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ON THE RATE OF CONVERGENCE OF PERIODIC SOLUTIONS IN PERTURBED AUTONOMOUS SYSTEMS AS THE PERTURBATION VANISHES

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ABSTRACT. We consider an autonomous system in $\mathbb{R}^n$ having a limit cycle x_0 of period $T > 0$ which is nondegenerate in a suitable sense, (see Definition 2.1). We then consider the perturbed system obtained by adding to the autonomous system a T-periodic, not necessarily differentiable, term whose amplitude tends to 0 as a small parameter $\varepsilon > 0$ tends to 0. Assuming the existence of a T-periodic solution x_ε of the perturbed system and its convergence to x_0 as $\varepsilon \to 0$, the paper establishes the existence of $\Delta_\varepsilon \to 0$ as $\varepsilon \to 0$ such that $\|x_\varepsilon(t + \Delta_\varepsilon) - x_0(t)\| \le \varepsilon M$ for some $M > 0$ and any $\varepsilon > 0$ sufficiently small. This paper completes the work initiated by the authors in [4] and [11]. Indeed, in [4] the existence of a family of T-periodic solutions x_ε of the perturbed system considered here was proved. While in [11] for perturbed systems in $\mathbb{R}^2$ the rate of convergence was investigated by means of the method considered in this paper.

1. Introduction.

Assume that the perturbed autonomous system

$$\dot{x} = f(x) + \varepsilon g(t, x, \varepsilon), \qquad x \in \mathbb{R}^n, \ \ t \in \mathbb{R}, \tag{1}$$

possesses a family of T-periodic solutions $\{x_\varepsilon\}_{\varepsilon \in (0,1]}$ such that

$$x_\varepsilon(t) \to x_0(t) \quad \text{as } \varepsilon \to 0 \tag{2}$$

uniformly with respect to $t \in \mathbb{R}$, where x_0 is a limit cycle of period $T > 0$ of the system

$$\dot{x} = f(x). \tag{3}$$

The following system is an example of (1) having a family of 2π-periodic solutions $\{x_\varepsilon\}_{\varepsilon \in (0,1]}$ satisfying (2), where the 2π-periodic limit cycle x_0 is represented by the

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circumference centered at the origin with radius 1.

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1\left(x_1^2 + x_2^2 - (1+\varepsilon)^2\right) + \varepsilon\left((1+\varepsilon)\sin(t-\sqrt{\varepsilon}) - x_1\right),\\ \dot{x}_2 &= -x_1 - x_2\left(x_1^2 + x_2^2 - (1+\varepsilon)^2\right).\end{aligned}$$

In fact, as it is easy to see, for $\varepsilon > 0$ it has the 2π-periodic solution

$$x_\varepsilon(t) = \begin{pmatrix} (1+\varepsilon)\sin(t-\sqrt{\varepsilon}) \\ (1+\varepsilon)\cos(t-\sqrt{\varepsilon}) \end{pmatrix}$$

which, for any $t \in \mathbb{R}$, converges to $x_0(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$ when $\varepsilon \to 0$. This example shows that the rate of convergence in (2) can be less than $\varepsilon > 0$, indeed

$$\frac{\|x_\varepsilon(t) - x_0(t)\|}{\varepsilon} = \frac{1}{\varepsilon}\left\|\begin{pmatrix} \sin(t-\sqrt{\varepsilon}) - \sin t \\ \cos(t-\sqrt{\varepsilon}) - \cos t \end{pmatrix} + \varepsilon\begin{pmatrix} \sin(t-\sqrt{\varepsilon}) \\ \cos(t-\sqrt{\varepsilon}) \end{pmatrix}\right\| \to \infty$$

as $\varepsilon \to 0$.

On the other hand the example also suggests that a suitable shift in time in x_ε gives convergence at the rate ε. In fact, we have that

$$\frac{\|x_\varepsilon(t+\sqrt{\varepsilon}) - x_0(t)\|}{\varepsilon} = 1 \quad \text{for any } t \in [0, 2\pi] \quad \text{and any } \varepsilon > 0. \tag{4}$$

In this paper we show that the situation described by the above example occurs in general. Namely we prove that, given a family of T-periodic solutions $\{x_\varepsilon\}_{\varepsilon\in(0,1]}$ to (1) satisfying (2), it is always possible to find a suitable family of shifts $\{\Delta_\varepsilon\}_{\varepsilon>0} \subset \mathbb{R}$ satisfying

$$\frac{\|x_\varepsilon(t+\Delta_\varepsilon) - x_0(t)\|}{\varepsilon} \le const \quad \text{for any } t \in [0, T] \text{ and any } \varepsilon > 0 \tag{5}$$

provided that the limit cycle x_0 is nondegenerate in the sense that the algebraic multiplicity of the characteristic multiplier +1 of

$$\dot{y} = f'(x_0(t))y \tag{6}$$

is equal to 1. In particular, our result implies that if x_0 is a nondegenerate cycle of (3) then the distance between the sets $x_\varepsilon([0,T])$ and $x_0([0,T])$ is of order $\varepsilon > 0$. Our result does not require differentiability of g, indeed here we assume that

$$f \in C^1(\mathbb{R}^n, \mathbb{R}^n) \quad \text{and} \quad g \in C(\mathbb{R} \times \mathbb{R}^n \times [0,1], \mathbb{R}^n). \tag{7}$$

This paper completes the existence and convergence results of T-periodic solutions x_ε of (1) proved in [4] under assumptions (7). In more general settings the existence of bifurcation of T-periodic solutions was proved by Fečan [3]. In fact, in [4] we have observed in Remark 3.4 that the rate of convergence of $x_\varepsilon([0,T])$ to $x_0([0,T])$ is of order ε^p with $0 < p < 1$. The convergence of $x_\varepsilon([0,T])$ to $x_0([0,T])$ at rate ε^1 was established in [11] for the case when $n = 2$, but instead of Δ_ε we had $\Delta_\varepsilon(t)$ in (5). The possibility of considering Δ_ε independent on time in this paper is due to the considerable simplification of the approach used in [11] that we have performed here.

The classical results on the existence and convergence at rate ε, of T-periodic solutions to equations of the form (1), where $\varepsilon > 0$ is small, are due to Malkin ([12], Statement p. 41) and Loud ([10], Theorem 1) where it is assumed that

$$f \in C^2(\mathbb{R}^n, \mathbb{R}^n) \quad \text{and} \quad g \in C^1(\mathbb{R} \times \mathbb{R}^n \times [0,1], \mathbb{R}^n). \tag{8}$$

The persistence of the limit cycle x0 for piecewise differential systems like (1) can be studied by means of methods due to Aĭzerman-Gantmaher [1], Kolovskiĭ [7], Lazer-McKenna [9] and Steĭnberg [14]. To our best knowledge [1] and [11] are the first papers that provide existence and convergence results of T-periodic solutions of (1) bifurcating from a limit cycle x_0 under assumptions (7).

The paper is organized as follows. Section 2 is devoted to our main result: Theorem 1 which states the validity of the inequality (5). In Section 3 we apply (5) for studying some further properties of convergence in (2), namely we investigate

$$\lim_{\varepsilon\to 0}\frac{x_\varepsilon(t+\Delta_\varepsilon)-x_0(t)}{\varepsilon} \tag{9}$$

by means of the first approximation. In particular, using only the eigenfunctions z of the adjoint system $\dot z = -(f'(x_0(t)))^* z$ and the function g we give conditions ensuring that $\lim_{\varepsilon\to 0}\|x_\varepsilon(\Delta_\varepsilon) - x_0(0)\|/\varepsilon = 0$ and we determine the signum of the angle between the vectors $z(t)$ and $x_\varepsilon(t+\Delta_\varepsilon) - x_0(t)$. In the smooth case (8) these results may be derived, as discussed in Remark 3.3, from the Loud's formula (48). Note that formula (48) is established under the condition that a suitably defined bifurcation function (45) has a simple zero, our result does not require such a condition.

2. A formula for the distance between the periodic solutions of the perturbed system and the limit cycle of the unperturbed one.

In this Section we establish our main result, namely the validity of (5). This result does not depend on the perturbation term g, indeed the only property we need is the following.

Definition 2.1. We say that the limit cycle x_0 of (3) is nondegenerate if the algebraic multiplicity[1] of the characteristic multiplier $+1$ of (6) is equal to 1.

In order to introduce the family $\{\Delta_\varepsilon\}$ that appears in (5) we define in what follows a suitable surface $S \in C(\mathbb{R}^{n-1}, \mathbb{R}^n)$. For this, let A_{n-1} be an arbitrary $n\times n-1$ matrix such that the $n\times n$ matrix $(\dot x_0(0), A_{n-1})$ is nonsingular and $\Omega(\cdot, t_0, \xi)$ is the solution of (3) satisfying $\Omega(t_0, t_0, \xi) = \xi$. The surface S is given by

$$\begin{aligned} S(v) &= \Omega(T,0,h(v)),\\ h(v) &= x_0(0) + A_{n-1}v. \end{aligned} \tag{10}$$

The following result shows that the surface S intersects x_0 transversally.

Lemma 2.2. *Assume $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Let x_0 be a nondegenerate T-periodic cycle of (3). Then $\dot x_0(0) \notin S'(0)(\mathbb{R}^{n-1})$.*

Proof. We argue by contradiction, thus we assume that there exists $v_0 \in \mathbb{R}^{n-1}$, $v_0 \neq 0$, such that $\dot x_0(0) = S'(0)v_0$. We have

$$\dot x_0(0) = S'(0)v_0 = \Omega'_\xi(T,0,x_0(0))A_{n-1}v_0, \tag{11}$$

where Ω'_ξ is the derivative of Ω with respect to the third variable.
On the other hand, (see [8], Theorem 2.1) $\Omega'_\xi(\cdot, 0, x_0(0))$ is a fundamental matrix to (6) and since $\dot x_0$ is a solution to (6), we have that

$$\Omega'_\xi(T,0,x_0(0))\dot x_0(0) = \dot x_0(T) = \dot x_0(0). \tag{12}$$

From (11) and (12) we conclude that $\dot x_0(0) = A_{n-1}v_0$, which means that the matrix $(\dot x_0(0), A_{n-1})$ is singular contradicting the definition of A_{n-1}. □

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[1] Let $Y(t)$ be the normalized fundamental matrix of (6). An eigenvalue ρ of $Y(T)$ is called the characteristic multiplier of (6). Since ρ is a root of the corresponding characteristic equation then it is possible to consider the multiplicity of this root, which is called *algebraic multiplicity* of the characteristic multiplier ρ. System (6) always has at least one characteristic multiplier $+1$ since from $\dot x_0(t) = f(x_0(t))$ we have $\ddot x_0(t) = f'(x_0(t))\dot x_0(t)$.

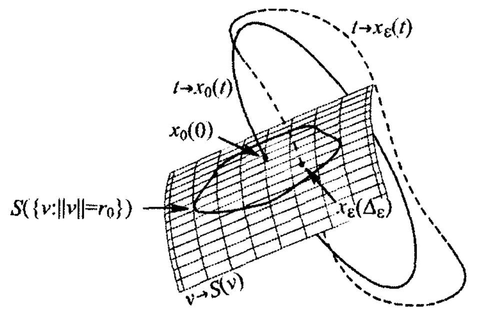

FIGURE 1.

As a consequence of the previous lemma we have the following result.

Corollary 2.3. *Assume the conditions of Lemma 2.2. Let* x_ε *be a* T*-periodic solution to perturbed system (1) satisfying (2) uniformly with respect to* $t \in \mathbb{R}$, *then there exists* $\varepsilon_0 > 0$ *and* $r_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0]$ *the equation* $x_\varepsilon(\Delta) = S(v)$ *has an unique solution* $(\Delta_\varepsilon, v_\varepsilon)$ *in* $[-r_0, r_0] \times \{v \in \mathbb{R}^{n-1} : \|v\| \le r_0\}$. *Moreover, the functions* $\varepsilon \to \Delta_\varepsilon$, $\varepsilon \to v_\varepsilon$ *are continuous at* $\varepsilon = 0$ *with* $\Delta_0 = 0$ *and* $v_0 = 0$.

Proof. Define the function $F \in C(\mathbb{R}^n \times [0,1], \mathbb{R}^n)$ as $F((t,v),\varepsilon) = x_\varepsilon(t) - S(v)$, then $F((0,0),0) = 0$. Moreover, F is continuously differentiable with respect to the first variable and $F'_{(t,v)}((0,0),0) = (\dot{x}_0(0), -S'(0))$ is nonsingular by Lemma 2.2. The conclusion follows from a generalized version, see ([6], Ch. X, § 2.1), of the classical implicit function theorem which requires $F \in C^1(\mathbb{R}^n \times [0,1], \mathbb{R}^n)$, while we do not have here the differentiability of F with respect to the second variable. □

Figure 1 illustrates the meaning of Corollary 2.3. We are now in the position to prove inequality (5).

Theorem 2.4. *Assume* $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $g \in C(\mathbb{R} \times \mathbb{R}^n \times [0,1], \mathbb{R}^n)$. *Let* x_ε *be a* T*-periodic solution to perturbed system (1) satisfying*

$$\|x_\varepsilon(t) - x_0(t)\| \to 0 \quad as \quad \varepsilon \to 0 \tag{13}$$

uniformly with respect to $t \in [0,T]$, *where* x_0 *is a nondegenerate* T*-periodic limit cycle of unperturbed system (3). Let* $\varepsilon_0 > 0$ *and* $\{\Delta_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]} \subset \mathbb{R}$ *be as in Corollary 2.3. Then there exists* $M > 0$ *such that*

$$\|x_\varepsilon(t + \Delta_\varepsilon) - x_0(t)\| \le M\varepsilon \quad \textit{for any } t \in [0,T] \textit{ and any } \varepsilon \in (0, \varepsilon_0]. \tag{14}$$

Proof. In the sequel $\varepsilon \in (0, \varepsilon_0]$ and $\tau \in [0,T]$. Consider the change of variables $\nu_\varepsilon(\tau) = \Omega(0, \tau, x_\varepsilon(\tau + \Delta_\varepsilon))$ in system (1). Observe that

$$x_\varepsilon(\tau + \Delta_\varepsilon) = \Omega(\tau, 0, \nu_\varepsilon(\tau)). \tag{15}$$

Taking the derivative in (15) with respect to τ we obtain

$$\dot{x}_\varepsilon(\tau + \Delta_\varepsilon) = f(\Omega(\tau, 0, \nu_\varepsilon(\tau))) + \Omega'_\xi(\tau, 0, \nu_\varepsilon(\tau))\dot{\nu}_\varepsilon(\tau). \tag{16}$$

On the other hand from (1) we have

$$\dot{x}_\varepsilon(\tau + \Delta_\varepsilon) = f(\Omega(\tau, 0, \nu_\varepsilon(\tau))) + \varepsilon g(\tau + \Delta_\varepsilon, \Omega(\tau, 0, \nu_\varepsilon(\tau)), \varepsilon). \tag{17}$$

From (16) and (17) it follows

$$\dot{\nu}_\varepsilon(\tau) = \varepsilon \left(\Omega'_\xi(\tau, 0, \nu_\varepsilon(\tau))\right)^{-1} g(\tau + \Delta_\varepsilon, \Omega(\tau, 0, \nu_\varepsilon(\tau)), \varepsilon),$$

and since

$$\nu_\varepsilon(0) = x_\varepsilon(\Delta_\varepsilon) = x_\varepsilon(T + \Delta_\varepsilon) = \Omega(T, 0, \nu_\varepsilon(T))$$

we finally obtain

$$\nu_\varepsilon(\tau) = \Omega(T, 0, \nu_\varepsilon(T)) + \varepsilon \int_0^\tau \left(\Omega'_\xi(s, 0, \nu_\varepsilon(s))\right)^{-1} g(s + \Delta_\varepsilon, \Omega(s, 0, \nu_\varepsilon(s)), \varepsilon) ds. \quad (18)$$

Since $\nu_\varepsilon(\tau) \longrightarrow x_0(0)$, for any $\tau \geq 0$, as $\varepsilon \longrightarrow 0$ we can write $\nu_\varepsilon(\tau)$ in the following form

$$\nu_\varepsilon(\tau) = x_0(0) + \varepsilon\mu_\varepsilon(\tau). \quad (19)$$

We now prove that the functions μ_ε are bounded on $[0, T]$ uniformly with respect to $\varepsilon \in (0, \varepsilon_0]$. For this, we first subtract $x_0(0)$ from both sides of (18), with $\tau = T$, obtaining

$$\begin{aligned} \varepsilon\mu_\varepsilon(T) &= \varepsilon\,\Omega'_\xi(T, 0, x_0(0))\mu_\varepsilon(T) + o(\varepsilon\mu_\varepsilon(T)) \\ &\quad + \varepsilon \int_0^T \left(\Omega'_\xi(s, 0, \nu_\varepsilon(s))\right)^{-1} g(s + \Delta_\varepsilon, \Omega(s, 0, \nu_\varepsilon(s)), \varepsilon) ds, \end{aligned} \quad (20)$$

where, from (19), $\frac{o(\varepsilon\mu_\varepsilon(T))}{\|\varepsilon\mu_\varepsilon(T)\|} \longrightarrow 0$ as $\varepsilon \rightarrow 0$. Since $x_\varepsilon(\Delta_\varepsilon) \in S\left(\{v \in \mathbb{R}^{n-1} : \|v\| \leq r_0\}\right)$ then by Corollary 2.3 there exists $v_\varepsilon \in \mathbb{R}^{n-1}$, $\|v_\varepsilon\| \leq r_0$, such that

$$x_\varepsilon(\Delta_\varepsilon) = \Omega(T, 0, x_0(0) + A_{n-1}v_\varepsilon) \quad (21)$$

and

$$v_\varepsilon \longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0. \quad (22)$$

Now by using (21) we can represent $\varepsilon\mu_\varepsilon(T)$ as follows

$$\begin{aligned} \varepsilon\mu_\varepsilon(T) &= \nu_\varepsilon(T) - x_0(0) = \Omega(0, T, x_\varepsilon(\Delta_\varepsilon)) - x_0(0) \\ &= \Omega(0, T, \Omega(T, 0, x_0(0) + A_{n-1}v_\varepsilon)) - x_0(0) = A_{n-1}v_\varepsilon. \end{aligned} \quad (23)$$

Therefore (20) can be rewritten as follows

$$\begin{aligned} A_{n-1}v_\varepsilon &= \Omega'_\xi(T, 0, x_0(0))A_{n-1}v_\varepsilon + o(A_{n-1}v_\varepsilon) \\ &\quad + \varepsilon \int_0^T \left(\Omega'_\xi(s, 0, \nu_\varepsilon(s))\right)^{-1} g(s + \Delta_\varepsilon, \Omega(s, 0, \nu_\varepsilon(s)), \varepsilon) ds. \end{aligned} \quad (24)$$

Let us show that there exists $M_1 > 0$ such that

$$\|v_\varepsilon\| \leq \varepsilon M_1, \qquad \text{for any} \quad \varepsilon \in (0, \varepsilon_0]. \quad (25)$$

Arguing by contradiction we assume that there exist sequences $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0, \varepsilon_0]$, $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$, such that $\|v_{\varepsilon_k}\| = \varepsilon_k c_k$, where $c_k \rightarrow \infty$ as $k \rightarrow \infty$. Let $q_k = \frac{v_{\varepsilon_k}}{\|v_{\varepsilon_k}\|}$, then from (24) we have

$$\begin{aligned} A_{n-1}q_k &= \Omega'_\xi(T, 0, x_0(0))A_{n-1}q_k + \frac{o(A_{n-1}v_{\varepsilon_k})}{\|v_{\varepsilon_k}\|} \\ &\quad + \frac{1}{c_k} \int_0^T \left(\Omega'_\xi(s, 0, \nu_{\varepsilon_k}(s))\right)^{-1} g(s + \Delta_{\varepsilon_k}, \Omega(s, 0, \nu_{\varepsilon_k}(s)), \varepsilon_k) ds, \end{aligned} \quad (26)$$

where $\frac{o(A_{n-1}v_{\varepsilon_k})}{\|v_{\varepsilon_k}\|} \longrightarrow 0$ as $k \longrightarrow \infty$, in fact $\frac{o(A_{n-1}v_{\varepsilon_k})}{\|v_{\varepsilon_k}\|} = \frac{o(A_{n-1}v_{\varepsilon_k})}{\|A_{n-1}v_{\varepsilon_k}\|} \cdot \frac{\|A_{n-1}v_{\varepsilon_k}\|}{\|v_{\varepsilon_k}\|}$. Without loss of generality we may assume that the sequence $\{q_k\}_{k\in\mathbb{N}}$ converges, let

$q_0 = \lim_{k\to\infty} q_k$ with $\|q_0\| = 1$. By passing to the limit as $k \to \infty$ in (26) we have that

$$A_{n-1}q_0 = \Omega'_\xi(T, 0, x_0(0))A_{n-1}q_0.$$

Therefore $A_{n-1}q_0$ is the initial condition of a T-periodic solutions to (6). On the other hand the cycle x_0 is nondegenerate, hence $A_{n-1}q_0$ is linearly dependent with $\dot{x}_0$ contradicting the choice of A_{n-1}. Thus (25) is true for some $M_1 > 0$. From (19) and the fact that $\nu_\varepsilon(0) = x_\varepsilon(\Delta_\varepsilon)$ we have

$$\begin{aligned} \|x_\varepsilon(\Delta_\varepsilon) - x_0(0)\| &= \varepsilon\|\mu_\varepsilon(0)\| \le \varepsilon\|\mu_\varepsilon(T)\| + \|\varepsilon\mu_\varepsilon(T) - \varepsilon\mu_\varepsilon(0)\| \\ &= \varepsilon\|\mu_\varepsilon(T)\| + \|\nu_\varepsilon(T) - \nu_\varepsilon(0)\|. \end{aligned} \tag{27}$$

From (18) we have that there exists $M_2 > 0$ such that

$$\|\nu_\varepsilon(T) - \nu_\varepsilon(0)\| \le \varepsilon M_2, \qquad \text{for any } \varepsilon \in (0, \varepsilon_0]. \tag{28}$$

Therefore combining (23) with (25) and taking into account (28) we have from (27) that

$$\|x_\varepsilon(\Delta_\varepsilon) - x_0(0)\| \le \varepsilon\|A_{n-1}\|M_1 + \varepsilon M_2, \qquad \text{for any } \varepsilon \in (0, \varepsilon_0].$$

Since

$$\begin{aligned} x_\varepsilon(t + \Delta_\varepsilon) - x_0(t) = {} & x_\varepsilon(\Delta_\varepsilon) - x_0(0) + \textstyle\int_0^t (f(x_\varepsilon(s + \Delta_\varepsilon)) - f(x_0(s)))ds \\ & + \varepsilon \textstyle\int_0^t g(s + \Delta_\varepsilon, x_\varepsilon(s + \Delta_\varepsilon), \varepsilon)ds \end{aligned}$$

and $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ then there exist a constant $M_3 \ge 0$ such that

$$\begin{aligned} \|x_\varepsilon(t + \Delta_\varepsilon) - x_0(t)\| \le {} & (\varepsilon\|A_{n-1}\|M_1 + \varepsilon M_2) \\ & + M_3 \textstyle\int_0^t \|x_\varepsilon(s + \Delta_\varepsilon) - x_0(s)\|ds + \varepsilon M_3, \end{aligned} \tag{29}$$

for any $\varepsilon \in (0, \varepsilon_0]$. By means of the Gronwall-Bellman lemma (see e.g. [2], Ch. II, § 11) inequality (29) implies

$$\|x_\varepsilon(t + \Delta_\varepsilon) - x_0(t)\| \le \varepsilon\,(\|A_{n-1}\|M_1 + M_2 + M_3)\,\mathrm{e}^{M_3T} \quad \text{for any } \varepsilon \in (0, \varepsilon_0].$$

and thus the proof is complete. □

Remark 2.5. Assume that the T-periodic solution x_ε of system (1) satisfies the property $\|x_\varepsilon(t) - \widetilde{x}(t)\| \to 0$, where $\widetilde{x}$ is a T-periodic nondegenerate limit cycle of (3). Let $\tau \in [0, T]$, define $x_0^\tau(t) := \widetilde{x}(t + \tau)$, then we have $\|x_\varepsilon(t + \tau) - x_0^\tau(t)\| \to 0$ as $\varepsilon \to 0$. Denote by S^τ the surface S corresponding to x_0^τ given by (10). Observe now that $\varepsilon_0 > 0$ of Corollary 2.3 can be chosen sufficiently small in such a way that it does not depend on the choice of $\tau \in [0, T]$ used in the definition of x_0^τ. Therefore for any $\tau \in [0, T]$ and any $\varepsilon \in (0, \varepsilon_0]$ Corollary 2.3 guarantees the existence of a Δ_ε^τ with the property

$$S^\tau(\{v \in \mathbb{R}^{n-1} : \|v\| \le r_0\}) \cap x_\varepsilon([0, T]) = \{x_\varepsilon(\Delta_\varepsilon^\tau + \tau)\}. \tag{30}$$

In conclusion, Theorem 2.4 can be proved by replacing in (14) Δ_ε by Δ_ε^τ, namely one has the following conclusion

$$\|x_\varepsilon(t + \tau + \Delta_\varepsilon^\tau) - x_0^\tau(t)\| \le M\varepsilon \quad \text{for any } t, \tau \in [0, T] \text{ and any } \varepsilon \in (0, \varepsilon_0]. \tag{31}$$

In particular, if for any $\varepsilon > 0$ sufficiently small there exists $\tau_\varepsilon \in [0, T]$ such that $x_\varepsilon(\tau_\varepsilon)$ belongs to S^{τ_ε} then we can take $\Delta_\varepsilon^{\tau_\varepsilon} = 0$ in (30) and (31) becomes

$$\|x_\varepsilon(t) - \widetilde{x}(t)\| \le M\varepsilon \quad \text{for any } t \in [0, T] \text{ and } \varepsilon > 0 \text{ sufficiently small.}$$

Remark 2.6. It can be checked, see e.g. ([5], formula 37), that the cycle x_0 of the example in the introduction is nondegenerate. Thus Theorem 2.4 applies to the example to ensure the existence of $\Delta_\varepsilon = (\sqrt{\varepsilon})$ for which the left hand side of (4) is uniformly bounded with respect to $t \in [0, 2\pi]$ and $\varepsilon > 0$ sufficiently small.

3. **Applications.** Let z be an eigenfunction of the adjoint system of (6)

$$\dot{z} = -(f'(x_0(t)))^* z, \tag{32}$$

which is not T-periodic. Here and in the following * denotes the transpose. We recall that an eigenfunction of a linear T-periodic system is a Floquet solution of this system, namely it is a solution z satisfying $z(t+T) = \rho z(t)$ for some $\rho \in \mathbb{R}$ and any $t \in \mathbb{R}$. Consider the scalar product

$$\left\langle z(t), \lim_{\varepsilon \to 0} \frac{x_\varepsilon(t + \Delta_\varepsilon) - x_0(t)}{\varepsilon} \right\rangle. \tag{33}$$

In this Section we provide several results about the convergence of $x_\varepsilon(t + \Delta_\varepsilon)$ to $x_0(t)$ in terms of (33). The main tool is the following scalar function

$$M_z^\perp(t) = \frac{\rho}{\rho - 1} \int_{t-T}^{t} \langle z(s), g(s, x_0(s), 0) \rangle \, ds, \tag{34}$$

where ρ is the characteristic multiplier of (32) corresponding to the eigenfunction z. The relationship between (33) and $M_z^\perp$ is shown by the following result.

Theorem 3.1. *Assume* $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $g \in C(\mathbb{R} \times \mathbb{R}^n \times [0,1], \mathbb{R}^n)$. *Let* x_ε *be a* T*-periodic solution to (1) such that*

$$\|x_\varepsilon(t + \Delta_\varepsilon) - x_0(t)\| \le M\varepsilon \quad \textit{for any } t \in [0, T] \textit{ and any } \varepsilon \in (0, \varepsilon_0], \tag{35}$$

where $\Delta_\varepsilon \to 0$ *as* $\varepsilon \to 0$, $M, \varepsilon_0 > 0$ *and* x_0 *is a nondegenerate limit cycle of (3). Let* z *be a not* T*-periodic eigenfunction of (32). Then*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \langle z(t), x_\varepsilon(t + \Delta_\varepsilon) - x_0(t) \rangle = M_z^\perp(t) \tag{36}$$

uniformly with respect to $t \in [0, T]$.

Proof. In the sequel $\varepsilon \in (0, \varepsilon_0]$, and $t, \tau \in [0, T]$. Let A be a nonsingular $n \times n$ matrix such that

$$z(0)^* A = (0, ..., 0, 1). \tag{37}$$

Let $Y(t)$ be the fundamental matrix of the linearized system (6) with initial condition $Y(0) = A$. Since A is nonsingular then the columns of $Y(t)$ are linearly independent. Let

$$Z(t) = (Y(t)^*)^{-1} \tag{38}$$

and $a_\varepsilon \in C([0, T], \mathbb{R}^n)$ is given by

$$a_\varepsilon(t) = Z(t)^* \frac{x_\varepsilon(t + \Delta_\varepsilon) - x_0(t)}{\varepsilon}.$$

Then we have

$$x_\varepsilon(t + \Delta_\varepsilon) - x_0(t) = \varepsilon Y(t) a_\varepsilon(t), \tag{39}$$

In what follows by $o(\varepsilon)$, $\varepsilon > 0$, we will denote a function, which may depend also on other variables, having the property that $\frac{o(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0$ uniformly with respect to these variables when they belong to any bounded set.

By subtracting (3), where $x(t)$ is replaced by $x_0(t)$, from (1), where $x(t)$ is replaced by $x_\varepsilon(t+\Delta_\varepsilon)$, we obtain

$$\begin{aligned}\dot{x}_\varepsilon(t+\Delta_\varepsilon)-\dot{x}_0(t) &= f'(x_0(t))(x_\varepsilon(t+\Delta_\varepsilon)-x_0(t))\\ &\quad+\varepsilon g(t+\Delta_\varepsilon, x_\varepsilon(t+\Delta_\varepsilon),\varepsilon)+o_t(\varepsilon),\end{aligned}\tag{40}$$

here $\varepsilon \to o_t(\varepsilon)$ is such that $o_{t+T}(\cdot)=o_t(\cdot)$ for any $t\in\mathbb{R}$. By substituting (39) into (40) we have

$$\varepsilon\dot{Y}(t)a_\varepsilon(t)+\varepsilon Y(t)\dot{a}_\varepsilon(t)=\varepsilon f'(x_0(t))Y(t)a_\varepsilon(t)+\varepsilon g(t+\Delta_\varepsilon,x_\varepsilon(t+\Delta_\varepsilon),\varepsilon)+o_t(\varepsilon).$$

Since $f'(x_0(t))Y(t)=\dot{Y}(t)$ the last relation can be rewritten as

$$\varepsilon Y(t)\dot{a}_\varepsilon(t)=\varepsilon g(t+\Delta_\varepsilon,x_\varepsilon(t+\Delta_\varepsilon),\varepsilon)+o_t(\varepsilon).\tag{41}$$

By means of Perron's lemma [13] (see also Demidovich ([2], Sec. III, §12) formula (37) implies that

$$z(t)^*Y(t)=(0,...,0,1)\quad\text{for any } t\in\mathbb{R}.\tag{42}$$

Therefore, applying $z(t)^*$ to both sides of (41) we have

$$\varepsilon(a_{\varepsilon,n})'(t)=\varepsilon z(t)^*g(t+\Delta_\varepsilon,x_\varepsilon(t+\Delta_\varepsilon),\varepsilon)+z(t)^*o_t(\varepsilon),$$

where $a_{\varepsilon,n}(t)$ is the n-th component of the vector $a_\varepsilon(t)$, and so

$$\begin{aligned}a_{\varepsilon,n}(t) &= a_{\varepsilon,n}(t_0)+\int_{t_0}^{t}\langle z(\tau),g(\tau+\Delta_\varepsilon,x_\varepsilon(\tau+\Delta_\varepsilon),\varepsilon)\rangle\,d\tau\\ &\quad+\int_{t_0}^{t}\left\langle z(\tau),\frac{o_\tau(\varepsilon)}{\varepsilon}\right\rangle d\tau.\end{aligned}\tag{43}$$

From (38) we have that $Z(0)^*Y(0)=I$. Therefore

$$([Z(0)]_n)^*A=(0,...,0,1),$$

where $[Z(0)]_n$ is the n-th column of $Z(0)$. Thus $[Z(0)]_n=z(0)$ and so $a_{\varepsilon,n}(t)$ satisfies

$$a_{\varepsilon,n}(t_0)=\rho a_{\varepsilon,n}(t_0-T)\quad\text{for any } t_0\in[0,T].\tag{44}$$

Solving (43)-(44) with respect to $a_{\varepsilon,n}(t_0)$ we obtain

$$\begin{aligned}a_{\varepsilon,n}(t_0) &= \frac{\rho}{\rho-1}\int_{t_0-T}^{t_0}\langle z(\tau),g(\tau+\Delta_\varepsilon,x_\varepsilon(\tau+\Delta_\varepsilon),\varepsilon)\rangle\,d\tau\\ &\quad+\frac{\rho}{\rho-1}\int_{t_0-T}^{t_0}\left\langle z(\tau),\frac{o_\tau(\varepsilon)}{\varepsilon}\right\rangle d\tau\quad\text{for any } t_0\in[0,T].\end{aligned}$$

On the other hand taking the scalar product of (39) with $z(t)$ and using (42) we obtain

$$\langle z(t),x_\varepsilon(t+\Delta_\varepsilon)-x_0(t)\rangle=\varepsilon a_{\varepsilon,n}(t)$$

and thus

$$\begin{aligned}\frac{1}{\varepsilon}\langle z(t),x_\varepsilon(t+\Delta_\varepsilon)-x_0(t)\rangle &= \frac{\rho}{\rho-1}\int_{t_0-T}^{t_0}\langle z(\tau),g(\tau,x_0(\tau),0),0)\rangle\,d\tau\\ &\quad+\frac{\rho}{\rho-1}\int_{t_0-T}^{t_0}\langle z(\tau),g(\tau+\Delta_\varepsilon,x_\varepsilon(\tau+\Delta_\varepsilon),\varepsilon)-g(\tau,x_0(\tau),0)\rangle\,d\tau\\ &\quad+\frac{\rho}{\rho-1}\int_{t_0-T}^{t_0}\left\langle z(\tau),\frac{o_\tau(\varepsilon)}{\varepsilon}\right\rangle d\tau.\end{aligned}$$

The proof is complete. □

Remark 3.2. It is of some interest to compare the scalar function $M_z^\perp$ introduced in (34) with the Malkin's bifurcation function (see [12], formula 3.13) that for system (1) takes the form

$$M_{z_0}(t) = \int_0^T \langle z_0(s), g(s-t, x_0(s), 0)\rangle \, ds, \tag{45}$$

where z_0 is a T-periodic solution of system (32). This bifurcation function was employed by Loud (see [10] formula 3.48) to study for system (1) the convergence of x_ε to x_0.

Remark 3.3. Under the regularity assumptions (8), Malkin in [12] and Loud in [10] proved that if

$$M_{z_0}(0) = 0 \text{ and } (M_{z_0})'(0) \neq 0, \tag{46}$$

then (14) is valid with $\Delta_\varepsilon = 0$. Furthermore, letting

$$y_0(t) = \lim_{\varepsilon \to 0} \frac{x_\varepsilon(t) - x_0(t)}{\varepsilon} \tag{47}$$

Malkin ([12], formulas 4.3-4.4) showed that y_0 is a T-periodic solution of

$$\dot{y} = f'(x_0(t))y + g(t, x_0(t), 0)$$

and Loud ([10], formula 1.3, Lemma 1 and formula for x at p. 510) up to a change of coordinate represented y_0 in the form

$$\begin{aligned} y_0(t) = \; & \Phi_0(t) \textstyle\int_0^t \Phi_0^{-1}(s) g(s, x_0(s), 0) ds + C\dot{x}_0(t) \\ & + \Phi_0(t) \begin{Vmatrix} 0 & \alpha^{-1}\beta(B-I)^{-1}B \\ 0 & -(B-I)^{-1}B \end{Vmatrix} \int_0^T \Phi_0^{-1}(s) g(s, x_0(s), 0) ds, \end{aligned} \tag{48}$$

where Φ_0 is a suitable fundamental matrix of the linearized system (6), $\alpha \in \mathbb{R}$, $\beta^* \in \mathbb{R}^{n-1}$, and B is a $n-1 \times n-1$ matrix defined by means of $\Phi_0(0)$ and $\Phi_0(T)$. Assuming that z is not a T-periodic eigenfunction of (32) the question if (47)-(48) imply (36) (with $\Delta_\varepsilon = 0$) was not addressed in the previous papers. This means that our Theorem 3.1 represents also a contribution to the case when f, g satisfy (8) without assuming (46).

In the sequel by using (36) of Theorem 3.1 and the properties of $M_z^\perp$ several results about the behavior of $\frac{x_\varepsilon(t+\Delta_\varepsilon) - x_0(t)}{\varepsilon}$ as $\varepsilon \to 0$ are given. By $\cos\angle(a,b) = \frac{\langle a,b\rangle}{\|a\|\cdot\|b\|}$ we denote the cosine of the angle between the vectors a, $b \in \mathbb{R}^n$.

Corollary 3.4. *Assume all the conditions of Theorem 3.1. Then for any eigenfunction z of (32), any $t \in [0,T]$ and sufficiently small $\varepsilon > 0$ we have*

$$\begin{aligned} &\textit{if} \quad M_z^\perp(t) > 0 \quad \textit{then} \quad \cos\angle\left(z(t), x_\varepsilon(t+\Delta_\varepsilon) - x_0(t)\right) > 0, \\ &\textit{if} \quad M_z^\perp(t) < 0 \quad \textit{then} \quad \cos\angle\left(z(t), x_\varepsilon(t+\Delta_\varepsilon) - x_0(t)\right) < 0. \end{aligned}$$

To prove Corollary 3.4 it is sufficient to observe that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \|z(t)\| \, \|x_\varepsilon(t+\Delta_\varepsilon) - x_0(t)\| \cos\angle\left(z(t), x_\varepsilon(t+\Delta_\varepsilon) - x_0(t)\right) = M_z^\perp(t) \tag{49}$$

obtained by substituting

$$\begin{aligned} &\langle z(t), x_\varepsilon(t+\Delta_\varepsilon) - x_0(t)\rangle \\ &\qquad = \|z(t)\| \, \|x_\varepsilon(t+\Delta_\varepsilon) - x_0(t)\| \cos\angle\left(z(t), x_\varepsilon(t+\Delta_\varepsilon) - x_0(t)\right) \end{aligned}$$

into formula (36).

The next result is a direct consequence of (49).

Corollary 3.5. *Assume all the conditions of Theorem 3.1. If there exists a not T-periodic eigenfunction z to (32) such that $M_z^\perp(t) \neq 0$ for any $t \in [0,T]$ then*

$$c_1\varepsilon \le \|x_\varepsilon(t+\Delta_\varepsilon) - x_0(t)\| \le c_2\varepsilon \quad \text{for any } t \in [0,T]$$

for some $0 < c_1 \le c_2$, and sufficiently small $\varepsilon > 0$.

Combining Theorem 2.4 and Theorem 3.1 we can derive the following fact.

Corollary 3.6. *Assume all the conditions of Theorem 2.4. Assume that $T > 0$ is the least period of x_0. If there exists a not T-periodic eigenfunction z to (32) such that $M_z^\perp(0) \neq 0$ then*

$$x_\varepsilon(t) \neq x_0(0) \quad \textit{for any} \ \ t \in [0,T]$$

provided that $\varepsilon > 0$ is sufficiently small.

Proof. Equivalently we prove that $x_\varepsilon(t) \neq x_0(0)$ for any $t \in [-T/2, T/2]$. Arguing by contradiction we assume that there exist sequences $\varepsilon_k \to 0$ and $[-T/2,T/2] \ni t_k \to t_0$ as $k \to \infty$ such that

$$x_{\varepsilon_k}(t_k) = x_0(0) \quad \text{for any } k \in \mathbb{N}. \tag{50}$$

We have that

$$t_k = \Delta_{\varepsilon_k} \text{ for } k \in \mathbb{N} \text{ sufficiently large,} \tag{51}$$

where Δ_{ε_k} are given by Corollary 2.3. Indeed, if (51) does not hold then from Corollary 2.3 we obtain that $0 < r_0 < |t_k| \le T/2$. Therefore, passing to the limit in (50) we have $x_0(t_0) = x_0(0)$ with $0 < |t_0| \le T/2$. This contradicts the fact that $T > 0$ is the least period of x_0 and so (51) holds true. Hence, from (50) we have that $x_{\varepsilon_k}(\Delta_{\varepsilon_k}) = x_0(0)$ for any $k \in \mathbb{N}$ sufficiently large and passing to the limit as $k \to \infty$ in (36) with $\varepsilon = \varepsilon_k$ we obtain $M_z^\perp(0) = 0$ contradicting our assumptions. □

Observe that Corollary 3.6, unlike the other Corollaries of this Section, requires that the Δ_ε in (35) are those given by Corollary 2.3. This is the reason for assuming the conditions of Theorem 2.4 in Corollary 3.6.

In order to establish sufficient conditions to ensure that the convergence rate of x_ε to x_0 as $\varepsilon \to 0$ is of order greater than ε^1 we need the following preliminary result.

Lemma 3.7. *Assume all the conditions of Theorem 3.1. Assume that system (32) has n linearly independent eigenfunctions. If $\varepsilon_0 > 0$ is sufficiently small, then for every $\varepsilon \in (0, \varepsilon_0]$ such that*

$$x_\varepsilon(\Delta_\varepsilon) \neq x_0(0)$$

there exists a not T-periodic eigenfunction z of (32) satisfying

$$|\cos\angle\left(z(0), x_\varepsilon(\Delta_\varepsilon) - x_0(0)\right)| \ge \alpha_*, \tag{52}$$

where $\alpha_* > 0$ does not depend on ε.

To prove Lemma 3.7 we need the following property of eigenfunctions of (32).

Lemma 3.8. *Assume that $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and let x_0 be a nondegenerate T-periodic limit cycle of (3). Assume that system (32) has n linearly independent eigenfunctions. Denote by $z_1, ..., z_{n-1}$ the eigenfunctions of (32) which are not T-periodic. Let z_0 be the T-periodic eigenfunction of (32) such that*

$$\langle \dot{x}_0(0), z_0(0)\rangle = 1. \tag{53}$$

Then the last column of $\left((z_1(t), ..., z_{n-1}(t), z_0(t))^{-1}\right)^*$ is $\dot{x}_0(t)$.

Proof. Let $(y_1(t), ..., y_n(t)) = \left((z_1(t), ..., z_{n-1}(t), z_0(t))^{-1}\right)^*$, $t \in \mathbb{R}$. We want to show that $y_n(t) = \dot{x}_0(t)$, $t \in \mathbb{R}$. Since

$$(y_1(t), ..., y_n(t))^* \, (z_1(t), ..., z_{n-1}(t), z_0(t)) = I$$

then

$$\langle y_n(t), z_i(t)\rangle = 0 \quad \text{for any } i \in \overline{1, n-1} \text{ and } \langle y_n(t), z_0(t)\rangle = 1, \tag{54}$$

whenever $t \in \mathbb{R}$. Let us show that

$$\langle \dot{x}_0(0), z_i(0)\rangle = 0, \qquad \text{for any} \quad i \in \overline{1, n-1}. \tag{55}$$

Indeed, let $i \in \overline{1, n-1}$. Since eigenfunctions z_i are not T-periodic then $\rho_i z_i(0) = z_i(T)$ for some $\rho_i \neq 1$ that gives $\rho_i \langle \dot{x}_0(0), z_i(0)\rangle = \langle \dot{x}_0(T), z_i(T)\rangle$. On the other hand Perron's lemma [13] implies that $\langle \dot{x}_0(0), z_i(0)\rangle = \langle \dot{x}_0(T), z_i(T)\rangle$, thus $\langle \dot{x}_0(0), z_i(0)\rangle = 0$ and so (55) holds true. But the vectors $z_1(0), z_2(0), ..., z_0(0)$ form a basis of $\mathbb{R}^n$, hence by (53), (54) and (55) we get $y_n(0) = \dot{x}_0(0)$. □

Proof of Lemma 3.7. Let $z_1, ..., z_{n-1}$ be linearly independent not T-periodic eigenfunctions of (32). Let us show that there exists $i_* \in \overline{1, n-1}$ such that the conclusion holds with z replaced by z_{i_*}. Assume the contrary, therefore there exist $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0, 1]$, $\{\alpha_k\}_{k\in\mathbb{N}} \subset [-1, 1]$ with $\varepsilon_k \to 0$ and $\alpha_k \to 0$ as $k \to \infty$ such that

$$\cos\angle\left(z_i(0), x_{\varepsilon_k}(\Delta_{\varepsilon_k}) - x_0(0)\right) = \alpha_k \quad \text{for any } i \in \overline{1, n-1}$$

and

$$x_{\varepsilon_k}(\Delta_{\varepsilon_k}) \neq x_0(0).$$

Let $v_k \in \mathbb{R}^{n-1}$ be such that $S(v_k) = x_{\varepsilon_k}(\Delta_{\varepsilon_k})$. Therefore we have

$$\frac{\langle z_i(0), S(v_k) - S(0)\rangle}{\|z_i(0)\| \cdot \|S(v_k) - S(0)\|} = \alpha_k \quad \text{for any } i \in \overline{1, n-1},$$

and so

$$\frac{\left\langle z_i(0), \frac{S(v_k) - S(0)}{\|v_k\|}\right\rangle}{\|z_i(0)\| \cdot \left\|\frac{S(v_k) - S(0)}{\|v_k\|}\right\|} = \alpha_k \quad \text{for any } i \in \overline{1, n-1}. \tag{56}$$

Passing to a subsequence if necessary, we may assume that $\frac{v_k}{\|v_k\|} \to q_0$ as $k \to \infty$, thus $\|q_0\| = 1$. Taking the limit as $k \to \infty$ in (56) we obtain

$$\langle z_i(0), S'(0)q_0\rangle = 0 \quad \text{for any } i \in \overline{1, n-1}. \tag{57}$$

Since $q_0 \neq 0$ then $S'(0)q_0 \neq 0$. Otherwise we would have $\Omega'_\xi(T, 0, x_0(0))A_{n-1}q_0 = 0$, where Ω'_ξ is nonsingular being a fundamental matrix to (6) (see [8], Theorem 2.1). But this means that $A_{n-1}q_0 = 0$ contradicting the definition of A_{n-1}. Denoting by z_0 the T-periodic eigenfunction of (32) satisfying (53) from (57) we obtain

$$(z_1(t), ..., z_{n-1}(t), z_0(t))^* \, S'(0)q_0 = (0, \cdots, 0, a)^T$$

with $a \neq 0$. From this formula we have

$$S'(0)q_0 = \left((z_1(t), ..., z_{n-1}(t), z_0(t))^*\right)^{-1} (0, \cdots, 0, a)^T$$

and by Lemma 3.8 we have $S'(0)q_0 = a\dot{x}_0(0)$ contradicting Lemma 2.2. □

By combining Theorem 3.1 and Lemma 3.7 we have the following result.

Corollary 3.9. *Assume all the conditions of Theorem 3.1. Assume that the linearized system (6) has n linearly independent eigenfunctions. Assume that* $M_z^{\perp}(0) = 0$ *for any not T-periodic eigenfunction z of the adjoint system (32). Then*

$$\lim_{\varepsilon\to 0} \frac{\|x_\varepsilon(\Delta_\varepsilon) - x_0(0)\|}{\varepsilon} = 0.$$

Proof. By contradiction assume that there exist $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,1]$, $\varepsilon_k \to 0$ as $k \to \infty$ and $c_* > 0$ such that

$$\frac{\|x_{\varepsilon_k}(\Delta_{\varepsilon_k}) - x_0(0)\|}{\varepsilon_k} \geq c_*. \tag{58}$$

From (58) we have that the assumptions of Lemma 3.7 are satisfied, thus there exists a not T-periodic eigenfunction z of (32) such that (52) holds, but this contradicts (49). The proof is complete. □

Corollaries 3.4 and 3.9 allow us to formulate the following result.

Corollary 3.10. *Assume all the conditions of Theorem 3.1. Assume that system (32) has n linearly independent eigenfunctions and let* $\varepsilon_0 > 0$ *sufficiently small. Then there exists a not T-periodic eigenfunction z of (32) such that either*

$$\cos\angle\left(z(0), x_\varepsilon(\Delta_\varepsilon) - x_0(0)\right) \neq 0 \quad \text{for any} \quad \varepsilon \in (0, \varepsilon_0]$$

or

$$\lim_{\varepsilon\to 0} \frac{\|x_\varepsilon(\Delta_\varepsilon) - x_0(0)\|}{\varepsilon} = 0.$$

Finally, observe that from Remark 3.3 it follows that under Malkin's or Loud's assumptions Corollaries 3.4, 3.5, 3.6, 3.9, 3.10 hold true with $\Delta_\varepsilon = 0$.

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