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ON THE SOLVABILITY OF NONLINEAR EQUATIONS IN BANACH SPACES

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0. Introduction

In this note we are concerned with the solvability of nonlinear systems of the form

$$(1) \qquad \begin{cases} 0 = f(x,y) \\ \\ 0 = g(x,y) \end{cases}$$

with f and g continuous maps defined on the closure of an open subset U of the product of two Banach spaces X, Y of the form

$$f(x,y) = y - \overline{f}(x,y)$$

and

$$g(x,y) = x - \overline{g}(x,y)$$

where $\overline{f} : \overline{U} \to Y$ and $\overline{g} : \overline{U} \to X$ are compact maps.

In our approach, instead of considering sufficient conditions for the existence of fixed points of the compact map

$$(\overline{g},\overline{f}) : \overline{U} \to X \times Y,$$

we follow closely the "alternative method" developed by Cesari and his co-workers (see [4] for an extensive bibliography) in solving equations arising from nonlinear perturbations of Fredholm operators. Roughly speaking, we "solve" the first equation in term of y as a "function" of x and hence we introduce the solution $S(x)$ in the second one. The main goal of this paper consists in dropping the assumption on the *unique solvability* of the first equation. Thus we consider the multivalued *solution* map

$$x \longmapsto S(x) = \{y \in Y : \quad f(x,y) = 0\}$$

whose graph, in the $X \times Y$ space is the *solution* set

$$S = \{(x,y) : \quad f(x,y) = 0\}$$

of the first equation. Then we seek zeros of the multivalued map $x \longmapsto T(x) = g(x,S(x))$ which are obviously the only solutions of (1).

When $X = \mathbb{R}$ (the real line), the above approach has been used implicitly by several authors (see for example [15, 1, 10, 2] and it will be briefly described below.

Let us make the following assumption

(i) U is locally bounded over X

(2) (ii) the equation $0 = f(x,y) = y - \overline{f}(x,y)$ has no solutions on ∂U

(iii) for some $x \in X$, $\deg(f(x,\cdot), U(x), 0) \neq 0$, where $U(x) = \{y \in Y : (x,y) \in U\}$.

Under the assumption (2), given any continuous map $g : S \to \mathbb{R}$ the multivalued map $T(x) = g(x,S(x))$ has the intermediate value property. Namely:

(3) if for some $x_0, x_1 \in \mathbb{R}$ we have that $T(x_0) \subset \mathbb{R}_+$ and $T(x_1) \subset \mathbb{R}_-$, then $0 \in T(x)$ for some $x \in [x_0, x_1]$.

Thus (3) gives a sufficient condition for the solvability of system (1).

The proof of the above assertion follows directly from the existence of a connected subset C of S joining $\{x_0\} \times S(x_0)$ and $\{x_1\} \times S(x_1)$. The last has been proved in [3] under assumption (2). Notice that the converse is also true. Indeed, if for each continuous function $g : S \to \mathbb{R}$, the map $T(x) = g(x,S(x))$ has the intermediate value property (between x_0 and x_1) then we can ensure the existence of a connected set C as above.

The natural extension of the intermediate value property of a continuous function of a real variable to higher dimensional spaces is the well known Borsuk-Ulam theorem (see [9, p. 21]). Suppose for the moment that the solution map is a singlevalued one. Then, by the Borsuk-Ulam theorem we can assert the existence of a solution of (1) provided that for the compact vector field

$$T(x) = g(x,S(x)) = x - \bar{g}(x,S(x))$$

the following holds

(4) there exists $r > 0$ such that for each x with $\|x\| = r$ $T(x) \neq tT(-x)$ for any $t \geq 0$.

In Theorem (1.1) we show that also in the case when the solution map is not singlevalued then a Borsuk-Ulam type condition is sufficient to ensure the existence of a solution of system (1). More precisely:

Let f and g be as in (1). If f verifies assumption (2) then the multivalued map $T(x) = g(x,S(x))$ has a zero in a ball $B \subset X$ provided that

(5) for each $x \in \partial B$ there exists a continuous functional x^* on X such that $x^*(T(x)) \subset \mathbb{R}_+$ and $x^*(T(-x)) \subset \mathbb{R}_-$.

Notice that if T is a singlevalued map then (4) and (5) are equivalent.

Moreover we show, in Theorem (1.2), that under assumptions (2) if $\bar{g} : S \to X$ is any continuous map, then the multivalued map $\bar{T}(x) = \bar{g}(x,S(x))$ has the fixed point property with respect to compact and convex sets. Namely, if $K \subset X$, is compact convex with $\bar{T}(K) \subset K$, then there exists $x \in K$ such that $x \in \bar{T}(x)$ and hence the equation

$$\begin{cases} y - \bar{f}(x,y) = 0 \\ x - \bar{g}(x,y) = 0 \end{cases}$$

has a solution (x,y) with $x \in K$.

Thus, Theorems (1.1) and (1.2) can be viewed from one side as an extension of Liapunov-Schmidt to the case of nonuniqueness of solutions of the equation $f(x,y) = 0$ and on the other side as an improvement to dimensions higher than one of the connectivity property described in [15]. This will be the object of a forthcoming paper.

Although several particular cases of Theorem (1.1) can be proved by direct degree arguments, our proof is of independent interest. Roughly speaking, we show that the solution map $x \longmapsto S(x)$ and hence also $x \longmapsto T(x)$ can be approximated *in graph* by a particular class of finite valued upper-semicontinuous maps to which the ordinary degree and the fixed point theory of continuous maps can be extended. This class of multivalued maps, called weighted maps (w-maps), was first introcuced by Darbo [5]. The existence of zeros for the multivalued map T will be obtained, using w-maps, in the same way as the zeros of convex valued vector fields can be obtained by approximation with singlevalued maps.

We would like to add, in passing, that the same method used in Theorem (1.1) could lead to a different approach to define some invariants of parametrized families of maps known in algebraic topology as *transfer homomorphism* and *generalized index* (see [7, 8, 12]). This was already observed in [8].

The paper is divided into two parts.

In the first part we state our main results (Theorem (1.1) and Theorem (1.4)) and some corollaries. Finally, we consider the particular case of equations arising from nonlinear perturbations of Fredholm operators.

The second part is devoted to the proof of Theorem (1.1). First of all, we establish some basic properties of the solution map $x \longmapsto S(x)$. Next, we show that the map f can be arbitrarily approximated by maps f' having, at each level x, a finite number of zeros. Then we discuss briefly the notion of w-maps and we show that the solution maps of our approximation are actually w-maps. In the last part we prove the Borsuk-Ulam theorem for w-maps and we use it in order to prove Theorem (1.1). We close this section by proving Theorem (1.2) and Corollary (1.3).

1. General results

Throughout this section X, Y will denote real Banach spaces and $B = B(0,r)$ the closed ball centered at zero with radius r. Furthermore, given any subset S of $X \times Y$ we denote by

$$S(x) = \{y \in Y : (x,y) \in S\}$$

$$S(A) = \{y \in Y : (x,y) \in S, \quad x \in A\} \qquad \text{for } A \subset X,$$

$$S_x = S \cap (\{x\} \times Y) \text{ and by } S_A = S \cap (A \times Y) \qquad \text{for } A \subset X.$$

Let U be a subset of $X \times Y$. We shall say that U is *locally bounded over* X if for any $(x,y) \in U$ there exists a neighborhood N of x such that U_N is a bounded subset of $X \times Y$.

Let $U \subset X \times Y$ be open and locally bounded over X. We shall say that $f : \overline{U} \to Y$ is a *parametrized compact vector field* if $f(x,y) = y - \overline{f}(x,y)$ with $\overline{f}$ continuous and $\overline{f}(D)$ relatively compact in Y for any bounded subset D of U. Given any parametrized compact vector field $f : \overline{U} \to Y$ we shall denote by

$$S^f = \{(x,y) \in \overline{U} : \quad f(x,y) = 0\}$$

$$\mathcal{D}^f = \{x \in X : \quad S^f_x \cap \partial U = \phi\}.$$

When no confusion will arise, we shall omit the superscript f in the above notations.

REMARK

The set valued map $x \multimap S(x) = S^f(x)$ from $\mathcal{D}$ into Y is compact valued. Furthermore, $\mathcal{D}$ is an open subset of X and the map $x \multimap S(x)$ from $\mathcal{D}$ into Y is upper-semicontinuous (u.s.c.)(see Proposition (2.1)).

If f is defined on all $X \times Y$, taking

$$\mathcal{D} = \left\{ x \in X \;\middle|\; \begin{array}{l} x \text{ is not bifurcation point from infinity of the} \\ \text{equation } f(x,y) = 0 \end{array} \right\}$$

we have that $\mathcal{D}$ is an open subset of X and the map $x \multimap S(x)$ is u.s.c. from $\mathcal{D}$ into Y (see Proposition (2.2)).

Let $A_1, A_2 \subset X$. We shall say that A_1 and A_2 are *strictly separated by an hyperplane* (s.s.h.) if there exists a continuous functional $x^* \in X^*$ (the dual of X) such that $x^*(x) > 0$ for all $x \in A_1$ and $x^*(x) < 0$ for all $x \in A_2$. We are now able to state the main results. For this the following assumption will be taken once and for all.

$$\text{(A)} \quad \begin{cases} \text{The set } U \subset X \times Y \text{ is open in } X \times Y \text{ and locally bounded over } X. \text{ The map} \\ f : U \to Y \text{ is a parametrized compact vector field such that } 0 \in \mathcal{D} \text{ and the} \\ \text{Leray-Schauder degree } \deg(f(0,\cdot),\ U(0),0) \neq 0. \end{cases}$$

For any parametrized compact vector field $g : U \to X$ let us define $T : \mathcal{D} \multimap X$ by $T(x) = g(x,S(x))$.

(1.1) *THEOREM*

Under assumption (A) suppose that:

(i) there exists $r > 0$ such that $B = B(0,r) \subset \mathcal{D}$

(ii) for any $x \in \partial B$ such that $0 \notin T(x)$, $T(x)$ and $T(-x)$ are strictly separated by an hyperplane.

Then there exists $x \in B$ such that $0 \in T(x)$ and hence the system

$$\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases} \tag{1.1}$$

has a solution in U.

REMARK

Let us note that Theorem (1.1) holds for g of the form $g(x,y) = x - \bar{g}(x,y)$, where $\bar{g}$ is a continuous map defined on S_B and such that $\bar{g}(S_B)$ is a relatively compact subset of X.

The following result states that if $\overline{g}$ is any continuous map defined on the solution set S of a map f verifying (A) and with values in X then the multivalued map $\overline{T}(x) = \overline{g}(x,S(x))$ has the fixed point property with respect to compact convex sets. Namely:

(1.2) *THEOREM*

Assume that (A) holds. Furthermore, let $K \subset X$ be a compact convex set such that $0 \in K \subset \mathcal{D}$. Let $\overline{g} : S \to X$ be any continuous map and let $\overline{T}(x) = \overline{g}(x,S(x))$, if $\overline{T}(x) \subset K$ for any $x \in K$, then $\overline{T}$ has a fixed point in K, hence the system.

$$\begin{cases} y = \overline{f}(x,y) \\ x = \overline{g}(x,y) \end{cases} \qquad (1.1)'$$

has a solution.

REMARK

The above theorem holds for maps $f : \overline{U} \to X$ of the form $f(x,y) = y - \overline{f}(x,y)$ where $\overline{f} : \overline{U} \to Y$ is a map such that

(i) $\overline{f}(x,\cdot) : \overline{U}(x) \to Y$ is compact for each x

(ii) $\overline{f}$ is continuous in x uniformly in y.

Let $g : \overline{U} \to X$ be a parametrized compact vector field. Under assumption (A) the following corollaries are consequences of Theorem (1.2) and they can be regarded, via the associated multivalued map $\overline{T}$, as an extension of Leray-Schauder and Krasnosel'skiǐ fixed point theorems for single-valued compact maps.

(1.3) *COROLLARY*

Suppose that (A) and (i) of Theorem (1.1) hold. Then (1.1)' has a solution in U_B provided that the following holds:

(L.S.) If $x \in \partial B$ and $tx \in \overline{T}(x) = \overline{g}(x,S(x))$ then $t \leq 1$.

(1.4) *COROLLARY*

Let X be a Hilbert space (we denote by $<\cdot,\cdot>$ its scalar product). Assume that (A) and (i) of Theorem (1.1) hold. Then system (1.1) has a solution in U_B provided that $x \in \partial B$ and $y \in T(x) = g(x,S(x))$ imply $<x,y> \geq 0$.

SOLVABILITY OF NONLINEAR PERTURBATIONS OF FREDHOLM OPERATORS

Although Theorem (1.1) covers as particular cases several coincidence problems we would like to specialize it to the case of solvability of equations arising from nonlinear perturbations of Fredholm operators.

In the real Hilbert space $H = L^2(\Omega)$, where Ω is a bounded domain in a finite dimensional real Euclidean space, let $L : D(L) \subset H \to H$ be a closed linear operator with dense domain and closed range. Assume that dim Ker $L < +\infty$ and Im $L = (\text{Ker } L)^{\perp}$ (the orthogonal complement). Hence H has an orthogonal decomposition $H = \text{Ker } L \oplus \text{Im } L$. For $u \in H$, we set $u = v + w = Pu + (I - P)u$ with $v \in \text{Ker } L$ and $w \in \text{Im } L$. L is therefore a one to one map of $D(L) \cap \text{Im } L$ onto Im L. Assume furthermore that the "inverse" $L^{-1} : \text{Im } L \to \text{Im } L$ is compact. Let $h : H \to H$ be a continuous and bounded map (that is h maps bounded sets into bounded sets). We are interested in the solvability of the nonlinear equation

$$Lu = h(u) \quad \text{in } H. \tag{1.2}$$

Using the *alternative method*, equation (1.2) can be converted into the following system.

$$\begin{cases} w = L^{-1}(I - P)h(v + w) & \text{auxiliary equation} \\ 0 = Ph(v + w) & \text{bifurcation equation.} \end{cases} \tag{1.3}$$

The solvability of (1.3) can now be handled in two different ways. Roughly speaking, first solve the auxiliary equation considered as an equation in w with v as parameter. Introduce then the solution map $S(v)$ in the bifurcation equation. Solve the resulting equation

$$0 \in Ph(v + S(v)).$$

Conversely, we could solve first the bifurcation equation considered as an equation in v with w as parameter. Introduce then the solution map $S(w)$ in the first equation and then solve

$$w \in L^{-1}(I - P)\ h(S(w) + w).$$

We state now two results on the solvability of equation (1.2). Theorem (1.5) below is related to the first approach described above and this can be viewed as an abstract formulation of the so called Landesman-Lazer type conditions.

In Theorem (1.6) we will follow the second approach on the solvability of (1.2).

(1.5) *THEOREM*

Let $S(v) = \{w \in \text{Im } L : Lw = (I - P)h(v + w)\}$. Assume that there exist $r > 0$ and $R > 0$ such that

(i)' $S(v) \subset B(0,R) \subset \text{Im } L$ for all $v \in B(0, r) \subset \text{Ker } L$

(ii)' $\deg(I - L^{-1}(I - P)h\big|_{\text{Im } L}, B(0,R), 0) \neq 0$

(iii)' for $v \in \text{Ker } L$ with $\|v\| = r$ either $0 \in Ph(v + S(v))$ or there exists $v' \in \text{Ker } L$ such that

$$\int_\Omega h(v + w)v' < 0 \qquad \text{for } w \in S(v)$$

$$\int_\Omega h(-v + w)v' > 0 \qquad \text{for } w \in S(-v)$$

Then equation (1.2) has a solution in H.

REMARK

(i)' is equivalent to say that the auxiliary equation has no bifurcation points from infinity in $B(0,r)$.

(i)' and (ii)' are certainly verified if h is uniformly bounded.

Proof

Let $V = \text{Ker } L$ and $W = \text{Im } L$. Since equation (1.2) is equivalent to system (1.3), via the obvious isomorphism $V \times W \to H$ we can rewrite system (1.3) as

$$\begin{cases} f(v,w) = 0 \\ g(v,w) = 0 \end{cases}$$

where

$$f(v,w) = w - L^{-1}(I - P)h(v + w)$$

and

$$g(v,w) = Ph(v + w).$$

Assumption (i)' implies that $B(0,r) \subset \mathcal{D}^f$, while (ii)' implies (A).

The last assumption in Theorem (1.1) is satisfied by the functional v^* defined on Ker L as

$$v^*(v) = \int_\Omega v\, v' . \blacksquare$$

(1.6) *THEOREM*

Set $S(w) = \{v \in \text{Ker } L : 0 = Ph(v + w)\}$. Assume that

(i)" there exists a continuous function $\rho : \text{Im } L \to \mathbb{R}_+$ such that

$$Ph(v + w) = 0 \quad \text{implies} \quad \|v\| \le \rho(w)$$

(this is equivalent to assuming that Ph does not have any bifurcation points from infinity, see the proof of Proposition (2.2)).

(ii)" $\deg\,(Ph|_{\text{Ker } L \,\cap\, B(0,\rho(0))},\ B(0,\ \rho(0)),\ 0) \neq 0$;

(iii)" there exists $r > 0$ such that if $w \in D(L)$ $\|Lw\| = r$ then either

$Lw = h(v+w)$ for some $v \in S(w)$ or there exists $w' \in \operatorname{Im} L$ such that

$$\int_\Omega Lw\, w' > \int_\Omega h(v+w)\, w' \qquad \text{whenever } v \in S(w) \tag{1.4}$$

and

$$-\int_\Omega Lw\, w' < \int_\Omega h(\bar{v}-w)\, w' \qquad \text{whenever } \bar{v} \in S(-w). \tag{1.5}$$

Then equation (1.2) has a solution in H.

Proof

Since L is a closed Fredholm operator, $D(L)$, endowed with the norm $|||u||| = \|Lu\| + \|Pu\|$, becomes a Banach space in which $L : D(L) \to H$ is bounded. Furthermore $D(L)$ splits in a topological direct sum of $W = D(L) \cap \operatorname{Im} L$ and $V = \operatorname{Ker} L$.

Equation (1.2) is equivalent to the following system

$$\begin{cases} f(v,w) = 0 \\ g(v,w) = 0 \end{cases}$$

where $f(v,w) = Ph(v,w)$ and $g(v,w) = w - L^{-1}(I-P)h(v+w)$. Under the assumptions (i)' and (ii)', $\mathcal{D}^f = W$ and both (A) and (i) in Theorem (1.1) are satisfied. Let $w \in W$ with $|||w||| = \|Lw\| = r$ such that $Lw \neq h(v,w)$ whenever $Ph(v+w) = 0$. Let w^* be defined by

$$w^*(z) = \int_\Omega Lz\, w' \qquad \text{for } z \in W.$$

By the boundedness of L it follows that $w^* \in W^*$. By (1.4),

$$\int_\Omega Lw\, w' > \int_\Omega (I-P)h(v+w)w'$$

since

$$Ph(v+w) = 0.$$

Hence

$$\int_{\Omega} [Lw - LL^{-1}\ (I - P)h(v + w)]\, w' > 0$$

and so

$$\int_{\Omega} Lg(v,w)w' = w^*\ (g(v,w)) > 0 \qquad \text{for all} \quad v \in S(w).$$

Similarly

$$w^*(g(v,\ -w)) < 0 \qquad \text{for all} \quad v \in S(-w).$$

This completes the proof. ■

2. BORSUK-ULAM THEOREM AND THE PROOFS

PROPERTIES OF THE "SOLUTION MAP" $x \multimap S(x)$

(2.1) *PROPOSITION*

Let U be an open subset of $X \times Y$, locally bounded over X. Let $f : \overline{U} \to Y$ be a parametrized compact vector field. Then $\mathcal{D}$ is an open subset of X and the map $x \multimap S(x)$ from $\mathcal{D}$ into Y is u.s.c. (that is, $S(x)$ is compact and if V is a neighborhood of $S(x)$ then $S(x') \subset V$ provided that x' is close enough to x.).

Proof

Let $x_0 \in \mathcal{D}$. We shall prove that if V is any open set with $S(x_0) \subset V$ then there exist neighborhoods N of x_0 in X and V' of $S(x_0)$ in V such that $N \times V' \subset U$ and $S(x) \subset V'$ for any $x \in N$. Indeed, given $y \in S(x_0)$ consider neighborhoods of the form $N_y \times V_y$ where N_y is a neighborhood of x_0 in $\mathcal{D}$ and V_y, a neighborhood of y in Y, such that

$$V_y \subset U(x_0) \cap V \qquad \text{and} \qquad N_y \times V_y \subset U.$$

By the compactness of S_{x_0}, we can choose a finite number of neighborhoods of the above type, say $N_1 \times V_1$, $N_2 \times V_2, \dots, N_r \times V_r$ which cover S_{x_0}. Let

$$N_0 = \bigcap_{i=1}^{r} N_i \quad \text{and} \quad V' = \bigcup_{i=1}^{r} V_i .$$

Clearly for each neighborhood N of x_0, with $N \subset N_0$ we have that $N \times V' \subset U$. It remains to show that

(*) there exists a neighborhood N of x_0 such that $S(x) \subset V'$ for all $x \in N$.

Without loss of generality we can assume that U_{N_0} is a bounded set. Suppose now that there are no $N \subset N_0$ for which (*) holds. Then, we can construct a bounded sequence $\{x_n, y_n\}$ such that $\{x_n\}$ converges to x_0, $y_n \notin V'$ and $y_n = \overline{f}(x_n, y_n)$. By the compactness of $\overline{f}$ we can assume (by passing to an appropriate subsequence) that $\{y_n\}$ converges to some $y_0 \in S(x_0)$, contradicting $S(x_0) \subset V'$. ■

(2.2) *PROPOSITION*

Let $f : X \times Y \to Y$ be a parametrized compact vector field. Let

$$\mathcal{D} = \left\{ x \in X \,\middle|\, \begin{array}{l} x \text{ is not a bifurcation point from infinity for the equation} \\ f(x,y) = 0 \end{array} \right\}.$$

Then $\mathcal{D}$ is an open subset of X and the map $x \multimap S(x)$ from $\mathcal{D}$ into Y is u.s.c.

Proof

Assume that x is not a bifurcation point from infinity for the equation $f(x,y) = 0$. Then there exist a positive number r_x and a neighborhood N_x of x such that $S(x') \subset B(0,r_x)$ for any $x' \in N_x$. In particular it follows that $\mathcal{D}$ is an open subset of X. Let W_i be a locally finite refinement of $\{N_x\}_{x \in \mathcal{D}}$. Hence, for any i, there exists x_i such that $W_i \subset N_{x_i}$. Let $r_i = r_{x_i}$, then for any $x' \in W_i$ we have that $S(x') \subset \mathring{B}(0,r_i)$. If s_i is the partition of unity subordinated to $\{W_i\}$, then the function

$$r(x) = \sum_j s_j(x) r_j$$

is continuous from $\mathcal{D}$ into $\mathbf{R}_+$ and so the following set

$$U = \{(x,y) \in X \times Y : x \in \mathcal{D} \text{ and } \|y\| < r(x)\}$$

is open in $X \times Y$ and locally bounded over X. The assertion follows from Proposition (1.1) applied to $f|_{\overline{U}}$ (that is, the restriction of f to the closure $\overline{U}$ of U). ∎

APPROXIMATION LEMMAS

In the following. U will denote an open subset of $X \times Y$ locally bounded over X and $f : \overline{U} \to Y$ a parametrized compact vector field.

(2.3) *LEMMA*

Let $X = \mathbb{R}^n$ and let $B = B(0,r) \subset \mathcal{D}$. Then for any neighborhood W of S_B^f in U there exists $\varepsilon > 0$ such that if $f_1 : \overline{U} \to Y$ verifies $\|f_1(p) - f(p)\| < \varepsilon$ for all $p \in \overline{U}_B$, then $S_B^{f_1} \subset W$.

Proof

Set $A = \overline{U}_B \setminus W$. Then A is closed and bounded in $X \times Y$. Since B is compact, the projection $\pi : B \times Y \to Y$ is a proper map. Hence $f = \pi - \overline{f}$ is a closed map being a compact perturbation of a proper map. Therefore

$$\inf_{(x,y) \in A} \{\|f(x,y)\|\} = \varepsilon > 0.$$

If $f_1 : \overline{U} \to Y$ is an ε-approximation of f in $\overline{U}_B$ then clearly $S_B^{f_1} \subset W$. ∎

For the proof of Lemma (2.4) we shall need the following result due to Kurland and Robbin [13, Theorem (6.1)].

THEOREM (K.R.)

Let P, M and N be manifolds with $\dim(M) = \dim(N)$. Then there is an open dense $G \subset C^\infty(P \times M, N)$, endowed with the fine C^∞-topology, such that each $f \in G$ has the property that the map $f(p,\cdot) : M \to N$ is locally finite to-one (that is, every point of M has a neighborhood U such that $f^{-1}(n) \cap U$ is finite for all $n \in N$). In particular the inverse image of each point is discrete.

REMARK

The proof of the above result involves an "Infinite codimensional Lemma" (see [13, p. 139] and [16, p. 150]) and transversality theory.

(2.4) *LEMMA*

Let $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$ and $B = B(0,r) \subset \mathcal{D}$. For any $\varepsilon > 0$, there exists a $f_1 : \overline{U} \to Y$ such that $f_1|_{\overline{U}_B}$ is an ε-approximation of $f|_{\overline{U}_B}$ and

(i) $S^{f_1}(x)$ is a finite subset of $U(x)$ for all $x \in B$;

(ii) $\deg(f(0,\cdot), U(0), 0) = \deg(f_1(0,\cdot), U(0), 0)$.

Proof

Let B_1 be a closed ball such that $B \subset \mathring{B}_1 \subset \mathcal{D}$. Then $\overline{U}_{B_1}$ is a bounded subset of $X \times Y$ which is contained in $B_1 \times B_2$ for some ball $B_2 \subset Y$. Let us still denote by f any continuous extension of f to all $K = B_1 \times B_2$. Let $\varepsilon > 0$, and $\varepsilon' = \min\{\varepsilon, \rho\}$ where

$$\rho = \inf_{x \in B_1} \operatorname{dist}(S_x, \partial U).$$

Since K is compact, f is a uniformly continuous map on K and so there exists a $\delta > 0$ such that $\|f(p) - f(p')\| < \varepsilon'/2$ for any $p, p' \in K$ with $\|p - p'\| < \delta$. Let us consider a finite covering of K consisting of balls D_i centered at the points $p_i \in K$ of radius $\delta/2$. Let $\{s_i\}$ be a C^∞-partition of unity subordinated to $\{D_i\}$ and let

$$f'(p) = \sum_i s_i(p) f(p_i).$$

Clearly $f'(p)$ is a C^∞-map on K and for $p \in K$ we have that

$$\|f(p) - f'(p)\| < \varepsilon'/2.$$

By Theorem (K.R.) there exists a C^∞-map $f_0 : \mathring{K} \to Y$ such that $f_0(x,\cdot)$ has only isolated zeros for any $x \in \mathring{B}$ and

$$\|f'(p) - f_0(p)\| < \varepsilon'/2$$

for all $p \in \mathring{K}$. Let f_1 be any continuous extension of $f_0|_{\overline{U}_B}$ to all $\overline{U}$. Then clearly f_1 is an ε-approximation of f which verifies (i). The assertion (ii) follows directly from the invariance property of the degree for small perturbations. ∎

WEIGHTED MAPS

Now we introduce a particular class of multivalued maps which is of fundamental importance in the proof of Theorem (1.1) and we will show that the solution map $S'(x)$ of the ε-approximation given by Lemma (2.4) is actually a w-map. The reason will be briefly described below.

Let X, Y be finite dimensional spaces. As we mentioned in the introduction in solving system (1.1) we seek zeros of the multivalued map $T(x) = g(x, S(x))$. Under the hypotheses of Theorem (1.1), T is u.s.c. and verifies the Borsuk-Ulam condition on the boundary ∂B of a ball in X. (see definition below).

Furthermore using Lemmas (2.3), (2.4) we can modify f to f' in such a way that $S'(x) = S^{f'}(x)$ is arbitrarily near to S and it is a finite valued map. Hence also T can be approximated by the finite valued u.s.c. map $T'(x) = g(x, S'(x))$. Clearly the fact that S' is a finite valued map is not sufficient to ensure the existence of zeros for T'. But the map S', being a "solution map", has another nice characteristic: namely, each point $y \in S(x)$ has an assigned multiplicity as solution of the equation $0 = f(x,y)$ and roughly speaking the multiplicity changes "nicely" with respect to x.

Such a class of finite valued u.s.c. maps has been introduced by G. Darbo [5, 6], under the name of weighted maps. Weighted maps form a good category of multivalued maps that enlarges that of single valued ones, remaining adequate for the fixed point and degree theory. Now since T' is arbitrarily close to T, it follows that also T' verifies the Borsuk-Ulam condition on ∂B. Then, by the Borsuk-Ulam theorem for w-maps, T' has a zero in B and this ensures the existence of a zero for T.

Let us recall briefly Darbo's results: let X, Y be topological Hausdorff spaces.

(2.5) *DEFINITION*

A finite valued u.s.c. map $F : X \multimap Y$ will be called *weighted map* (shortly w-map) if to each x and $y \in F(x)$ a multiplicity or weight $m(y, F(x)) \in \mathbb{Z}$ is assigned in such a way that the following property holds

(a) if U is an open set in Y with $\partial U \cap F(x) = \phi$, then

$$\sum_{y \in F(x) \cap U} m(y, F(x)) = \sum_{y' \in f(x') \cap U} m(y', F(x'))$$

whenever x' is close enough to x.

(2.6) *REMARK*

The number $i(F(x), U) = \sum_{y \in F(x) \cap U} m(y, F(x))$ will be called the index or multiplicity of $F(x)$ in U. Notice that (a) in Definition (2.5) is a local invariance property of the index. It states that the index of $F(x')$ in U for x' close enough to x coincides with the index of $F(x)$ in U whenever $\partial U \cap F(x) = \phi$. In particular if X is connected the number $i(F(x), Y)$ does not depend on $x \in X$. In this case the number $i(F) = i(F(x), Y)$ will be called the index of the weighted map F.

Actually as defined by Darbo a wieighted map is an equivalence class of maps verifying (a). But Definition (2.5) is more adequate to our purposes and all the results proved in [5] hold also in this context.

Let X, Y, Z denote topological Hausdorff spaces. The following properties have been proved in [5] (see also [11, 14]).

1) The sum of two w-maps $F, G : X \multimap Y$ defined as the u.s.c. map $F + G(x) = F(x) \cup G(x)$ with multiplicities given by $m(y, F + G(x)) = m(y, F(x)) + m(y, G(x))$, is a w-map. In an analogous form is defined λF for $\lambda \in \mathbb{Z}$ (where we pose $m(y, F(x)) = 0$ whenever $y \notin F(x)$).

2) Given any two w-maps $F : X \multimap Z$ and $G : Y \multimap W$, the product $F \times G : X \times Y \multimap Z \times W$ defined as the u.s.c. map $F \times G(x,y) = \{(z,w) : z \in F(x) \text{ and } w \in G(y)\}$ with $m((z,w), (F \times G)(x,y)) = m(z, F(x)) \cdot m(w, G(y))$ is a w-map.

3) If $F : X \multimap Y$ and $G : Y \multimap Z$ are w-maps, then the composition $G \circ F : x \multimap Z$ becomes a w-map by assigning multiplicities

$$m(z, G \circ F(x)) = \sum_{\substack{y \in F(x) \\ z \in G(y)}} m(z, G(y)) \cdot m(y, F(x)).$$

Moreover, if X and Y are connected then

$$i(G \circ F) = i(G) \cdot i(F).$$

4) Any continuous singlevalued map $f : X \to Y$ can be considered as a w-map by assigning multiplicity 1 to each $f(x)$.

It follows from properties 1), 2), 3), 4) that the category having as objects Hausdorff spaces and as morphisms w-maps is an additive category containing as subcategory that of singlevalued continuous maps. The notion of homotopy in this category, so called σ-homotopy in [5, 14], is defined in the same way as for continuous maps, that is, two weighted maps $F, G : X \multimap Y$ are σ-homotopic if there exists a weighted map $H : X \times [0,1] \multimap Y$ such that $H(x,0) = F(x)$ and $H(x,1) = G(x)$. Notice that if X is connected then F is σ-homotopic to G implies that $i(F) = i(G)$. Furthermore, in [5] Darbo constructed a homology functor $H = \{H_n\}_{n \geq 0}$ defined in this category, compatible with the σ-homotopy, and such that it verifies the Eilenberg-MacLane axioms for a homology theory.

Hence the restriction of H to the category consisting of continuous maps between compact absolute-neighborhood retracts coincides with the ordinary singular homology functor. We would like to add in passing that by means of this functor, Darbo extended the Lefschetz fixed-point theorem to w-maps from a compact A.N.R. into itself (see [6]). Let X, Y be Banach spaces.

(2.7) *LEMMA*

Let $f : \overline{U} \subset X \times Y \to Y$ be a parametrized compact vector field defined in the clousure of an open and locally bounded set U. If $f(x,\cdot)$ has only isolated zeros, then the map $x \multimap S(x)$ is a w-map from $\mathcal{D}^f$ into Y with $i(S) = \deg(f(x,\cdot), U(x), 0)$.

Proof

By Proposition (2.1), the map $x \multimap S(x)$ is an u.s.c. finite valued map from $\mathcal{D}$ into Y. Hence it is enough to show that to every $y \in S(x)$ we can assign an integer $m(y, S(x))$ with the property described in Definition (2.5). For this, let $y \in S(x)$. Since y is an isolated zero of $f(x,\cdot): U(x) \to Y$, we define $m(y, S(x))$ to be the multiplicity of y as a zero of $f(x,\cdot)$ (that is $m(y, S(x)) = \deg(f(x,\cdot), \Omega, 0)$ where Ω is an open neighborhood of y such that $\Omega \cap S(x) = \{y\}$. By the excision property of the degree $m(y, S(x))$ does not depend on the particular choice of Ω). We will see that $m(\cdot, S(\cdot))$ verifies property (a) of Definition (2.5). Let W be an open subset of Y such that $S(x) \cap \partial W = \phi$. Then by the uppersemicontinuity of S, there exists a ball $B(x,r)$ such that for any $x' \in B(x,r)$ we have $S(x') \cap \partial W = \phi$. Without loss of generality, we can assume that $B(x,r) \times W \subset U$. For $x' \in B(x,r)$, let $H : W \times [0,1] \to Y$ be defined by $H(y,t) = f(tx + (1-t)x', y)$. Since $tx + (1-t)x' \in B(x,r)$ for all $t \in [0,1]$, H is an admissible homotopy between $f(x,\cdot)|_W$ and $f(x',\cdot)|_W$. This and the additivity of the degree imply that

$$\sum_{y \in S(x) \cap W} m(y, S(x)) = \deg(f(x,\cdot), W, 0) = \deg(f(x',\cdot), W, 0)$$

$$= \sum_{y \in S(x') \cap W} m(y, S(x')). \blacksquare$$

BORSUK-ULAM THEOREM FOR W-MAPS

In the following we shall extend, in a suitable form, the classical Borsuk-Ulam theorem for continuous map to the context of weighted maps.

Let $B \subset X$ be a closed ball centered at the origin. We shall say that an u.s.c. map $F : B \to Y$ verifies the *Borsuk-Ulam property on* ∂B if

(B.U.) for each $x \in \partial B$, $F(x)$ and $F(-x)$ are strictly separated by a hyperplane.

(2.8) *THEOREM*

Let B be the unit ball in $\mathbb{R}^n$. Let $F : B \multimap \mathbb{R}^n$ be a w-map with $i(F) \neq 0$. If F verifies (B.U.) on ∂B then there exists $x \in \mathring{B}$ such that $0 \in F(x)$.

Proof

Notice that it sufficies to show that if $\widetilde{F}$ denotes the restriction of F to ∂B then the homomorphism induced by $\tilde{F}: \partial B \multimap \mathbb{R}^n \setminus \{0\}$ in the n-th homology group is not trivial (that is $\tilde{F}_*: H_{n-1}(\partial B) \to H_{n-1}(\mathbb{R}^n \setminus \{0\})$ is different from the zero map). In fact, if $0 \notin F(B)$ we get that $H: \partial B \times I \to \mathbb{R}^n \setminus \{0\}$ defined by $H(x,t) = F(tx)$ is a σ-homotopy between $\tilde{F}$ and the "constant" w-map $G(x) = F(0)$. Hence $\tilde{F}_* = G_* = 0$. We shall show that $\tilde{F}_*$ is not trivial by constructing a σ-homotopy between $\tilde{F}$ and $i(F)f$ where $f: \partial B \to \mathbb{R}^n \setminus \{0\}$ is a singlevalued odd continuous map. This will prove the theorem since the oddness of f implies that $f_*: H_{n-1}(\partial B) \approx \mathbb{Z} \to H_{n-1}(\mathbb{R}^n \setminus \{0\}) \approx \mathbb{Z}$ is a multiplication by an odd number and hence, if $i(F) \neq 0$, we have that $(i(F)f)_* = i(F)f_* \neq 0$. For this, let us observe that the (B.U.)-condition in $\mathbb{R}^n$ states that

$$(*)\quad \begin{array}{l} \text{for each } x \in \partial B \text{ there exists } y \in \partial B \text{ such that} \\ \qquad \langle y,z \rangle > 0 \text{ for all } z \in \tilde{F}(x) \\ \qquad \langle y,z \rangle < 0 \text{ for all } z \in \tilde{F}(-x). \end{array}$$

For $y \in \partial B$, let $V_y = \{x \in \partial B : (*) \text{ holds}\}$. Since $co(\tilde{F}(x))$ and $co(\tilde{F}(-x))$ are compact and since $\tilde{F}$ is u.s.c. it follows that V_y is an open subset of ∂B for each y. By the (B.U.)-condition we have that $\{V_y\}_{y \in \partial B}$ is covering of ∂B. Let $\{V_{y_i}\}$, $0 \le i \le m$, be a subcovering of $\{V_y\}_{y \in \partial B}$ and let $s_i : \partial B \to [0,1]$, $0 \le i \le m$ be the partition of the unity subordinated to $\{V_{y_i}\}$.

Set $f(x) = \sum_{i=0}^{m} (s_i(x) - s_i(-x))y_i$. Then $f: \partial B \to \mathbb{R}^n$ is an odd continuous map. Consider $H: \partial B \times I \multimap \mathbb{R}^n$ defined as the composition

$$\partial B \times I \xrightarrow{\Delta \times Id} \partial B \times \partial B \times I \xrightarrow{\tilde{F} \times f \times Id} \mathbb{R}^n \times \mathbb{R}^n \times I \xrightarrow{g} \mathbb{R}^n$$

where $\Delta: \partial B \to \partial B \times \partial B$ is the diagonal map and

$$g(x, y, t) = tx + (1-t)y.$$

Clearly, H is a w-map (as composition of w-maps). Furthermore, $H(x,0) = i(F)f(x)$ and $H(x,1) = \tilde{F}(x)$ for all $x \in \partial B$. Actually we shall prove that the image of H is contained in $\mathbb{R}^n \setminus \{0\}$. In fact, if $0 \in H(x,t)$ then there exist $z \in \tilde{F}(x)$ and $t \in [0,1]$ such that

$$tz = -(1-t)f(x) = -(1-t)\left(\sum_{i=0}^{m} s_i(x)y_i - \sum_{i=0}^{m} s_i(-x)y_i\right).$$

Clearly, $t \neq 1$. Then for $\mu = -\frac{t}{1-t} < 0$ we have that

$$\mu z = \sum_{i=0}^{m} s_i(x)y_i - \sum_{i=0}^{m} s_i(-x)y_i .$$

Hence

$$(*) \qquad \mu \|z\|^2 = \sum_{i=0}^{m} s_i(x) \langle y_i, z\rangle - \sum_{i=0}^{m} s_i(-x) \langle y_i, z\rangle.$$

But, if $\langle y_i, z\rangle < 0$ then $x \notin V_{y_i}$ and so $s_i(x) = 0$. On the other hand,

$$\text{if } \langle y_i, z\rangle > 0 \text{ then } -x \notin V_{y_i} \text{ and so } s_i(-x) = 0.$$

This implies that the right-hand side of equality (*) is always **positive**, which is impossible. Hence $H : \partial B \times I \multimap \mathbb{R}^n \setminus \{0\}$ is a σ-homotopy between $\tilde{F}$ and $i(F)f : \partial B \to \mathbb{R}^n \setminus \{0\}$. This achieves the proof. ■

REMARK

Notice that the degree of a w-map $F : \partial B \multimap \mathbb{R}^n \setminus \{0\}$ verifying the (B.U.)-condition is not necessarily odd but it depends on the parity of $i(F)$. By degree we mean the unique $d \in \mathbb{Z}$ such that

$$F_*(\text{generator } H_{n-1}(\partial B)) = d \cdot \text{generator } H_{n-1}(\mathbb{R}^n \setminus \{0\}).$$

MORE ON THE BORSUK-ULAM CONDITION

Now we shall give a more geometric formulation of the (B.U.)-condition. This will be used in proving that an u.s.c. map close enough to one that verifies (B.U.) also has the (B.U.)-property.

First of all notice that if A, B are compact subsets of X, then A, B are s.s.h. (strictly separated by a hyperplane) if and only if $\overline{\text{co}}A$, $\overline{\text{co}}B$ are s.s.h. where $\overline{\text{co}}A$ denotes the closed convex hull of A.

Let $A \subset X$ be bounded and convex. We shall denote by

$$K(A) = \{\lambda x : \lambda \in [0,1] \text{ and } x \in A\}$$

the bounded convex cone of A with vertex at the origin. Notice that $K(A)$ is compact whenever A is compact.

(2.9) *PROPOSITION*

Let $A, B \subset X$ be compact. A, B are s.s.h. if and only if

$$\begin{cases} K(\overline{coA}) \cap \overline{coB} = \phi \\ K(\overline{coB}) \cap \overline{coA} = \phi . \end{cases} \qquad (2.1)$$

Proof

The "only if" part is trivial. To show the converse, let us notice that by Hahn-Banach separation property from (2.1) it follows that there exist $\varphi, \psi \in X^*$ and $\alpha, \beta \in \mathbb{R}$ such that

$$\varphi(K(\overline{coA})) \subset (\alpha,+\infty), \qquad \varphi(\overline{coB}) \subset (-\infty,\alpha)$$

and

$$\psi(K(\overline{coB})) \subset (\beta,+\infty), \qquad \psi(\overline{coA}) \subset (-\infty,\beta).$$

Since $0 \in K(\overline{coA}) \cap K(\overline{coB})$ it follows that $\alpha, \beta < 0$. If we consider $\theta = \alpha\psi - \beta\varphi$, it is easy to show that A, B are strictly separated by the hyperplane defined by θ. ■

(2.10) *PROPOSITION*

$F : B \multimap X$ verifies the (B.U.)-property if and only if the multivalued map $\widetilde{F} : B \multimap X$ defined by

$$\widetilde{F}(x) = K(\overline{co}F(x)) - \overline{co}F(-x)$$

has no zeros on ∂B.

Proof

Let $x \in \partial B$. Clearly

$$0 \notin \tilde{F}(x) \quad \text{if and only if} \quad K(\overline{co}F(x)) \cap \overline{co}F(-x) = \phi.$$

Analogously,

$$0 \notin \tilde{F}(-x) \quad \text{if and only if} \quad K(\overline{co}F(-x)) \cap \overline{co}F(x) = \phi.$$

Now the assertion follows from Proposition (2.9).■

(2.11) *PROPOSITION*

If $F : B \multimap X$ is a multivalued compact vector field then $\tilde{F}$ is u.s.c. and proper.

Proof

The uppersemicontinuity of $\tilde{F}$ follows easily from the definition. In order to show that $\tilde{F}$ is a proper map, let us observe that if $F(x) = x - \bar{F}(x)$ with $\bar{F}$ u.s.c. and compact, then

$$\tilde{F}(x) = K(x - \overline{co}\bar{F}(x)) + x + \overline{co}\bar{F}(-x).$$

Let C be compact subset of X and let $\{x_n\}$ be a sequence in $\tilde{F}^{-1}(C)$. Let $\{y_n\}$, $y_n \in C \cap \tilde{F}(x_n)$. By passing to a subsequence, if necessary, we can assume that $y_n \to y \in C$. By the definition of $\tilde{F}$, we can write $y_n = \lambda_n(x_n - u_n) + x_n + v_n$ with $u_n \in \overline{co}\bar{F}(x_n)$ and $v_n \in \overline{co}\bar{F}(-x_n)$. Since $\overline{co}\bar{F}(B)$ is a precompact subset of X and since $\lambda_n \in [0,1]$, passing to subsequences if it is necessary, we can assume that $\lambda_n \to \lambda \in [0,1]$, $u_n \to u$, $v_n \to v$. Then

$$x_n = \frac{y_n + \lambda_n u_n - v_n}{\lambda_n + 1}$$

converges to some $x \in B$. By the uppersemicontinuity of $\tilde{F}$ and since $y_n \in \tilde{F}(x_n)$ and $y_n \to y$, we have that $y \in \tilde{F}(x)$ so $x \in \tilde{F}^{-1}(C)$. ■

(2.12) *REMARK*

Let $F, F' : X \multimap Y$ be u.s.c. maps and $\varepsilon > 0$. Denote by GrF the graph of F and εA the set of points of distance less than ε from A.

The equivalence of the following statements is easy to check

(i) $GrF' \subset \varepsilon\, GrF$

(ii) $\rho^+(GrF'.\ GrF) < \varepsilon$ where ρ^+ is the upper Hausdorff separation

(iii) for each $x \in X$, $F'(x) \subset \varepsilon F(\varepsilon x)$.

Here $X \times Y$ is considered with the norm $\|(x,y)\| = \sup(\|x\|, \|y\|)$.

DEFINITION

We shall say that an u.s.c. multivalued map $F': X \multimap Y$ is an *ε-approximation of* $F : X \multimap Y$ if one and hence any one of the above statements is satisfied.

(2.13) *PROPOSITION*

Let $F : B \multimap X$ be a compact vector field satisfying the (B.U.)-property. Then there exists $\varepsilon > 0$ such that any ε-approximation $F' : B \multimap X$ of F satisfies the (B.U.)-property.

<u>*Proof*</u>

By Proposition (2.10), we have that $\tilde{F}$ has no zeros on ∂B. Actually, using the properness of $\tilde{F}$, we shall show that there exists $\varepsilon' > 0$ such that

$$0 \notin \varepsilon' \tilde{F}(\varepsilon' x) \quad \text{for all } x \in \partial B. \tag{2.2}$$

In fact, assuming the contrary, there exist sequences $\{\varepsilon_n\}$, $\{x_n\}$ with $\varepsilon_n \to 0$ $x_n \in \partial B$ and such that $0 \in \varepsilon_n \tilde{F}(\varepsilon_n x_n)$, that is, there exist $\{x_n'\}$ and $\{y_n\}$ with $x_n' \in \partial B$, $\|x_n' - x_n\| < \varepsilon_n$, $\|y_n\| < \varepsilon_n$, $y_n \in \tilde{F}(x_n')$. Since $y_n \to 0$, by the properness of $\tilde{F}$, it follows that $\{x_n'\}$ has a convergent subsequence to some $x \in \partial B$, and so $0 \in \tilde{F}(x)$. Contradicting the fact that $\tilde{F}$ has no zeros of ∂B. Let $\varepsilon = \frac{1}{2}\varepsilon'$.

If $F' : B \multimap X$ is an ε-approximation of F we have that

$$F'(x) \subset \varepsilon F(\varepsilon x) \subset \varepsilon F(\varepsilon' x). \tag{2.3}$$

Let us notice that if $A, B \subset X$ then the following relations hold

$$\begin{aligned} &(i) \quad \overline{co}\varepsilon A \subset \varepsilon\overline{co}A \\ &(ii) \quad K(\varepsilon A) \subset \varepsilon K(A) \\ &(iii) \quad \varepsilon A - \varepsilon B \subset 2\varepsilon(A - B). \end{aligned} \tag{2.4}$$

From (2.3) and (2.4) it follows easily that for each $x \in \partial B$ we have

$$\tilde{F}'(x) = K(\overline{co}F'(x)) - \overline{co}F'(-x) \subset \varepsilon'\tilde{F}(\varepsilon' x).$$

Hence $0 \notin \tilde{F}'(x)$ for all $x \in \partial B$ and so the assertion follows from Proposition (2.10). ■

Proof of Theorem (1.1)

Let us consider the multivalued map $T : B \multimap X$ defined by $T(x) = g(x, S(x)) = x - \overline{g}(x, S(x))$. By Proposition (2.1), T is u.s.c.. Moreover, since $\overline{g}$ is compact, T is a multivalued compact vector field. Hence T is proper. Suppose now that the system (1.1) has no solutions (x,y) with $x \in B$. Then $0 \notin T(x)$ for all $x \in B$ and so, as in the proof of Proposition (2.13), there exists $\varepsilon_1 > 0$ such that

(*) $$0 \notin \varepsilon_1 T(\varepsilon_1 x) \quad \text{for all } x \in B.$$

On the other hand from the assumptions in Theorem (1.1), it follows that T verifies the (B.U.)-property on ∂B and so, by Proposition (2.13), there exists ε_2 such that any u.s.c. multivalued map $T' : B \multimap X$ with $GrT' \subset \varepsilon_2 GrT$ has the (B.U.)-property on ∂B. Let $\delta = \min\{\varepsilon_1, \varepsilon_2\}$ and let $V \subset U$ be defined by

$$V = \{(x,y) \in U : (x, g(x,y)) \in \delta GrT\}.$$

Clearly V is an open set being the inverse image of the open set δGrT by a continuous map. Furthermore, V is a neighborhood of S_B. We will divide the rest of the proof in three steps.

<u>1.st step.</u> $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$.

Let ε' be given by Lemma (2.3), that is any ε'-approximation f_1 of f has the property that $S_B^{f_1} \subset V$. By Lemmas (2.4) and (2.7), there exists a continuous map $f_1 : V \to \mathbb{R}^n$ which ε'-approximates f on $\overline{V}_B$ and such that the multivalued map $x \longmapsto S'(x) = S^{f_1}(x)$ from B into $\mathbb{R}^n$ is a w-map with

$$i(S'(x)) = \deg(f(0,\cdot), V(0), 0) = \deg(f(0,\cdot), U(0), 0) \neq 0$$

and such that $S'_B \subset V$. By properties 3) and 4) following Remark (2.6), the multivalued map

$$T'(x) = g(x, S'(x))$$

is a w-map with $i(T') = i(S') \neq 0$. Furthermore, since $S'_B \subset V$, we have that $GrT' \subset \delta GrT$. By our choice of δ it follows that T' verifies the (B.U.)-property on ∂B and hence by Theorem (2.8) there exists $x \in \mathring{B}$ such that $0 \in T'(x)$. Hence $0 \in \delta T(\delta x)$, contradicting (*).

<u>2.nd step.</u> $X = \mathbb{R}^n$ and Y any Banach space.

Since U_B is a bounded subset of $X \times Y$ and f is a parametrized compact vector field, there exists an ε'-approximation $\overline{f}_1$ of $\overline{f}$ on $\overline{U}_B$ with range contained in a finite dimensional subspace Y_1 of Y. Set $f_1 = Id - \overline{f}_1$. By Lemma (2.3) and the homotopy property of the degree we have that

$$S_B^{f_1} \subset V \cap X \times Y_1 = V_1,$$

and

$$\deg(f_1(0,\cdot), V(0), 0) = \deg(f(0,\cdot), V(0), 0) \neq 0.$$

Hence $S_B^{f_1} \neq \phi$ and so V_1 is a nonempty subset of $X \times Y_1$ which is locally bounded over X. Furthermore, by the reduction property of the degree for

$$f_2 = f_1\big|_{V_1}$$

we have that

$$\deg(f_2(0,\cdot), V_1(0), 0) = \mathrm{def}(f_1(0,\cdot), V(0), 0) \neq 0.$$

Let $g_2 = g\big|_{V_1}$. It is clear that the pair (g_2, f_2) satisfies the assumptions of Theorem (1.1). Hence, by the 1^{st} step, the multivalued map

$$T_2(x) = g_2(x, S^{f_2}(x))$$

has a zero in B. But, $S_B^{f_2}(x) \subset V$ and so $\mathrm{Gr}T_2 \subset \delta\,\mathrm{Gr}T$, contradicting (*).

<u>3^{rd} step</u>. X, Y any Banach spaces.

Let $\bar{g}_1 : \bar{U}_B \to X$ be a finite dimensional ε-approximation of $\bar{g}$ on $\bar{U}_B$. Let X_1 denote any finite dimensional subspace of X containing the range of $\bar{g}_1$. Set

$$g_1 = \mathrm{Id} - \bar{g}_1\Big|_{X_1 \times Y \cap \bar{U}_B}$$

and

$$f_1 = f\Big|_{X_1 \times Y \cap \bar{U}_B} .$$

Let $T' : B' = B \cap X_1 \multimap X_1$ be defined by $T'(x) = g_1(x, S(x))$. Since the multivalued map T', considered as a map from B' into X, is an ε-approximation of the restriction of T to B', it follows, from Proposition (2.13), that T' has the (B.U.)-property on $\partial B'$. Therefore the pair (f_1, g_1) verifies the assumption (ii) of Theorem (1.1). Since (i) and (A) are direct consequences of the definition of f_1, by the 2^{nd} step we have that T' has a zero in $B' \subset B$. But this contradicts (*). ∎

<u>*Proof of Theorem* (1.2)</u>

It is easy to see that under the assumptions (A) the map $\bar{f} : \bar{U}_K \to Y$ is compact. On the other hand since K is an absolute retract and S_K is closed in $X \times Y$ the map $\bar{g} : S_K \to K$ can be extended to a continuous map defined on all of $X \times Y$ with values in K that we still denote by $\bar{g}$.

Now let $r : X \to K$ be any retraction and let

$$U' = \{(x,y) \in X \times Y : (r(x), y) \in U\}.$$

Then U' is an open subset of $X \times Y$ which is locally bounded over X. Let us consider the maps

$$f : \overline{U}' \to Y,$$

$$g : \overline{U}' \to X$$

defined by

$$f(x,y) = y - \overline{f}(r(x), y),$$

$$g(x,y) = x - \overline{g}(x,y).$$

Since $\overline{f}(\overline{U}_K)$ is precompact and $\overline{g}(X \times Y) \subset K$ we have that f and g are parametrized compact vector fields. We shall see that f and g verify the hypotheses of Theorem (1.1). First of all notice that under our assumptions $\mathcal{D} = X$. Moreover, by the generalized homotopy invariance of degree

$$\deg(f(0,\cdot), U'(0), 0) = \deg(f(x,\cdot), U'(x), 0) = \deg(Id - \overline{f}(x,\cdot), U(x), 0)$$

for each $x \in K$. Therefore by (A) , $\deg(f(0,\cdot), U'(0), 0)$ is different from zero.

Let $B = B(0,r)$ be such that $K \subset \overset{\circ}{B}$. Let us show that the multivalued map $T : B \multimap X$ defined by

$$T(x) = g(x, S(x))$$

verifies the (B.U.)-property on ∂B. By Proposition (2.9) this is equivalent to prove that $K(\overline{co}\ T(x)) \cap \overline{co}\ T(-x) = \phi$ for all $x \in \partial B$. Indeed, if this is not true then there exist $\lambda \in [0,1]$ $y_1 \in \overline{co}\ \overline{g}(x, S(x)) \subset K$ and $y_2 \in \overline{co}\ \overline{g}(-x, S(-x)) \subset K$ such that

$$\lambda(x - y_1) = -x - y_2 .$$

Then

$$(\lambda + 1)\|x\| = \|\lambda y_1 - y_2\| < (\lambda + 1)r.$$

Contradicting $x \in \partial B$. Thus (ii) of Theorem (1.1) is verified and so Theorem (1.2) follows. ■

Proof of Corollary (1.3)

Let $p : X \to B$ be the canonical retraction of the space X into the unit ball B of X. The system

$$\begin{cases} y = \overline{f}(x,y) \\ x = p\overline{g}(x,y) \end{cases}$$

verifies the *hypothesis of Theorem* (1.2) with respect to the compact convex set $K = \overline{co}(p(\overline{T}(B))) \subset B$ and hence has a solution $(\overline{x},\overline{y})$ with $\overline{x} \in B$. Now because of the assumption (L.S) of Corollary (1.3) $\|g(\overline{x},\overline{y})\| \leq 1$. Therefore $p\overline{g}(\overline{x},\overline{y}) = \overline{g}(\overline{x},\overline{y})$ and hence $(\overline{x},\overline{y})$ is also a solution of (1.1)'. ■

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