis operator of Krasnosel'skii-Pokrovskii type whose output $w(t)$ belongs to a Banach space E_3 for every $t \geq 0$ and $(Px_0, w_0) \in E_0 \times E_3$, represents the initial state of the hysteresis operator, where $P : E_1 \rightarrow E_0$ is a bounded linear operator. The nonlinear operators $f_i, i = 1, 2$, are assumed to be T -periodic in the first variable and satisfying suitable conditions that will be precised later.

Although the approach in this paper closely follows that in [10], both the assumptions and the proofs of the relevant results are substantially different. In fact, in the present case due to the lack of compactness of $A_i^{-1}, i = 1, 2$, we are led to conditions on $e^{-A_i t}$ and $f_i, i = 1, 2$, ensuring the continuity and the condensivity, with respect to a suitably defined measure of compactness, of an operator $V_\epsilon^T : E_1 \times K \times E_2 \rightarrow E_1 \times K \times E_2$, $K \subset E_3$ compact, whose fixed points represent the initial conditions of periodic solutions to (S). In fact, firstly the problem of finding T -periodic solutions to (S) is converted into the problem of finding fixed points of $V_\epsilon^T, \epsilon > 0$. Then to show the existence of fixed points of V_ϵ^T , for $\epsilon > 0$ sufficiently small, we introduce for system (S) when $\epsilon = 0$ an averaging

operator $\Psi : Q \cap (E_1 \times K) \rightarrow E_1 \times K$, which turns out to be continuous and condensing. Here Q is a nonempty, open set of $E_1 \times E_3$. Moreover, for $\epsilon > 0$ sufficiently small, we prove that there exist linear admissible homotopies between V_ϵ^T and the averaging operator Ψ . Finally, under the assumption that the topological degree of $I - \Psi$ is different from zero, we infer the existence of a fixed point (x_0, w_0, y_0) of V_ϵ^T representing the initial condition of a T -periodic solution $(x_\epsilon, w_\epsilon, y_\epsilon)$ of system (S). Furthermore, we prove that for any sequence $\epsilon_n \rightarrow 0$ there exists a subsequence of $(x_{\epsilon_n}, w_{\epsilon_n}, y_{\epsilon_n})$ converging to some (x_*, w_*, y^*) as $n \rightarrow \infty$, where (x_*, w_*) is a fixed point of the averaging operator and y^* is a T -periodic solution of the third equation of (S) corresponding to $\epsilon = 1$ and (x_*, w_*) .

A similar approach was employed in Kamenskii et al. [11] and [12] to established the existence and the behavior of periodic solutions for singularly perturbed systems of semilinear differential inclusions with analytic and C_0 -semigroups respectively. Topological approaches based on the averaging method are widely employed to deal with singularly perturbed inclusions in finite dimensional spaces: Dontchev et al. [4]; Grammel [8]; Kamenskii et al. [14]; Quincampoix et al. [28] and some related optimal periodic control problems: Gaitsgory [6], [7] and Grammel [9]. Several papers are also devoted to the study of the existence of periodic and almost periodic solutions for ordinary and partial differential equations with hysteresis of various type, we cite among many others: Brokate et al. [2], [3]; Fečkan [5]; Koptová [15]; A.M. Krasnosel'skii [16] and A.M. Krasnosel'skii et al. [17]; M.A. Krasnosel'skii et al. [18]; Krejčí [21], [22], [23], [24]; Macki et al. [26]; Rachinskii [29]; Visintin [30] and Zubov [31]. The paper is organized as follows. In Section 2 we state the problem, we formulate the assumptions and we give some definitions and preliminary results. In Section 3 we formulate and we prove our main result: Theorem 1.

2. Formulation of the problem, assumptions and definitions

In this paper we consider the following system

$$\begin{cases} x'(\tau) + A_1 x(\tau) = f_1(\tau/\varepsilon, x(\tau), y(\tau), w(\tau)), \\ w(\tau) = \Gamma[0, Px_0, w_0]Px(\tau), \\ \varepsilon y'(\tau) + A_2 y(\tau) = f_2(\tau/\varepsilon, x(\tau), y(\tau), w(\tau)), \end{cases} \quad (S)$$

where ε is a small positive parameter, $-A_1$ and $-A_2$ are generators of C_0 -semigroups of linear operators $e^{-A_1 t}$ and $e^{-A_2 t}$, acting in the Banach spaces E_1 and E_2 respectively, see for instance Pazy [27], Γ is an hysteresis nonlinearity of Krasnosel'ski-Pokrovskii type, (Px_0, w_0) is the initial state of the hysteresis operator belonging to $E_0 \times E_3$, E_0 and E_3 Banach spaces, $P : E_1 \rightarrow E_0$ is a bounded linear operator. The nonlinear operators $f_i, i = 1, 2$ are assumed T -periodic in the first variable.

We aim at solving the following problem.

Problem: to give conditions under which, for $\varepsilon > 0$ sufficiently small, system (S) admits εT -periodic solutions and to study their behavior as $\varepsilon \rightarrow 0$.

Our main result: Theorem 1 of Section 3 provides the solution of this problem under the following conditions:

F_1) The nonlinear operators $f_i : \mathbf{R}_+ \times E_1 \times E_2 \times E_3 \rightarrow E_i, i = 1, 2$, are continuous and satisfy a Lipschitz condition with respect to the variables $(x, y, w) \in E_1 \times E_2 \times E_3$, that is for any $R > 0$ there exists $L(R) > 0$ such that

$$\begin{aligned} & \|f_i(t, x_1, y_1, w_1) - f_i(t, x_2, y_2, w_2)\| \\ & \leq L(R)(\|x_1 - x_2\| + \|y_1 - y_2\| + \|w_1 - w_2\|), \end{aligned}$$

$i = 1, 2, \|x_j\| \leq R, \|y_j\| \leq R, \|w_j\| \leq R, j = 1, 2, t \in [0, T]$.

F_2) For every $R > 0$ there exists a positive constant ρ_R such that

$$\|f_i(t, x, y, w)\| \leq \rho_R(1 + \|x\| + \|y\|),$$

where $\|w\| \leq R, x \in E_1, y \in E_2, i = 1, 2$, and $t \in [0, T]$.

F_3) There exist nonnegative constants $m_{ij}, i, j = 1, 2$, such that for any pair of bounded sets $\Omega_1 \subset E_1, \Omega_2 \subset E_2$ and any compact set $K \subset E_3$ the following inequalities hold

$$\chi_{E_1}(f_i([0, T], \Omega_1, \Omega_2, K)) \leq m_{i1} \chi_{E_1}(\Omega_1) + m_{i2} \chi_{E_2}(\Omega_2),$$

where $\chi_{E_i}, i = 1, 2$, denotes the Hausdorff measure of noncompactness in the space $E_i, i = 1, 2$, respectively.

A) There exist positive constants $\gamma_i, i = 1, 2$, such that

$$\|e^{-A_i t}\|(\alpha) \leq e^{-\gamma_i t}, t > 0, i = 1, 2,$$

and

$$M) \rho(\Lambda^{-1}M) < 1, \text{ with } \Lambda^{-1} = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}^{-1}, M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}.$$

Here and in the following $\rho(H)$ denotes the spectral radius of the matrix H and $\|L\|(\alpha) := \chi(LS)$, where L is a bounded linear operator from a Banach space E into itself and S is the unit sphere in E .

The operator $\Gamma : \mathbf{R}_+ \times E_0 \times E_3 \times C([t_0, t], E_0) \rightarrow C([t_0, t], E_3)$ is a Krasnosel'ski-Pokrovskii hysteresis operator. For every fixed $t_0 \geq 0$ the operator $\Gamma[t_0, \cdot, \cdot](\cdot)$ is defined on the set

$$\begin{aligned} D_{t_0}(\Gamma) &= \{(x_*, w_0, x^*) \in E_0 \times E_3 \times C([t_0, t], E_0) : \\ & (x_*, w_0) \in \Omega_\Gamma, x^*(t_0) = x_*\}, \end{aligned}$$

where Ω_Γ is the domain of all the possible states of the hysteresis operator. For the structure of the set Ω_Γ we refer to M.A. Krasnosel'skii et al. [19].

We assume that the operator Γ satisfies the following conditions:

C_1) For any $t_0 \geq 0$ the operator $\Gamma[t_0, \cdot, \cdot](\cdot)$ acts from $E_0 \times E_3 \times C([t_0, t], E_0)$ to $C([t_0, t], E_3)$, $t \geq t_0$, and it is constant on constant functions.

C_2) For any $t_0 \geq 0$ and any $(x_*, w_0, x^*) \in D_{t_0}(\Gamma)$ we have that

$$\Gamma[t_0, x_*, w_0]x^*(t_0) = w_0.$$

C₃) The nonlinearity Γ satisfies, for $t_0 \leq t_1 \leq t$, the semigroup property

$$\Gamma[t_0, x_*, w_0]x^*(t) = \Gamma[t_1, x^*(t_1), \Gamma[t_0, x_*, w_0]x^*(t_1)]x^*(t).$$

C₄) Let $t_0 \geq 0$, $(x_*, w_0, x^*) \in D_{t_0}(\Gamma)$, $v(t) = x^*(t + t_0 - t_1)$ for $t \geq t_1$. Then $(x_*, w_0, v) \in D_{t_1}(\Gamma)$ and

$$\Gamma[t_0, x_*, w_0]x^*(t) = \Gamma[t_1, x_*, w_0]v(t - t_0 + t_1), \quad t \geq t_0.$$

C₅) Let $t_0 \geq 0$, $\beta > 0$, $(x_*, w_0, x^*) \in D_{t_0}(\Gamma)$, $v(t) = x^*(\beta t + (1 - \beta)t_0)$ for $t \geq t_0$. Then $(x_*, w_0, v) \in D_{t_0}(\Gamma)$ and

$$\Gamma[t_0, x_*, w_0]v(t) = \Gamma[t_0, x_*, w_0]x^*(\beta t + (1 - \beta)t_0), \quad t \geq t_0.$$

C₆) For any $t_0 \geq 0$, the operator Γ satisfies the following Lipschitz condition: there exists $k > 0$ such that

$$\begin{aligned} \max_{s \in [t_0, t]} \|\Gamma[t_0, x_1, w_1]x^1(s) - \Gamma[t_0, x_2, w_2]x^2(s)\| &\leq k(\|x_1 - x_2\|_{E_0} \\ &+ \|w_1 - w_2\|_{E_3} + \|x^1 - x^2\|_{C([t_0, t], E_0)}), \end{aligned}$$

*

whenever $t \geq t_0$.

C₇) There exist a nonempty convex compact set $K \subset E_3$, a nonempty open set $Q \subset E_1 \times E_3$ such that for any $(x_0, w_0) \in Q \cap (E_1 \times K)$, $t_0 \in \mathbb{R}_+$, we have $(Px_0, w_0) \in \Omega_T$ and $\Gamma[t_0, Px_0, w_0]Px^0(t) \in K$, $t \geq t_0$, for any $x^0 \in C([t_0, t], E_1)$, such that $x^0(t_0) = x_0$.

By the change of variable $\tau/\varepsilon = t$ system (S) takes the form

$$\begin{cases} \bar{x}'(t) + \varepsilon A_1 \bar{x}(t) = \varepsilon f_1(t, \bar{x}(t), \bar{y}(t), \bar{w}(t)), \\ \bar{w}(t) = \Gamma[0, Px_0, w_0]P\bar{x}(t), \\ \bar{y}(t) + A_2 \bar{y}(t) = f_2(t, \bar{x}(t), \bar{y}(t), \bar{w}(t)). \end{cases} \quad (S')$$

Denoting $(\bar{x}, \bar{w}, \bar{y})$ again by (x, w, y) we can write

$$\begin{cases} x'(t) + \varepsilon A_1 x(t) = \varepsilon f_1(t, x(t), y(t), w(t)), \\ w(t) = \Gamma[0, Px_0, w_0]Px(t), \\ y'(t) + A_2 y(t) = f_2(t, x(t), y(t), w(t)). \end{cases} \quad (1)$$

Following M.A. Krasnosel'skii et al. [20], we say that $(x_\varepsilon, w_\varepsilon, y_\varepsilon)$ is a mild solution of (1) on the interval $[0, T]$ with the initial conditions

$$x(0) = x_0, \quad w(0) = w_0, \quad y(0) = y_0 \quad (2)$$

if $x_\varepsilon, w_\varepsilon, y_\varepsilon$ are continuous functions, defined on the interval $[0, T]$ with values in E_1, E_3, E_2 respectively, satisfying the equations

$$x_\varepsilon(t) = e^{-\varepsilon A_1 t} x_0 + \int_0^t e^{-\varepsilon A_1(t-s)} \varepsilon f_1(s, x_\varepsilon(s), y_\varepsilon(s), w_\varepsilon(s)) ds, \quad (3)$$

$$w_\varepsilon(t) = \Gamma[0, Px_0, w_0]Px_\varepsilon(t), \quad (4)$$

$$y_\varepsilon(t) = e^{-A_2 t} y_0 + \int_0^t e^{-A_2(t-s)} f_2(s, x_\varepsilon(s), y_\varepsilon(s), w_\varepsilon(s)) ds. \quad (5)$$

By a T-periodic solution of (1) we mean a continuous T-periodic function $(x_\varepsilon, w_\varepsilon, y_\varepsilon)$ satisfying (3), (4) and (5).

Definition 1. (The quasitranslation operator) Let $K \subset E_3$ be the nonempty convex compact set of C₇). For any $x_0 \in E_1, w_0 \in K, y_0 \in E_2$ and any $\varepsilon > 0$ we assume that there exists an unique solution $(x_\varepsilon, w_\varepsilon, y_\varepsilon)$ on $[0, T]$ of system (3), (4), (5) with initial condition (x_0, w_0, y_0) . The quasitranslation operator $V_\varepsilon^T: E_1 \times K \times E_2 \rightarrow E_1 \times K \times E_2$ is defined as follows

$$\begin{aligned} V_\varepsilon^T(x_0, w_0, y_0) &= \\ &= \left((I - e^{-\varepsilon A_1 T})^{-1} \int_0^T e^{-\varepsilon A_1(T-s)} \varepsilon f_1(s, x_\varepsilon(s), y_\varepsilon(s), w_\varepsilon(s)) ds, \right. \\ &\quad \left. w_\varepsilon(T), (I - e^{-A_2 T})^{-1} \int_0^T e^{-A_2(T-s)} f_2(s, x_\varepsilon(s), y_\varepsilon(s), w_\varepsilon(s)) ds \right). \end{aligned}$$

Observe that, without loss of generality, we can assume that the semi-groups of linear operators $e^{-A_i t}, t \geq 0$, satisfy the estimates

$$\|e^{-A_i t}\| \leq ce^{-dt}, \quad t \geq 0, \quad i = 1, 2, \quad (6)$$

for some $d > 0$. In fact, it is possible to reduce ourselves to this case by adding and subtracting suitable linear operators to A_1 and A_2 , and observing that the resulting nonlinearities still satisfy F₁) - F₃). This also shows that the linear operators $(I - e^{-\varepsilon A_1 T})^{-1}$ and $(I - e^{-A_2 T})^{-1}$ are defined.

Remark 1 In Proposition 2 we will prove that our assumptions guarantee that system (3), (4) and (5) has a unique solution defined on $[0, T]$ for every $x_0 \in E_1, w_0 \in K, y_0 \in E_2$ and $\epsilon > 0$.

The following result holds true.

Proposition 1 Any fixed point (x_0, w_0, y_0) of the operator V_ϵ^T represents the initial conditions $x(0) = x_0, w(0) = w_0$ and $y(0) = y_0$ of a T -periodic solution $(x_\epsilon, w_\epsilon, y_\epsilon)$ of system (1).

Proof. Let $(x_0, w_0, y_0) = V_\epsilon^T(x_0, w_0, y_0)$, then

$$\begin{aligned} x_0 &= (I - e^{-\epsilon A_1 T})^{-1} \int_0^T e^{-\epsilon A_1(T-s)} \epsilon f_1(s, x_\epsilon(s), y_\epsilon(s), w_\epsilon(s)) ds, \\ w_0 &= \Gamma[0, Px_0, w_0] P x_\epsilon(T), \\ y_0 &= (I - e^{-A_2 T})^{-1} \int_0^T e^{-A_2(T-s)} f_2(s, x_\epsilon(s), y_\epsilon(s), w_\epsilon(s)) ds, \end{aligned}$$

where $(x_\epsilon, w_\epsilon, y_\epsilon)$ satisfy (3), (4) and (5). Therefore,

$$\begin{aligned} x_0 &= e^{-\epsilon A_1 T} x_0 + \int_0^T e^{-\epsilon A_1(T-s)} \epsilon f_1(s, x_\epsilon(s), y_\epsilon(s), w_\epsilon(s)) ds = x_\epsilon(T), \\ w_0 &= \Gamma[0, Px_0, w_0] P x_\epsilon(T) = w_\epsilon(T), \\ y_0 &= e^{-A_2 T} y_0 + \int_0^T e^{-A_2(T-s)} f_2(s, x_\epsilon(s), y_\epsilon(s), w_\epsilon(s)) ds = y_\epsilon(T). \end{aligned}$$

Denote by $\tilde{x}_\epsilon, \tilde{w}_\epsilon, \tilde{y}_\epsilon$ the T -periodic extensions of the functions $x_\epsilon, w_\epsilon, y_\epsilon$ respectively. Let $t \in [T, 2T]$, then by C_3 we have

$$\begin{aligned} \Gamma[0, Px_0, w_0] P \tilde{x}_\epsilon(t) &= \Gamma[T, P \tilde{x}_\epsilon(T), \Gamma[0, Px_0, w_0] P \tilde{x}_\epsilon(T)] P \tilde{x}_\epsilon(t) \\ &= \Gamma[T, Px_0, w_0] P \tilde{x}_\epsilon(t) = \Gamma[0, Px_0, w_0] P x_\epsilon(t - T) = w_\epsilon(t - T) = \tilde{w}_\epsilon(t). \end{aligned}$$

By repeating the same arguments for $t \in [(n+1)T, (n+2)T], n \in N$, we obtain

$$\tilde{w}_\epsilon(t) = \Gamma[0, Px_0, w_0] P \tilde{x}_\epsilon(t)$$

for all $t \geq 0$. Now, by using standard arguments, see e.g. M.A. Krasnoselskii et al. [20], it is easy to show that $\tilde{x}_\epsilon, \tilde{y}_\epsilon$ satisfy (3) and (5) for all $t \geq 0$.

Remark 2 The viceversa is also true. In fact, let $(x_\epsilon, w_\epsilon, y_\epsilon)$ be a T -periodic solution of (1). Let $x_\epsilon(0) = x_0, w_\epsilon(0) = w_0, y_\epsilon(0) = y_0$, then (x_0, w_0, y_0) is a fixed point of the operator V_ϵ^T . Hence, the problem of finding T -periodic solutions of system (1) is equivalent to that of finding fixed points of the operator V_ϵ^T .

Let us consider the measure of noncompactness X in $E_1 \times E_3 \times E_2$ defined by

$$X(\Omega) = (\chi_{E_1}(\Omega_1), \chi_{E_3}(\Omega_3), \chi_{E_2}(\Omega_2)),$$

with $\Omega \subset E_1 \times E_3 \times E_2$ bounded, $\Omega_i = \text{pr}_i(\Omega)$ and $\text{pr}_i : E_1 \times E_3 \times E_2 \rightarrow E_i, i = 1, 2, 3$, are the natural projections. We have the following result.

Proposition 2 Assume that $A), M), F_1) - F_3), C_1) - C_7)$ are satisfied. Then, for $\epsilon > 0$, the operator V_ϵ^T is well-defined, continuous and condensing with respect to the measure of noncompactness X .

Proof. We first show that V_ϵ^T is X -condensing. For this, let $\Omega \subset E_1 \times K \times E_2$ be a bounded set for which

$$X(\Omega) \leq X(V_\epsilon^T(\Omega)). \tag{7}$$

We will show that Ω is a relatively compact set. Let $Z_\epsilon(\Omega)$ be the set of solutions of (3), (4), (5) with initial conditions $(x_0, w_0, y_0) \in \Omega$. Let

$$(\varphi_1(t), \varphi_3(t), \varphi_2(t)) = (\chi_{E_1}(\{x_\epsilon(t)\}), \chi_{E_3}(\{w_\epsilon(t)\}), \chi_{E_2}(\{y_\epsilon(t)\})),$$

where $(x_\epsilon, w_\epsilon, y_\epsilon) \in Z_\epsilon(\Omega)$. Then from $A)$, and $F_3)$ we obtain the estimates

$$\varphi_1(t) \leq e^{-\epsilon \gamma_1 t} \chi_{E_1}(\Omega_1) + \int_0^t e^{-\epsilon \gamma_1(t-s)} \epsilon (m_{11} \varphi_1(s) + m_{12} \varphi_2(s)) ds \tag{8}$$

$$\varphi_2(t) \leq e^{-\gamma_2 t} \chi_{E_2}(\Omega_2) + \int_0^t e^{-\gamma_2(t-s)} (m_{21} \varphi_1(s) + m_{22} \varphi_2(s)) ds, \tag{9}$$

where $\Omega_i = \text{pr}_i(\Omega) \subset E_i, i = 1, 2$. Let

$$\Lambda_\epsilon = \begin{pmatrix} \epsilon \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}, \quad \tilde{X}(\Omega) = \begin{pmatrix} \chi_{E_1}(\Omega_1) \\ \chi_{E_2}(\Omega_2) \end{pmatrix},$$

$$\varphi(t) = \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}, \quad M_\varepsilon = \begin{pmatrix} \varepsilon m_{11} & \varepsilon m_{12} \\ m_{21} & m_{22} \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}, \quad M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}.$$

Then (8), and (9) can be rewritten as

$$\varphi(t) \leq e^{-\Lambda_\varepsilon t} \tilde{X}(\Omega) + \int_0^t e^{-\Lambda_\varepsilon(t-s)} M_\varepsilon \varphi(s) ds. \tag{10}$$

By the Gronwall inequality we get

$$\varphi(t) \leq e^{(-\Lambda_\varepsilon + M_\varepsilon)t} \tilde{X}(\Omega). \tag{11}$$

Furthermore, we have that

$$X(V_\varepsilon^T(\Omega)) =$$

$$= \begin{pmatrix} \chi_{E_1}(\{(I - e^{-\varepsilon A_1 T})^{-1} \int_0^T e^{-\varepsilon A_1(T-s)} \varepsilon f_1(s, x_\varepsilon(s), y_\varepsilon(s), w_\varepsilon(s)) ds\}), \\ \chi_{E_2}(\{w_\varepsilon(T)\}), \\ \chi_{E_2}(\{(I - e^{-A_2 T})^{-1} \int_0^T e^{-A_2(T-s)} f_2(s, x_\varepsilon(s), y_\varepsilon(s), w_\varepsilon(s)) ds\}). \end{pmatrix}$$

Since $w_\varepsilon(t) \in K$ for all $t \in [0, T]$, then $\chi_{E_2}(\{w_\varepsilon(T)\}) = 0$ and from (7) it follows that $\chi_{E_2}(\Omega_2) = 0$. Moreover, from A) and F_3) it follows

$$\chi_{E_1}(\{(I - e^{-\varepsilon A_1 T})^{-1} \int_0^T e^{-\varepsilon A_1(T-s)} \varepsilon f_1(s, x_\varepsilon(s), y_\varepsilon(s), w_\varepsilon(s)) ds\})$$

$$\leq \frac{1}{(1 - e^{-\varepsilon \eta T})} \int_0^T e^{-\varepsilon \eta(T-s)} \varepsilon (m_{11} \varphi_1(s) + m_{12} \varphi_2(s)) ds,$$

and

$$\chi_{E_2}(\{(I - e^{-A_2 T})^{-1} \int_0^T e^{-A_2(T-s)} f_2(s, x_\varepsilon(s), y_\varepsilon(s), w_\varepsilon(s)) ds\})$$

$$\leq \frac{1}{(1 - e^{-\gamma_2 T})} \int_0^T e^{-\gamma_2(T-s)} (m_{21} \varphi_1(s) + m_{22} \varphi_2(s)) ds.$$

Denote the left hand sides of these two inequalities by ξ_1 and ξ_2 respectively and let $\xi = (\xi_1, \xi_2)$ thus

$$\xi \leq (I - e^{-\Lambda T})^{-1} \int_0^T e^{-\Lambda_\varepsilon(T-s)} M_\varepsilon \varphi(s) ds.$$

Using inequality (11) we get

$$\xi \leq (I - e^{-\Lambda_\varepsilon T})^{-1} \int_0^T e^{-\Lambda_\varepsilon(T-s)} M_\varepsilon e^{(-\Lambda_\varepsilon + M_\varepsilon)s} \tilde{X}(\Omega) ds$$

$$= (I - e^{-\Lambda_\varepsilon T})^{-1} (e^{(-\Lambda_\varepsilon + M_\varepsilon)T} - e^{-\Lambda_\varepsilon T}) \tilde{X}(\Omega). \tag{12}$$

We now show that all the elements of the matrix $e^{(-\Lambda_\varepsilon + M_\varepsilon)T} - e^{-\Lambda_\varepsilon T}$ are nonnegative. For this consider

$$e^{(-\Lambda_\varepsilon + M_\varepsilon)T} - e^{-\Lambda_\varepsilon T} = \int_0^T e^{-\Lambda_\varepsilon(T-s)} M_\varepsilon e^{(-\Lambda_\varepsilon + M_\varepsilon)s} ds, \tag{13}$$

since the elements of the matrix $e^{-\Lambda_\varepsilon(T-s)} M_\varepsilon$ are nonnegative, it is sufficient to prove that $e^{(-\Lambda_\varepsilon + M_\varepsilon)t}$, $t \geq 0$, is a matrix with nonnegative elements.

To this aim, let $z_0 = (1, 0) \in \mathbf{R}^2$, and define a sequence of functions $z_n(t)$, $n \geq 1$, as follows

$$z_n(t) = e^{-\Lambda_\varepsilon t} z_0 + \int_0^t e^{-\Lambda_\varepsilon(t-s)} M_\varepsilon z_{n-1}(s) ds. \tag{14}$$

Since the elements of z_0 , $e^{-\Lambda_\varepsilon(t-s)}$ and M_ε are nonnegative, then the components of $z_n(t)$ are nonnegative for all $t \in [0, T]$. Passing to the limit as $n \rightarrow \infty$ in (14) we obtain

$$z(t) = e^{-\Lambda_\varepsilon t} z_0 + \int_0^t e^{-\Lambda_\varepsilon(t-s)} M_\varepsilon z(s) ds, \quad t \in [0, T],$$

where $z = \lim_{n \rightarrow \infty} z_n$. Then

$$z(t) = e^{(-\Lambda_\varepsilon + M_\varepsilon)t} z_0.$$

Since $z_0 = (1, 0)$, the right hand side of this equality is the first column of the matrix $e^{(-\Lambda_\varepsilon + M_\varepsilon)t}$. On the other hand, the components of

$z_n(t)$ are nonnegative, hence the first column of the matrix $e^{(-\Lambda_\epsilon + M_\epsilon)t}$ is a vector with nonnegative elements. The nonnegativity of the second column of $e^{(-\Lambda_\epsilon + M_\epsilon)t}$ can be proved as before by replacing z_0 by the vector $z^0 = (0, 1)$. Thus, from (13) it follows that the elements of the matrix $e^{(-\Lambda_\epsilon + M_\epsilon)T} - e^{-\Lambda_\epsilon T}$ are nonnegative.

From (7) it follows that $\bar{X}(\Omega) \leq \xi$. Since the elements of the matrix

$$(I - e^{-\Lambda_\epsilon T})^{-1}(e^{(-\Lambda_\epsilon + M_\epsilon)T} - e^{-\Lambda_\epsilon T})$$

are nonnegative, we have

$$\xi \leq (I - e^{-\Lambda_\epsilon T})^{-1}(e^{(-\Lambda_\epsilon + M_\epsilon)T} - e^{-\Lambda_\epsilon T})\xi. \tag{15}$$

We now show that

$$\rho((I - e^{-\Lambda_\epsilon T})^{-1}(e^{(-\Lambda_\epsilon + M_\epsilon)T} - e^{-\Lambda_\epsilon T})) < 1. \tag{16}$$

Let

$$C_\epsilon^{-1} = (I - e^{-\Lambda_\epsilon T})^{-1}, \quad D_\epsilon = e^{(-\Lambda_\epsilon + M_\epsilon)T} - e^{-\Lambda_\epsilon T}.$$

Proposition 5 of Kamenskii et al. [12] shows that condition (16) is equivalent to the exponential stability of $-C_\epsilon + D_\epsilon$, where

$$-C_\epsilon + D_\epsilon = (-I + e^{-\Lambda_\epsilon T} + e^{(-\Lambda_\epsilon + M_\epsilon)T} - e^{-\Lambda_\epsilon T}) = e^{(-\Lambda_\epsilon + M_\epsilon)T} - I.$$

Since $\Lambda_\epsilon^{-1}M_\epsilon = \Lambda^{-1}M$, by condition M) the spectral radius of $\Lambda_\epsilon^{-1}M_\epsilon$ satisfies the following inequality

$$\rho(\Lambda_\epsilon^{-1}M_\epsilon) < 1.$$

Hence, the matrix $-\Lambda_\epsilon + M_\epsilon$ is exponentially stable. Furthermore,

$$\sigma(e^{(-\Lambda_\epsilon + M_\epsilon)T}) = e^{\sigma((- \Lambda_\epsilon + M_\epsilon)T)} = e^{T\sigma(-\Lambda_\epsilon + M_\epsilon)},$$

where $\sigma(H)$ denotes the spectrum of the matrix H . Let λ be an element of $\sigma(e^{(-\Lambda_\epsilon + M_\epsilon)T})$, then $\lambda = e^{T(\lambda_1 + i\lambda_2)}$, where $\lambda_1 + i\lambda_2 \in \sigma(-\Lambda_\epsilon + M_\epsilon)$.

Since $\lambda_1 < 0$, we have

$$\|\lambda\| = e^{T\lambda_1} < 1,$$

thus $\rho(e^{(-\Lambda_\epsilon + M_\epsilon)T}) < 1$ and so $Re(\sigma(e^{(-\Lambda_\epsilon + M_\epsilon)T} - I)) < 0$, namely the matrix $-C_\epsilon + D_\epsilon$ is exponentially stable and inequality (16) is proved.

Let $\mathcal{T}_\epsilon = (I - e^{-\Lambda_\epsilon T})^{-1}(e^{(-\Lambda_\epsilon + M_\epsilon)T} - e^{-\Lambda_\epsilon T})$, thus the vector $\xi = (\xi_1, \xi_2)$ with nonnegative coordinates satisfies the inequality

$$\xi \leq \mathcal{T}_\epsilon \xi. \tag{17}$$

From (16) we have that the eigenvalues of \mathcal{T}_ϵ belongs to the unit circle. Therefore, (17) implies that $\xi = 0$ and from (7) it follows that $\bar{X}(\Omega) = 0$ that is Ω is relatively compact.

Finally, the continuity of the operator V_ϵ^T follows from the Lipschitz conditions F_1), C_6) and the estimate (6). This completes the proof.

In Theorem 1 of Section 3 we will formulate conditions which ensure the existence of T -periodic solutions for system (1) in terms of the averaging operator introduced by Definition 2 below. In order to define the averaging operator we need the following conditions.

D_1) For any $x \in E_1, w \in K$ and $\lambda \in [0, 1]$ there exists an unique solution $\hat{y}_{x,w}^\lambda$ of the integral equation

$$y(t) = e^{-\Lambda_2 t} (I - e^{-\Lambda_2 T})^{-1} \lambda \int_0^T e^{-\Lambda_2(T-s)} f_2(s, x, y(s), w) ds + \lambda \int_0^t e^{-\Lambda_2(t-s)} f_2(s, x, y(s), w) ds, \quad t \geq 0.$$

D_2) Let $G \subset E_1 \times E_3$ be a bounded set and let

$$\hat{y}_G^{[0,1]} = C_T(E_2) \cap \{ \hat{y}_{x,w}^\lambda : (x, w, \lambda) \in G \times [0, 1] \}.$$

For any nonempty, bounded set $G \subset E_1 \times E_3$ the set $\hat{y}_G^{[0,1]}$ is bounded.

Here and in the sequel $C_T(E_i)$ denotes the Banach space of the T -periodic functions with values in the Banach space E_i .

Definition 2. (The averaging operator) Given a nonempty open set $Q \subset E_1 \times E_3$, the averaging operator $\Psi : Q \cap (E_1 \times K) \rightarrow E_1 \times K$ is defined as follows

$$\Psi(x, w) = (F(x, w), w),$$

where

$$F(x, w) = A_1^{-1} \frac{1}{T} \int_0^T f_1(s, x, y(s), w) ds, \quad y = \hat{y}_{x,w}^1.$$

280 A. GUDOVICH, M. KAMENSKII, P. NISTRI AND M. QUINCAMPOIX

Let $\Omega \subset E_1 \times K$ be a bounded set. We define the measure of noncompactness β in $E_1 \times K$, as follows

$$\beta(\Omega) = (\chi_{E_1}(\Omega_1), \chi_{E_3}(\Omega_3)),$$

where $\Omega_i = \text{pr}_i(\Omega)$, $i = 1, 3$.

We can prove the following result.

Proposition 3 *Assume that conditions A), M), $F_1) - F_3)$, $D_1)$, $D_2)$ are satisfied. Then the operator Ψ is continuous and condensing with respect to the measure of noncompactness β .*

Proof. Let $\Omega \subset E_1 \times K$ be a bounded set for which

$$\beta(\Omega) \leq \beta(\Psi(\Omega)). \quad (18)$$

We will show that Ω is a relatively compact set. For this, let

$$\eta_1 = \chi_{E_1}(\Omega_1), \quad \psi(t) = \chi_{E_2}(\{y(t), u \in \hat{y}_\Omega^1\}), \quad \eta_2 = \sup_{t \in [0, T]} \psi(t).$$

Since

$$\Psi(\Omega) = \left\{ \left(A_1^{-1} \frac{1}{T} \int_0^T f_1(s, x, y(s), w) ds, w \right) : (x, w) \in \Omega, u \in \hat{y}_\Omega^1 \right\},$$

from $F_3)$ and A) we have

$$\begin{aligned} & \chi_{E_1}(\{A_1^{-1} \frac{1}{T} \int_0^T f_1(s, x, y(s), w) ds\}) \\ & \leq \|A_1^{-1}\|^{(x)} \frac{1}{T} \int_0^T m_{11} \chi_{E_1}(\Omega_1) + m_{12} \chi_{E_2}(\{y(s)\}) ds \\ & \leq \frac{1}{\gamma_1} (m_{11} \eta_1 + m_{12} \eta_2). \end{aligned}$$

Hence, by using inequality (18), we get

$$\eta_1 \leq \frac{m_{11}}{\gamma_1} \eta_1 + \frac{m_{12}}{\gamma_1} \eta_2. \quad (19)$$

Moreover, from $F_3)$ and A) we have

$$\begin{aligned} \psi(t) & \leq e^{-\gamma_2 t} (1 - e^{-\gamma_2 T})^{-1} \int_0^T e^{-\gamma_2(T-s)} (m_{21} \eta_1 + m_{22} \eta_2) ds \\ & \quad + \int_0^t e^{-\gamma_2(t-s)} (m_{21} \eta_1 + m_{22} \eta_2) ds = \frac{m_{21}}{\gamma_2} \eta_1 + \frac{m_{22}}{\gamma_2} \eta_2; \end{aligned}$$

and then

$$\eta_2 \leq \frac{m_{21}}{\gamma_2} \eta_1 + \frac{m_{22}}{\gamma_2} \eta_2. \quad (20)$$

From (19) and (20) we get that the vector $\eta = (\eta_1, \eta_2)$ with nonnegative coordinates satisfies the inequality

$$\eta \leq \Lambda^{-1} M \eta. \quad (21)$$

From M) it follows that the eigenvalues of $\Lambda^{-1} M$ belong to the unit circle. Therefore (21) implies that $\eta = 0$ and so $\eta_1 = \chi_{E_1}(\Omega_1) = 0$, that is Ω_1 is relatively compact. The relative compactness of the set Ω_3 follows from the compactness of K .

3. Main result

In order to prove our main result we need a stronger condition on the semigroup $e^{-A_1 t}$, $t \geq 0$. Specifically, we assume that

$$A_1) \quad \|e^{-A_1 t}\| \leq e^{-\gamma_1 t}, \quad t \geq 0, \quad \gamma_1 > 0.$$

That is the semigroup $e^{-A_1 t}$ is contractive.

Given $\varepsilon > 0$ and any bounded set $\Omega \subset E_1 \times K \times E_2$, we recall that the solution set $Z_\varepsilon(\Omega)$ is defined as follows

$$\begin{aligned} Z_\varepsilon(\Omega) & = \{(x_\varepsilon, w_\varepsilon, y_\varepsilon) \in C([0, T], E_1 \times E_3 \times E_2) : (x_\varepsilon, w_\varepsilon, y_\varepsilon) \text{ is a solution} \\ & \text{of system (3), (4), (5) with initial conditions } (x_0, w_0, y_0) \in \Omega\}. \end{aligned}$$

We are now in the position to prove the main result of this paper.

Theorem 1. Assume that $A), A_1), F_1) - F_3), C_1) - C_T), D_1), D_2), M)$ are satisfied and assume also that for every nonempty bounded set $\Omega \subset E_1 \times K \times E_2$ the set $Z_\epsilon(\Omega)$ is bounded by a constant independent of ϵ . Assume that there exists an open ball $B \subset Q$, such that the equation $\Psi(x, w) = (x, w)$ has no solution on $\partial(B \cap (E_1 \times K))$ and

$$\text{deg}(I - \Psi, B \cap (E_1 \times K), 0) \neq 0.$$

Then, for sufficiently small $\epsilon > 0$, system (1) has at least one T -periodic solution $(x_\epsilon, w_\epsilon, y_\epsilon) \in C_T(E_1) \times C_T(E_3) \times C_T(E_2)$. Moreover, for every sequence $\epsilon_n \rightarrow 0$, as $n \rightarrow \infty$, the sequence $\{(x_{\epsilon_n}, w_{\epsilon_n}, y_{\epsilon_n})\}$ is relatively compact and the limit points (x_*, w_*, y^*) of this sequence are such that (x_*, w_*) is a fixed point of the operator Ψ and $y^* = \hat{y}_{x_*, w_*}^1$.

Proof. Define the auxiliary operator $F_2 : E_1 \times K \times E_2 \rightarrow E_2$ as follows

$$F_2(x, w, y) = \bar{y}(T^1),$$

where $\bar{y}(\cdot)$ is the solution of

$$y'(t) = e^{-A_2 t} y + \lambda \int_0^t e^{-A_2(t-s)} f_2(s, x, y(s), w) ds, \tag{22}$$

corresponding to $\lambda = 1$.

From $D_2)$ it follows that there exists $r > 0$ such that $\|y\|_{C_T(E_2)} < r$ for all $y \in \hat{y}_{(B \cap (E_1 \times K))}^1$. Let $B_{E_2}(0, r) = \{y \in E_2 : \|y\|_{E_2} < r\}$.

As shown in Borisovich et al. [1], there exists a continuous compact operator $\hat{\Psi} : (B \cap (E_1 \times K)) \rightarrow E_1 \times K$ which is linearly homotopic to $\hat{\Psi}$ on $(B \cap (E_1 \times K))$.

Consider the operator $\Psi^* : (B \cap (E_1 \times K)) \times B_{E_2}(0, r) \rightarrow E_1 \times K \times E_2$ defined by

$$\Psi^*(x, w, y) = (\hat{\Psi}_1(x, w), \hat{\Psi}_2(x, w), F_2(x, w, y)),$$

where $\hat{\Psi}_1(x, w), \hat{\Psi}_2(x, w)$, are the projections of $\hat{\Psi}(x, w)$ on the spaces E_1 and E_3 respectively.

We now prove that, for $\epsilon > 0$ sufficiently small, V_ϵ^T and Ψ^* are linearly homotopic on $\partial((B \cap (E_1 \times K)) \times B_{E_2}(0, r))$.

Assume the contrary, then there exist sequences $\epsilon_n \rightarrow 0, \lambda_n \rightarrow \lambda_0, \lambda_n \in [0, 1]$ and $\{(x_n, w_n, y_n)\} \in \partial((B \cap (E_1 \times K)) \times B_{E_2}(0, r))$ such that

$$x_n = \lambda_n (I - e^{-\epsilon_n A_1 T})^{-1} \int_0^T e^{-\epsilon_n A_1 (T-s)} \epsilon_n v_1^n(s) ds + (1 - \lambda_n) \hat{\Psi}_1(x_n, w_n), \tag{23}$$

$$w_n = \lambda_n w^n(T) + (1 - \lambda_n) \hat{\Psi}_2(x_n, w_n), \tag{24}$$

$$y_n = \lambda_n (I - e^{-A_2 T})^{-1} \int_0^T e^{-A_2 (T-s)} v_2^n(s) ds + (1 - \lambda_n) (e^{-A_2 T} y_n + \int_0^T e^{-A_2 (T-s)} v_2^n(s) ds), \tag{25}$$

where

$$v_i^n(s) = f_i(s, x^n(s), y^n(s), w^n(s)), \quad i = 1, 2, \quad \bar{v}_2^n(s) = f_2(s, x_n, \bar{y}^n(s), w_n),$$

and

$$(x^n(s), w^n(s), y^n(s)), s \in [0, T],$$

is the solution of (3), (4), (5) with initial condition (x_n, w_n, y_n) corresponding to ϵ_n and $\bar{y}^n(s)$ is the solution of (22), corresponding to $\lambda = 1, x = x_n, w = w_n, y = y_n$.

Since the sequence $\{(x_n, w_n, y_n)\}$ is bounded in $E_1 \times E_3 \times E_2$, the sequence of functions $\{(x^n, w^n, y^n)\}$ is bounded in $C([0, T], E_1) \times C([0, T], E_3) \times C([0, T], E_2)$. Assumption $F_2)$ implies that the sequences $\{v_1^n\}, \{v_2^n\}$ and $\{\bar{v}_2^n\}$ are also bounded. Note that from the boundedness of the sequence $\{(x_n, w_n)\}$ it follows that the set $\hat{\Psi}(x_n, w_n)$ is relatively compact.

We are now able to prove the relative compactness of $\{(x^n, w^n, y^n)\}$ in $C_T(E_1) \times C_T(E_3) \times C_T(E_2)$. Firstly we consider $\chi_{E_1}(\{x^n(t)\})$, from $F_3)$ we have

$$\begin{aligned} \chi_{E_1}(\{v_1^n(t)\}) &\leq m_{11} \chi_{E_1}(\{x^n(t)\}) + m_{12} \chi_{E_2}(\{y^n(t)\}) \\ &\leq m_{11} \sup_{t \in [0, T]} \chi_{E_1}(\{x^n(t)\}) + m_{12} \sup_{t \in [0, T]} \chi_{E_2}(\{y^n(t)\}). \end{aligned} \tag{26}$$

By using the results of Kamenskii et al. [13] we have that for any $\delta > 0$ there exist a set $e_\delta \subset [0, T]$, a compact set $K_\delta \subset E_1$ and a sequence $\{g^n\}$

in $L^\infty([0, T], E_1)$ such that $\text{meas}(e_\delta) < \delta$, $g^n(s) \in K_\delta$, $s \in [0, T]$, and

$$\|v_1^n(t) - g^n(t)\| \leq m_{11} \sup_{t \in [0, T]} \chi_{E_1}(\{x^n(t)\}) + m_{12} \sup_{t \in [0, T]} \chi_{E_2}(\{g^n(t)\}) + \delta, \tag{27}$$

for $t \in [0, T] \setminus e_\delta$.

We define now a sequence $\{\tilde{x}_n\} \subset E_1$ as follows

$$\tilde{x}_n = \lambda_n(I - e^{-\epsilon_n A_1 T})^{-1} \int_0^T e^{-\epsilon_n A_1(T-s)} \epsilon_n g^n(s) ds + (1 - \lambda_n) \hat{\Psi}_1(x_n, w_n),$$

whenever $n \geq 1$.

Let us prove the relative compactness of the sequence $\{\tilde{x}_n\}$ in E_1 . For this we consider the set

$$K_1^\delta = \left\{ \int_0^T e^{-\epsilon_n A_1(T-s)} g^n(s) ds \right\}.$$

Since the semigroup $e^{-A_1 t}$ is of class C_0 then the set

$$S = \bigcup_n \bigcup_{s \in [0, T]} e^{-\epsilon_n A_1(T-s)} K_\delta$$

is relatively compact in E_1 . Furthermore

$$\int_0^T e^{-\epsilon_n A_1(T-s)} g^n(s) ds \subset T \overline{\text{co}} S, \quad n \in \mathbb{N},$$

hence the set K_1^δ is relatively compact. By means of Theorem 4.1.5 in Kamenskii et al. [13] we have that

$$\epsilon_n(I - e^{-\epsilon_n A_1 T})^{-1} x \rightarrow \frac{1}{T} A_1^{-1} x$$

for all $x \in E_1$ and thus

$$\bigcup_n \epsilon_n(I - e^{-\epsilon_n A_1 T})^{-1} K_1^\delta$$

is a relatively compact set. In conclusion

$$\{\tilde{x}_n\} \subseteq \lambda_n \bigcup_n \epsilon_n(I - e^{-\epsilon_n A_1 T})^{-1} K_1^\delta + (1 - \lambda_n) \hat{\Psi}_1(x_n, w_n)$$

is a relatively compact sequence in E_1 .

Now we want to estimate $\|x_n - \tilde{x}_n\|$. By using (27) for n sufficiently large we obtain

$$\begin{aligned} \|x_n - \tilde{x}_n\| &\leq \frac{\epsilon_n}{1 - e^{-\epsilon_n \gamma_1 T}} \int_0^T e^{-\epsilon_n \gamma_1(T-s)} \|v_1^n(s) - g^n(s)\| ds \\ &\leq \frac{\epsilon_n}{1 - e^{-\epsilon_n \gamma_1 T}} \left[T(m_{11} \sup_{t \in [0, T]} \chi_{E_1}(\{x^n(t)\}) \right. \\ &\quad \left. + m_{12} \sup_{t \in [0, T]} \chi_{E_2}(\{g^n(t)\}) + \delta \right] + M\delta \\ &\leq \frac{1}{\gamma_1 T(1 - \delta)} \left[T(m_{11} \sup_{t \in [0, T]} \chi_{E_1}(\{x^n(t)\}) \right. \\ &\quad \left. + m_{12} \sup_{t \in [0, T]} \chi_{E_2}(\{g^n(t)\}) + \delta \right] + M\delta \\ &= \frac{1}{1 - \delta} \left[\frac{m_{11}}{\gamma_1} \sup_{t \in [0, T]} \chi_{E_1}(\{x^n(t)\}) \right. \\ &\quad \left. + \frac{m_{12}}{\gamma_1} \sup_{t \in [0, T]} \chi_{E_2}(\{g^n(t)\}) \right] + \frac{\delta}{1 - \delta} \left(\frac{1}{\gamma_1} + \frac{M}{\gamma_1 T} \right). \end{aligned}$$

Since $\delta > 0$ is arbitrary we have

$$\chi_{E_1}(\{x_n\}) \leq \frac{m_{11}}{\gamma_1} \sup_{t \in [0, T]} \chi_{E_1}(\{x^n(t)\}) + \frac{m_{12}}{\gamma_1} \sup_{t \in [0, T]} \chi_{E_2}(\{g^n(t)\}). \tag{28}$$

Moreover, since $e^{-A_1 t}$ is a C_0 -semigroup and

$$\left\| \int_0^t e^{-A_1(t-s)} \epsilon_n v_1^n(s) ds \right\| \leq \epsilon_n M T \rightarrow 0$$

as $n \rightarrow \infty$, we obtain $\chi_{E_1}(\{x^n(t)\}) = \chi_{E_1}(\{x_n\})$ and

$$\chi_{E_1}(\{x^n(t)\}) \leq \frac{m_{11}}{\gamma_1} \sup_{t \in [0, T]} \chi_{E_1}(\{x^n(t)\}) + \frac{m_{12}}{\gamma_1} \sup_{t \in [0, T]} \chi_{E_2}(\{g^n(t)\}). \tag{29}$$

Consider now $\chi_{E_2}(\{g^n(t)\})$, we have that

$$\begin{aligned} \chi_{E_2}(\{g^n(t)\}) &\leq e^{-\gamma_2 t} \chi_{E_2}(\{g_n\}) \\ &\quad + \int_0^t e^{-\gamma_2(t-s)} (m_{21} \chi_{E_1}(\{x_n\}) + m_{22} \chi_{E_2}(\{g^n(s)\})) ds. \end{aligned}$$

Using the Gronwall inequality we get

$$\begin{aligned} \chi_{E_2}(\{y^n(t)\}) &\leq e^{(-\gamma_2+m_{22})t} \chi_{E_2}(\{y_n\}) \\ &+ \int_0^t e^{(-\gamma_2+m_{22})(t-s)} m_{21} \chi_{E_1}(\{x_n\}) ds. \end{aligned} \tag{30}$$

Analogously, for $\chi_{E_2}(\{g^n(t)\})$ we get

$$\begin{aligned} \chi_{E_2}(\{g^n(t)\}) &\leq e^{(-\gamma_2+m_{22})t} \chi_{E_2}(\{y_n\}) \\ &+ \int_0^t e^{(-\gamma_2+m_{22})(t-s)} m_{21} \chi_{E_1}(\{x_n\}) ds. \end{aligned} \tag{31}$$

Let us now estimate the measure of noncompactness of the right side of (25). Consider the first term, by using (30) we obtain

$$\begin{aligned} P_1 &= \chi_{E_2}(\{(I - e^{-A_2 T})^{-1} \int_0^T e^{-A_2(T-s)} g_2^n(s) ds\}) \\ &\leq (1 - e^{-\gamma_2 T})^{-1} \int_0^T e^{-\gamma_2(T-s)} (m_{21} \chi_{E_1}(\{x_n\}) \\ &+ m_{22} e^{(-\gamma_2+m_{22})s} \chi_{E_2}(\{y_n\})) \\ &+ \frac{m_{21} m_{22}}{\gamma_2 - m_{22}} (1 - e^{(-\gamma_2+m_{22})s}) \chi_{E_1}(\{x_n\}) ds \\ &= (1 - e^{-\gamma_2 T})^{-1} \left(\frac{m_{21}}{\gamma_2 - m_{22}} (1 - e^{(-\gamma_2+m_{22})T}) \chi_{E_1}(\{x_n\}) \right. \\ &\left. + (e^{(-\gamma_2+m_{22})T} - e^{-\gamma_2 T}) \chi_{E_2}(\{y_n\}) \right). \end{aligned}$$

For the second term of the right side of (25) by using (31) we obtain

$$\begin{aligned} P_2 &= \chi_{E_2}(\{e^{-A_2 T} y_n + \int_0^T e^{-A_2(T-s)} v_2^n(s) ds\}) \\ &\leq e^{-\gamma_2 T} \chi_{E_2}(\{y_n\}) \\ &+ \int_0^T e^{-\gamma_2(T-s)} (m_{21} \chi_{E_1}(\{x_n\}) + m_{22} (e^{(-\gamma_2+m_{22})s} \chi_{E_2}(\{y_n\}) \\ &+ \frac{m_{21}}{\gamma_2 - m_{22}} (1 - e^{(-\gamma_2+m_{22})s}) \chi_{E_1}(\{x_n\}))) ds \\ &= \frac{m_{21}}{\gamma_2 - m_{22}} (1 - e^{(-\gamma_2+m_{22})T}) \chi_{E_1}(\{x_n\}) + e^{(-\gamma_2+m_{22})T} \chi_{E_2}(\{y_n\}). \end{aligned}$$

From the inequality (25) it follows that $\chi_{E_2}(\{y_n\}) \leq \max\{P_1, P_2\}$. Suppose that $\chi_{E_2}(\{y_n\}) \leq P_1$. In this case

$$\begin{aligned} \chi_{E_2}(\{y_n\}) &\leq (1 - e^{-\gamma_2 T})^{-1} \left(\frac{m_{21}}{\gamma_2 - m_{22}} (1 - e^{(-\gamma_2+m_{22})T}) \chi_{E_1}(\{x_n\}) \right. \\ &\left. + (e^{(-\gamma_2+m_{22})T} - e^{-\gamma_2 T}) \chi_{E_2}(\{y_n\}) \right). \end{aligned}$$

If we multiply this inequality by $(1 - e^{-\gamma_2 T})$, we get

$$\begin{aligned} \chi_{E_2}(\{y_n\}) &\leq \frac{m_{21}}{\gamma_2 - m_{22}} \chi_{E_1}(\{x_n\}), \\ \chi_{E_2}(\{y_n\}) &\leq \frac{m_{21}}{\gamma_2} \chi_{E_1}(\{x_n\}) + \frac{m_{22}}{\gamma_2} \chi_{E_2}(\{y_n\}). \end{aligned} \tag{32}$$

Analogously, assuming that $\chi_{E_2}(\{y_n\}) \leq P_2$ we have

$$\begin{aligned} \chi_{E_2}(\{y_n\}) &\leq \frac{m_{21}}{\gamma_2 - m_{22}} (1 - e^{(-\gamma_2+m_{22})T}) \chi_{E_1}(\{x_n\}) + e^{(-\gamma_2+m_{22})T} \chi_{E_2}(\{y_n\}), \\ (1 - e^{(-\gamma_2+m_{22})T}) \chi_{E_2}(\{y_n\}) &\leq \frac{m_{21}}{\gamma_2 - m_{22}} (1 - e^{(-\gamma_2+m_{22})T}) \chi_{E_1}(\{x_n\}), \end{aligned}$$

Thus $\chi_{E_2}(\{y_n\})$ satisfies again (32). By (30), (32) we obtain

$$\begin{aligned} \chi_{E_2}(\{g^n(t)\}) &\leq e^{(-\gamma_2+m_{22})t} \frac{m_{21}}{\gamma_2 - m_{22}} \chi_{E_1}(\{x_n\}) \\ &+ \frac{m_{21}}{\gamma_2 - m_{22}} (1 - e^{(-\gamma_2+m_{22})t}) \chi_{E_1}(\{x_n\}), \\ \chi_{E_2}(\{g^n(t)\}) &\leq \frac{m_{21}}{\gamma_2 - m_{22}} \chi_{E_1}(\{x_n\}). \end{aligned}$$

Consequently, we have

$$\sup_{t \in [0, T]} \chi_{E_2}(\{y^n(t)\}) \leq \frac{m_{21}}{\gamma_2} \sup_{t \in [0, T]} \chi_{E_1}(\{x^n(t)\}) + \frac{m_{22}}{\gamma_2} \sup_{t \in [0, T]} \chi_{E_2}(\{y^n(t)\}). \tag{33}$$

Condition M) and inequalities (29) and (33) produce

$$\chi_{E_1}(\{x^n(t)\}) = 0, \chi_{E_2}(\{y^n(t)\}) = 0, t \in [0, T]. \tag{34}$$

Since $w^n(T) \in K$, then $\{w_n\}$ is relatively compact. Thus, all the sequences $\{x_n\}$, $\{w_n\}$, $\{y_n\}$ are relatively compact. Without loss of generality we can assume that $x_n \rightarrow x_0$, $w_n \rightarrow w_0$, $y_n \rightarrow y_0$. Then x^n converges to x_0 uniformly on $[0, T]$. In fact,

$$\|x^n(t) - x_0\| \leq \|e^{-\epsilon_n A_1 t} x_n - e^{-\epsilon_n A_1 t} x_0\| + \|e^{-\epsilon_n A_1 t} x_0 - x_0\| + \epsilon_n M T.$$

Since $e^{-A_1 t}$ is a C_0 -semigroup, the right side of this inequality converges to zero uniformly with respect to t .

Consider now the sequence $\{w^n\} \subset C_T(E_3)$. We have

$$\|w^n(t) - w_0\| \leq k(\|Px_n - Px_0\|_{E_3} + \|w_0 - w_n\|_{E_3} + \|Px_n - Px_0\|_{C_T(E_3)}).$$

Since $x^n \rightarrow x_0$, and $w_n \rightarrow w_0$, then w^n converges to w_0 uniformly on $[0, T]$.

Finally, consider the sequences $\{y^n\}$ and $\{y^n^*\}$. From F_3 and (34) it follows that $\chi_{E_2}(\{v_2^n(t)\}) = 0$, for any $t \in [0, T]$. Then, Corollary 5.1.1 of Kamenskii et al. [13] implies that the sequence $\{y^n^*\}$ is relatively compact in $C([0, T], E_2)$, i.e. $y^n^* \rightarrow y^0$. Since the function f_2 satisfy Lipschitz condition, we get

$$y^0(t) = e^{-A_2 t} y_0 + \int_0^t e^{-A_2(t-s)} f_2(s, x_0, y^0(s), w_0) ds.$$

Analogously, $y^n \rightarrow y^*$, with $y^*(t) = e^{-A_2 t} y_0 + \int_0^t e^{-A_2(t-s)} f_2(s, x_0, y^*(s), w_0) ds$. The Lipschitz condition F_1 imply that $y^0 \equiv y^*$. Therefore, from (25) as $n \rightarrow \infty$ we obtain

$$\begin{aligned} y_0 &= \lambda_0(I - e^{-A_2 T})^{-1} \int_0^T e^{-A_2(T-s)} f_2(s, x_0, y^0(s), w_0) ds + \\ &+ (1 - \lambda_0)(e^{-A_2 t} y_0 + \int_0^T e^{-A_2(T-s)} f_2(s, x_0, y^0(s), w_0) ds). \end{aligned}$$

Replacing $\int_0^T e^{-A_2(T-s)} f_2(s, x_0, y^0(s), w_0) ds$ by $y^0(T) - e^{-A_2 T} y_0$ we obtain $y_0 = y^0(T)$, i.e. $y^0 = y_{x_0, w_0}^1$.

Consider now (23), Theorem 4.1.5 of Kamenskii et al. [13] ensures that

$$\epsilon(I - e^{-\epsilon A_1 T})^{-1} \int_0^T e^{-\epsilon A_1(T-s)} v(s) ds \rightarrow A_1^{-1} \frac{1}{T} \int_0^T v(s) ds \text{ as } \epsilon \rightarrow 0$$

for every $v \in C([0, T], E_1)$. Since f_1 is a Lipschitz function, v_1^n converges uniformly to v_1^0 , where $v_1^0(s) = f_1(s, x_0, y^0(s), w_0)$. Hence,

$$(I - e^{-\epsilon_n A_1 T})^{-1} \int_0^T e^{-\epsilon_n A_1(T-s)} \epsilon_n v_1^n(s) ds \rightarrow A_1^{-1} \frac{1}{T} \int_0^T v_1^0(s) ds.$$

Since $y^0 = y_{x_0, w_0}^1$, it follows that

$$A_1^{-1} \frac{1}{T} \int_0^T v_1^0(s) ds = F(x_0, w_0).$$

Furthermore, from (23) as $n \rightarrow \infty$ we get

$$x_0 = \lambda_0 F(x_0, w_0) + (1 - \lambda_0) \hat{\Psi}_1(x_0, w_0), \tag{35}$$

Finally, from (24), as $n \rightarrow \infty$, we obtain

$$w_0 = \lambda_0 w_0 + (1 - \lambda_0) \hat{\Psi}_2(x_0, w_0). \tag{36}$$

Since the set $\partial((B \cap (E_1 \times K)) \times B_{E_2}(0, r))$ is closed then (x_0, w_0, y_0) belongs to $\partial((B \cap (E_1 \times K)) \times B_{E_2}(0, r))$. On the other hand, by our choice of r it follows that $y_0 \notin \partial(B_{E_2}(0, r))$. Therefore $(x_0, w_0) \in \partial(B \cap (E_1 \times K))$ which is a contradiction with (35), (36) and the choice of B .

Consider now the homotopy on $\partial((B \cap (E_1 \times K)) \times B_{E_2}(0, r))$ defined as follows

$$\Psi_\lambda^*(x, w, y) = (\hat{\Psi}_1(x, w), \hat{\Psi}_2(x, w), e^{-A_2 T} y + \lambda \int_0^T e^{-A_2(T-s)} \bar{v}_2(s) ds),$$

where $\bar{v}_2(s) = f_2(s, x, \bar{y}(s), w)$ and $\bar{y}(s)$ is the solution of (22).

For $\lambda = 0$ we have

$$\Psi_0^*(x, w, y) = (\hat{\Psi}_1(x, w), \hat{\Psi}_2(x, w), e^{-A_2 T} y)$$

and for $\lambda = 1$ we have

$$\Psi_1^*(x, w, y) = \Psi^*(x, w, y).$$

Define the auxiliary operators $\Psi_0^0 : (B \cap (E_1 \times K)) \times E_2 \rightarrow E_1 \times K \times E_2$ and $D : E_1 \times E_3 \times E_2 \rightarrow E_1 \times E_3 \times E_2$ as follows

$$\Psi_0^0(x, w, y) = (\hat{\Psi}_1(x, w), \hat{\Psi}_2(x, w), 0), \quad D(x, w, y) = (0, 0, e^{-A_2 T} y).$$

Observing that

$$I - \Psi_0^* = (I - \Psi_0^0)(I - D),$$

by the properties of the topological degree for condensing operators, (see e.g. [13]), we have

$$\begin{aligned} & | \deg(I - \Psi_0^*, (B \cap (E_1 \times K)) \times B_{E_2}(0, r), 0) | \\ &= | \deg(I - \Psi_0^0, (B \cap (E_1 \times K)) \times B_{E_2}(0, r), 0) | \\ & \cdot | \deg(I - D, (B \cap (E_1 \times K)) \times B_{E_2}(0, r), 0) |. \end{aligned}$$

By the reduction property of the topological degree we obtain that

$$\begin{aligned} & | \deg(I - \Psi_0^0, (B \cap (E_1 \times K)) \times B_{E_2}(0, r), 0) | = | \deg(I - \hat{\Psi}, B \cap (E_1 \times K), 0) | \\ &= | \deg(I - \Psi, B \cap (E_1 \times K), 0) | \neq 0, \end{aligned}$$

and

$$\begin{aligned} & | \deg(I - D, (B \cap (E_1 \times K)) \times B_{E_2}(0, r), 0) | \\ &= | \deg(I - e^{-A_2 T}, B_{E_2}(0, r), 0) | \neq 0. \end{aligned}$$

Consequently,

$$| \deg(I - \Psi_0^*, (B \cap (E_1 \times K)) \times B_{E_2}(0, r), 0) | \neq 0.$$

Since Ψ_0^* , Ψ^* , and V_ε^T are homotopic on $\partial((B \cap (E_1 \times K)) \times B_{E_2}(0, r))$ we finally obtain

$$| \deg(I - V_\varepsilon^T, (B \cap (E_1 \times K)) \times B_{E_2}(0, r), 0) | \neq 0.$$

Hence, for sufficiently small $\varepsilon > 0$ there exists at least one T-periodic solution $(x_\varepsilon, w_\varepsilon, y_\varepsilon)$ of (1). Now, to conclude the proof we consider a sequence $\varepsilon_n \rightarrow 0$. Let $\{(x_{\varepsilon_n}, w_{\varepsilon_n}, y_{\varepsilon_n})\}$ be the sequence of the corresponding T-periodic solutions of system (1). Letting $x_{\varepsilon_n}(0) = x_n$, $w_{\varepsilon_n}(0) = w_n$, $y_{\varepsilon_n}(0) = y_n$, we have

$$\begin{aligned} x_n &= e^{-\varepsilon_n A_1 T} x_n + \int_0^T e^{-\varepsilon_n A_1(T-s)} \varepsilon_n f(s, x_{\varepsilon_n}(s), y_{\varepsilon_n}(s), w_{\varepsilon_n}(s)) ds, \\ w_n &= \Gamma[0, P x_n, w_n] P^T x_{\varepsilon_n}(T), \\ y_n &= e^{-A_2 T} y_n + \int_0^T e^{-A_2(T-s)} f_2(s, x_{\varepsilon_n}(s), y_{\varepsilon_n}(s), w_{\varepsilon_n}(s)) ds. \end{aligned}$$

By repeating the arguments of the first part of this proof we show the existence of a subsequence of $\{(x_{\varepsilon_n}, w_{\varepsilon_n}, y_{\varepsilon_n})\}$ converging to (x_*, w_*, y^*) , where (x_*, w_*) is a fixed point of the operator Ψ and $y^* = \hat{y}_{x_*, w_*}^1$.

Acknowledgement: This work has been partially supported by INDAM-GNAMPA and RFBR grants 06-01-72552, 07-01-00035.

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