Robust LQ design by a sliding manifold approach

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Abstract

A sliding manifold based control strategy is proposed for linear time invariant perturbed systems. By using the theory of singular perturbation a feedback control is designed to fulfill a LQ perfirmance criterion. It exhibits strong robustness properties with respect to exogenous disturbances and plant parametric uncertainties. The resulting control signal remains, after a fast transient, in a neighbourhood of the well–defined equivalent control. Then the control can take into account bounds on the available control signals and avoid the peaking phenomenon of high gain systems. An example is reported to show the effectiveness of the proposed strategy compared to the classic LQ design.

1 INTRODUCTION

We consider a regulation problem for a linear, time-invariant, completely controllable control system. We solve the problem by means of a sliding manifold approach based on the singular perturbation theory [1], [2]. More specifically by means of a suitable function, which depends only on the data of the considered problem, we define a sliding manifold and we introduce a dynamical feedback control which turns out to be the solution of a differential equation containing a small parameter $\varepsilon > 0$. The control after a fast transient, depending on its initial value, in which it approaches the well-defined *equivalent control* [3] remains close to the latter in the uniform topology. This approach has been introduced in [4] with the purpose of eliminating the chattering phenomenon in variable structure control systems and developed in [5] in the case of linear tracking problems. We would like to point out that, using this approach, the only fast dynamics is that of the control law, while all the state variables follow the slow dynamics. Then the relationship between the sliding mode obtained for $\varepsilon = 0$ and the states corresponding to $\varepsilon > 0$, which is prescribed by the singular perturbation theory, turns out to be different from that obtained by using high–gain techniques, in which the state is decomposed into two groups of variables, the one "fast", the second "slow" (see [6]). By a suitable selection of the design parameters the control strategy assures closed-loop performances equivalent to those obtained via the classic LQ design technique. Moreover the robustness properties are those typical of high gain controllers although the peaking phenomenon, one of the main drawbacks of the high gain policy, is avoided.

2 THE LQ REGULATION PROBLEM

Consider the linear, time invariant dynamic system

$$\dot{x} = Ax + Bu, \; x \in \mathbb{R}^n, \; u \in \mathbb{R}^m. \tag{1}$$

Now we state the regulation problem as follows.

Given $\beta > 0$, $\delta > 0$, and $x_0 \in \mathbb{R}^n$, it is required to design a feedback control law such that the solution $x(t)$ of system (1) with $x(0) = x_0$ satisfies

$$\|x(t)\| \leq \delta + Ae^{-\beta t} \tag{2}$$

for any $t \in [0, \infty)$, where A is a constant depending on the data.

Assume that B is a matrix of full rank m. Therefore, there is a nonsingular $n \times n$ matrix T such that

$$TB = \begin{pmatrix} 0 \\ B_2 \end{pmatrix}; \; B_2 \in \mathbb{R}^{m \times m} \tag{3}$$

Furthermore, if we put

$$z = Tx \tag{4}$$

then we can rewrite system (1) in the form

$$\dot{z} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} z + \begin{pmatrix} 0 \\ B_2 \end{pmatrix} u \tag{5}$$

where

$$A_{11} \in \mathrm{R}^{(n-m)\times(n-m)};\ A_{22} \in^{m\times m} \mathrm{R}$$

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix};\ z_1 \in \mathrm{R}^{n-m},\ z_2 \in \mathrm{R}^{m}. \qquad (6)$$

The regulation problem can be stated as follows. We solve an LQ regulation problem with performance index

$$\int_0^{\infty} (x'Qx + u'Ru)dt \qquad (7)$$

where Q and R are suitable weighting matrices. From the classical LQ theory we know that the solution to problem (1), (7) gives a control law $u = -Kx$ and a closed loop dynamic matrix

$$C_x = A - BK \qquad (8)$$

The aim of the proposed approach is to obtain a behaviour that reproduces the one of the classical LQ method and moreover guarantees robustness properties stronger than those of the LQ.

To solve this problem we propose a sliding manifold approach based on the singular perturbation theory. For this, define a function $s : \mathrm{R}^n \times [0,\infty) \to \mathrm{R}^m$ as follows

$$s(z,t) = H\left(-z + e^{Ct}z_0\right), \qquad (9)$$

where $z_0 = z(0), H = (H_1 \quad H_2)$ with H_1, H_2 $m \times (n-m)$ and $m \times m$ matrices respectively and C is given by

$$C = TC_xT^{-1} \qquad (10)$$

The matrix H will be chosen later in a suitable way. Observe that $s(z_0, 0) = 0$.

Define the related sliding manifold $\mathcal{S}$ as follows

$$\mathcal{S} = \{(z,t) \in \mathrm{R}^n \times \mathrm{R}_+ : s(z,t) = 0\}. \qquad (11)$$

Thus $(z_0, 0) \in \mathcal{S}$.

For any $\varepsilon > 0$, we form the system of differential equations

$$\dot{z} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} z + \begin{pmatrix} 0 \\ B_2 \end{pmatrix} u \qquad (12)$$

$$\varepsilon\dot{u} = (H_1 \quad H_2)\left(-\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} z - \begin{pmatrix} 0 \\ B_2 \end{pmatrix} u + Ce^{Ct}z_0\right) \qquad (13)$$

We have the following theorem, that can be deduced from the main result in [5]

Theorem 1 *Let δ, β and γ be given positive numbers. Assume that*

(i) $\quad Re\lambda(H_2B_2) \geq \beta;$

(ii) $\quad Re\lambda(A_{12}H_2^{-1}H_1 - A_{11}) \geq \beta + \gamma;$

Then there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0]$, the solution $(z(t,\varepsilon), u(t,\varepsilon))$ to (12) − (13) satisfying $(z(0,\varepsilon), u(0,\varepsilon)) = (z_0, u_0)$ whenever $u_0 \in \mathrm{R}^m$ is such that

$$\|z(t) - e^{Ct}z_0\| \leq \delta + Ae^{-\beta t} \qquad (14)$$

$$u(t,\varepsilon) = \frac{1}{\varepsilon}H\left(z(t,\varepsilon) + e^{Ct}z_0\right) + u_0 \qquad (15)$$

with $t \in [0,\infty)$ and A is a positive constant depending on the data.

Remark 2.1. It has been shown in [5] that it is always possible to choose H_1 in such a way that assumption (ii) is satisfied. Furthermore, since $\det(B_2) \neq 0$ we can also choose H_2 to satisfy assumption (i).

Remark 2.2. It is important to point out a property of the solution $(\tilde{z}, \tilde{u})$ of the reduced system

$$\dot{\tilde{z}} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tilde{z} + \begin{pmatrix} 0 \\ B_2 \end{pmatrix} \tilde{u} \qquad (16)$$

$$0 = (H_1 \quad H_2)\left(-\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tilde{z} - \begin{pmatrix} 0 \\ B_2 \end{pmatrix} \tilde{u} + Ce^{Ct}z_0\right) \qquad (17)$$

It is possible to show that $\tilde{z}(t) = e^{Ct}z_0$ and $\tilde{u}(t) = -K\tilde{z}(t)$.

Remark 2.3. By using the properties of the Hamilton matrix associated to the Riccati solution of the LQ problem, it is possible to show that the matrix C can be computed directly in the transformed space state by solving an LQ problem with weighting matrices $Q_z = TQT^{-1}$, $R_z = TRT^{-1}$.

Remark 3.3. Notice that the control law (15) is quite different from a high gain law. Infact we have $u(0,\varepsilon) = u_0$, while a high gain strategy would give the large initial peak $u(0) = Hz(0)/\varepsilon$.

3 ROBUSTNESS PROPERTIES

In this section we consider the perturbed system

$$\dot{x} = (A + \Delta A(t,x))x + (B + \Delta B(t,x))u \qquad (18)$$
$$x(0) = x_0$$

where $x \in \mathrm{R}^n$ and $u \in \mathrm{R}^m$ are the state and input vectors respectively, A and B are the nominal dynamic and input matrices respectively. In particular, B is assumed to have full rank, $\mathrm{rank}B = m$ and, for the sake of simplicity, to be of the form (3). Let moreover the pair (A, B) be controllable. Suppose the perturbation matrices satisfy the following conditions

1. $\Delta A : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $\Delta B : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous functions in $\mathbb{R} \times \mathbb{R}^n$.

2. They belong to the range of B, i.e. there are matrices $P(t,x)$ and $S(t,x)$ such that

$$\begin{aligned} \Delta A(t,x) &= BP(t,x), \\ \Delta B(t,x) &= BS(t,x) \end{aligned} \quad (19)$$

3. The input perturbations do not cause the input matrix to loose rank, i.e.

$$\text{rank}(B + \Delta B(t,x)) = m \quad \text{for all } (t,x) \quad (20)$$

In particular, condition (20) is necessary for the equivalent control to be well defined. A sufficient condition for condition (20) to hold is

$$\|S(t,x)\| = \alpha < 1, \quad \forall (t,x) \quad (21)$$

where $\|\cdot\|$ is the 1, 2, ∞ or Frobenius norm.

In order to apply Theorem 1 we must replace condition (i) with

$$Re\lambda\left(H_2(B_2 + \Delta B_2(t,x))\right) \geq \beta > 0 \quad \forall (t,x) \quad (22)$$

Now we state that condition (22) can be always satisfied by selecting the matrix H_2 such that the matrix H_2B_2 is symmetric, positive definite and with condition number $\chi(H_2B_2) \leq 1/\alpha$.

To prove this statement we utilize the following theorem [7]

Theorem 2 *The matrix $A = A_0 + E$, with A_0 Hurwitz-stable, is Hurwitz-stable if*

$$\bar{\sigma}(E) < \frac{1}{\bar{\sigma}(P)} \quad (23)$$

where P is the solution of the Lyapunov equation

$$A_0'P + PA_0 = -2I \quad (24)$$

In our case if we let $A_0 = -H_2B_2$ we have $P = (H_2B_2)^{-1}$ and

$$\bar{\sigma}(-H_2B_2S(t,x)) < \alpha\bar{\sigma}(H_2B_2) \leq \sigma_{\min}(H_2B_2) = \frac{1}{\bar{\sigma}(P)}.$$

Finally it is easy to compute the equivalent control

$$\begin{aligned} u_{eq} = &-\left(H_2(B_2 + \Delta B_2(t,\tilde{x}))\right)^{-1} H\left((A + \right. \\ &\left. \Delta A(t,\tilde{x}))\tilde{x} - Ce^{Ct}x_0\right) \end{aligned} \quad (25)$$

and the reduced system

$$\begin{pmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ -H_2^{-1}H_1A_{11} & -H_2^{-1}H_1A_{12} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ H_2^{-1}H_1 & I \end{pmatrix} Ce^{Ct}x_0 \quad (26)$$

Note that the reduced system does not depend on perturbations, which are utterly absorbed by the control action.

Example 2.1. Consider the simplified model of a position servo system ([8], Example 2.4)

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix} u(t) \\ &= Ax(t) + bu(t), \; x(0) = x_0 \end{aligned} \quad (27)$$

where $x(t) = (\theta(t), \dot{\theta}(t))'$, $\alpha = \beta/J$, $\kappa = k/J$, J is the moment of inertia of all the rotating parts, β the coefficient of viscous friction, k the motor constant, $u(t)$ the input voltage to the motor.

The nominal values are:

$$\alpha = 4.6s^{-1}, \; \kappa = 0.787\text{rad}/(V\cdot s^2), \; x_0 = (0.1\;0)' \quad (28)$$

The optimization criterion is

$$\min \int_0^\infty (\theta^2(t) + \rho u^2(t))dt \quad (29)$$

with $\rho = 0.00002$ rad^2/V^2. The classical LQ solution gives

$$u(t) = -kx(t), \; k = (223.6\;18.69) \quad (30)$$

Now let

$$C = A - bk \quad (31)$$

and the related function $s(x,t)$ (9)

$$s(x,t) = H(\exp(Ct)x_0 - x) \quad (32)$$

A possible selection is of the matrix H satisfying assumptions (i) and (ii) of theorem 1 is $H = (3 \quad 1)$.

In Fig. 1 the behaviour of the angular position and the input voltage are depicted for the two cases $\epsilon = 0.01$ and $\epsilon = 0.002$, and compared to the classical LQ solution. Note that the smaller ϵ is the quicker is the transient bringing the control close to the *equivalent control* (a).

The main feature of the proposed approach is its ability to reject disturbances and to compensate for parameter variations. Then the combined inertia of load and armature of the motor has been changed to 2/3 and 3/2 of its nominal value and further simulations have been carried out. In Fig. 2 the effect of the perturbation on $\theta(t)$ and $u(t)$ are shown for the LQR design, while only the behaviour of the control with the sliding manifold approach is illustrated in Fig. 3, as the angle θ assumes the same value shown in Fig. 1 (c) without any appreciable variation in the three cases.

4 REFERENCES

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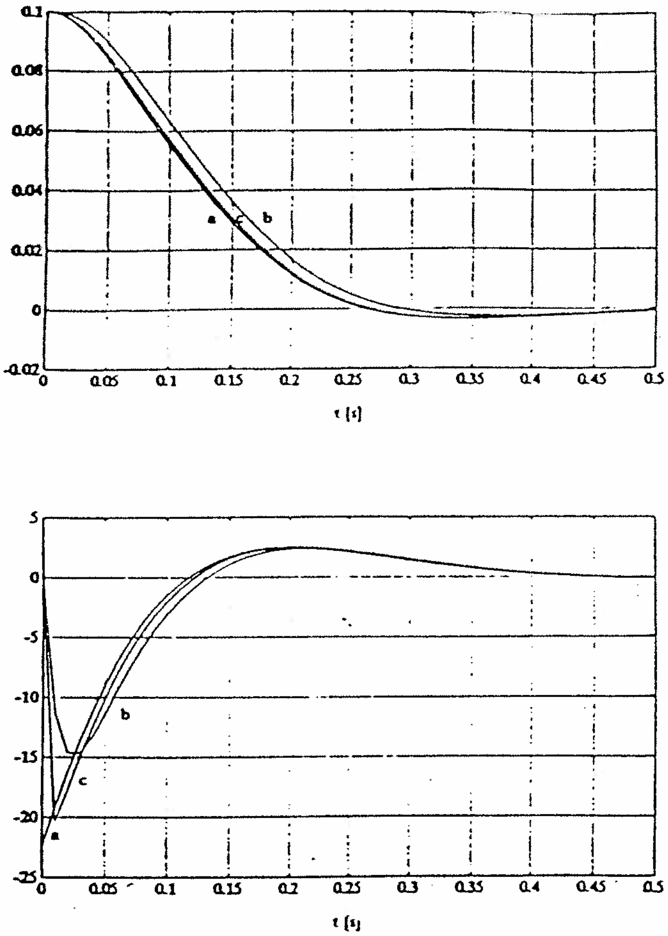

Figure 1: (a) LQR design; (b) sliding manifold approach, $\epsilon = 0.01$; (c) $\epsilon = 0.002$.

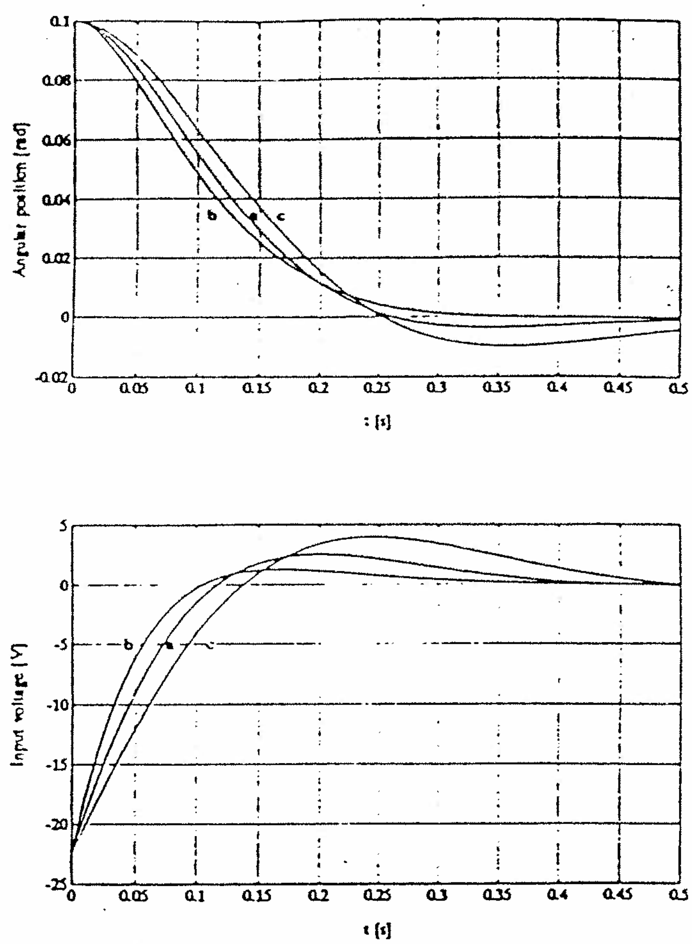

Figure 2: LQR design: effect of parameter variation; (a) nominal load; (b) inertial load 2/3 of nominal; (c) inertial load 3/2 of nominal

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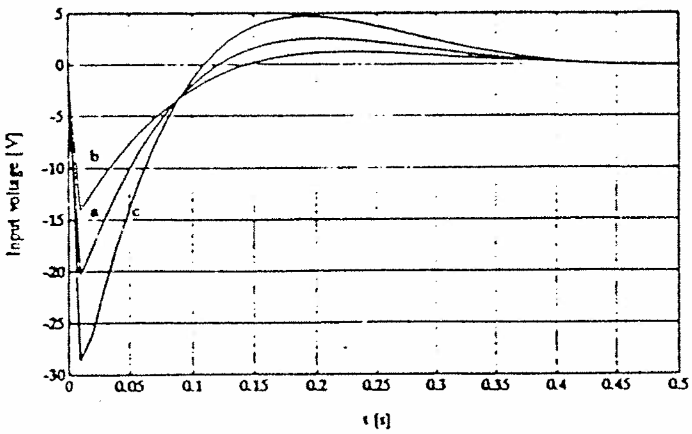

Figure 3: Sliding manifold approach: effect of parameter variation; (a) nominal load; (b) inertial load 2/3 of nominal; (c) inertial load 3/2 of nominal