

SYNCHRONIZATION PROBLEMS FOR UNIDIRECTIONAL FEEDBACK COUPLED NONLINEAR SYSTEMS

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Abstract. In this paper we consider three different synchronization problems consisting in designing a nonlinear feedback unidirectional coupling term for two (possibly chaotic) dynamical systems in order to drive the trajectories of one of them, the slave system, to a reference trajectory or to a prescribed neighborhood of the reference trajectory of the second dynamical system: the master system. If the slave system is chaotic then synchronization can be viewed as the control of chaos; namely the coupling term allows to suppress the chaotic motion by driving the chaotic system to a prescribed reference trajectory. Assuming that the entire vector field representing the velocity of the state can be modified, three different methods to define the nonlinear feedback synchronizing controller are proposed: one for each of the treated problems. These methods are based on results from the small parameter perturbation theory of autonomous systems having a limit cycle, from nonsmooth analysis and from the singular perturbation theory respectively. Simulations to illustrate the effectiveness of the obtained results are also presented.

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1 Introduction

In recent years in the literature on dynamical system analysis a considerable attention has been devoted to the problem of synchronization of coupled nonlinear dynamical systems (see e.g. [1], [3], [11], [19], [22], [27]). One of the most effective methods for solving such problem consists in designing a feedback coupling term which drives the trajectories of one of the two systems (the so-called slave system) to a prescribed reference trajectory of the second one (named master system). Examples of such approach can be found, for instance, in [21] where the coupling term is represented by a linear feedback of the tracking error. In [25] a bidirectional linear coupling term is proposed to synchronize two chaotic systems. An approach to synchronization based on the classical notion of observers, when the state is not fully available, can be found in [8] and [15]. In many cases, when one deals with nonlinear

chaotic dynamical systems, the interest is that of steering any trajectory of the chaotic system to an equilibrium point or to a limit cycle of the same system or of another coupled system, see [5], [10], [13] and [28]. For an adaptive control approach using a linear reference model we refer to [26].

In this paper, under the condition that the entire vector field of the velocity can be modified, we aim at designing a nonlinear feedback unidirectional coupling term, based on the state model system, in such a way that all the trajectories of the slave system are steered to a prescribed reference trajectory of the master system. Roughly speaking, the coupling term makes stable a prescribed trajectory of a dynamical system with respect to the trajectories of an other dynamical system by coupling these systems by means of a suitably defined nonlinear feedback coupling term.

Following the linear feedback approach of [21] and [25], we provide examples of how it can be possible, by means of a rigorous application of different mathematical tools, to define a nonlinear unidirectional feedback coupling term in order to determine a prescribed dynamical behavior to some classes of nonlinear dynamical systems. Actually, as pointed out in [12], sometimes the carelessness application of the mathematical tools employed to solve synchronization problems can lead to incorrect results.

The problems that we will treat in this paper are illustrated in the sequel. The first one is the problem of the synchronization of the phase of a limit cycle of an autonomous system with that of the limit cycle of the same period of another autonomous system. The feedback design is based on classical results due to Malkin [14] on the existence of periodic solutions of an autonomous system perturbed by a small parameter nonautonomous term and on their behavior when the perturbation disappears, namely when the parameter tends to zero. Many authors, see for instance [2], [17] and [20] and the extensive references therein, have considered the problem of the control of the balance between the phases of the subsystems state variables oscillations by coupling the subsystems in different ways, i.e. by suitably balancing the energy due to the interaction. In particular in [2] a dynamic feedback coupling term for phase locking of non identical oscillators is presented.

The second and third problem consist in the synchronization of the trajectories of the slave system to a reference trajectory of the master. To solve these problems we adopt two different feedback laws based on a sliding manifold approach. First, we define a static discontinuous feedback coupling term with a gain depending on the bounded set of the initial conditions for the trajectories of the slave system. It is defined by means of the signum function of the tracking error, that is by the signum of the difference between a trajectory of the slave system and the reference trajectory. By means of a suitably defined nondifferentiable Liapunov function and its subdifferentiability properties [7] we can prove that any trajectory originating from a given bounded set converges to the reference trajectory in an estimated finite time which depends on the feedback gain. It is worth to observe that, since the right hand side of the slave system is discontinuous with respect to the state,

it is necessary to introduce a suitable concept of solution for this system, in fact we consider solutions in the sense of Filippov [9]. The discontinuity along the reference curve $y_0(t), t \geq 0$, of the feedback law makes the coupled system robust against modelling errors and external disturbances, in the sense that the tracking error tends to zero in finite time also in presence of modelling imprecision and disturbances, if the gain is sufficiently large. Moreover, observe that it is possible to get the reference trajectory at any prescribed speed by suitably increasing the gain. Since the implementation of the associated switchings across the reference curve is necessarily imperfect, in practice switching is not instantaneous and the value of the tracking error $e(t) = x(t) - y_0(t), t \geq 0$, is not perfectly known; this leads to the chattering phenomenon which is the main drawback of this feedback law, namely the trajectory of the slave system rapidly oscillates around the reference trajectory. This is a quite undesirable effect, in fact each component of the signum of the tracking error in the coupling term switches very fast between the positive and negative value of the corresponding gain, which is not a feasible behavior for the physical implementation of the control law. In the framework of synchronization of chaotic system a feedback of this type has been used in [28], where the chattering phenomenon has been also emphasized.

To avoid the undesirable chattering phenomenon we then consider a dynamic feedback, as introduced in [4], defined by means of a differential equation involving the slave system and the reference trajectory. This equation depends also on a small parameter and it satisfies, under general conditions, all the assumptions of the classical singular perturbation theory on infinite intervals [16]. As we will see this ensures that any trajectory of the slave system approaches the reference trajectory within any prescribed error.

Finally, we present a simulation for each of the considered problems which illustrates the effectiveness of the obtained results. Precisely, for the first problem we consider, both for the master and slave system, the same FitzHugh-Nagumo type equation which has an asymptotically stable limit cycle and by implementing our method we synchronize the phase of the slave with that of the master. For the second and third problem we have considered a chaotic neural network as the slave system [6], and as master system a neural network which possesses a globally asymptotically exponentially stable periodic solution which is taken as reference trajectory [18].

The paper is organized as follows. In Section 2 we treat the problem of the phase synchronization of two self-oscillating nonlinear dynamical systems. In Section 3 for two nonautonomous systems we steer any trajectory of one of these two systems to a prescribed trajectory of the second one by means of a high gain discontinuous feedback of the tracking error. In Section 4 we consider the same systems and we design a dynamical feedback coupling term which drives any trajectory of the slave system to any prescribed neighborhood of the reference trajectory. Finally, in Section 5 we present the simulations which illustrate the meaning of the obtained results.

2 Phase synchronization of limit cycles

In this section we consider two autonomous systems

$$\dot{x} = f(x), \quad (1)$$

where $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$, and

$$\dot{y} = g(y), \quad (2)$$

where $g \in C^2(\mathbb{R}^n, \mathbb{R}^n)$. We assume that they have orbitally stable limit cycles $x_0(t)$ and $y_0(t)$, respectively, of the same period T . According to the terminology adopted in the literature for the problem that we will treat here, we will refer to systems (1) and (2) as slave and master system respectively. We are interested in the phase synchronization of the slave system to that of the master system by adding a coupling term to (1). Specifically, we will show that for every $\mu > 0$ we can define a function $\Phi_\mu : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that the system

$$\dot{x} = f(x) + \Phi_\mu(x, y_0(t)), \quad t \geq 0,$$

has a unique asymptotically stable T -periodic solution $x_\mu(t)$ in a neighborhood of the set $\{x_0(t) : t \in [0, T]\}$ satisfying the property

$$\left| \int_0^T |x_\mu(\tau) - y_0(\tau)|^2 d\tau - \min_{s \in [0, T]} \int_0^T |x_0(\tau + s) - y_0(\tau)|^2 d\tau \right| < \mu. \quad (3)$$

For this we assume that the Floquet multiplier equal to 1 of the linearized systems around $x_0(t)$ and $y_0(t)$ is simple and that the others $n - 1$ are inside of the unit open circle. We consider the following system

$$\begin{aligned} \dot{x} = f(x) + \varepsilon \left(|x - y_0(t)|^2 - \right. \\ \left. - \min_{s \in [0, T]} \frac{1}{T} \int_0^T |x_0(\tau + s) - y_0(\tau)|^2 d\tau - \delta \right) f(x), \end{aligned} \quad (4)$$

where ε and δ are positive scalar parameters.

We can prove the following result.

Theorem 1 *Assume, that the equation*

$$\int_0^T |x_0(\tau + \theta) - y_0(\tau)|^2 d\tau = \min_{s \in [0, T]} \int_0^T |x_0(\tau + s) - y_0(\tau)|^2 d\tau \quad (5)$$

has an unique solution $\theta_0 \in [0, T]$. Then for every $\mu > 0$ there exists $\delta_\mu > 0$ such that for every $\delta \in [0, \delta_\mu]$ there is $\varepsilon_\delta > 0$ for which the following results hold for $\varepsilon \in (0, \varepsilon_\delta)$.

1) *System (4) possesses a unique asymptotically stable T -periodic solution $x_\mu(t)$ such that*

$$x_\mu(t) \in \mathcal{N}_{\delta_\mu}(x_0), \text{ for any } t \in [0, T],$$

where $\mathcal{N}_{\delta_\mu}(x_0) = \left\{ x \in \mathbb{R}^n : \inf_{t \in [0, T]} |x - x_0(t)| < \delta_\mu \right\}$ denotes the δ_μ -neighborhood in \mathbb{R}^n of the limit cycle $x_0(t)$,

2) the solution $x_\mu(t)$ satisfies property (3).

To prove this theorem we need the following result due to I. G. Malkin [14], which is one of the main tools for the study of the synchronization of coupled systems (see [3]). Consider the system

$$\dot{x} = f(x) + \varepsilon \gamma(t, x) \quad (6)$$

where $\gamma \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, assume that γ is T -periodic with respect to time. Then system

$$\dot{z} = -(f'(x_0(t)))^* z$$

has a T -periodic solution $z^*(t)$ such that

$$\langle z^*(t), \dot{x}_0(t) \rangle = 1, \quad \text{for any } t \in [0, T]. \quad (7)$$

Let us introduce the function $F: \mathbb{R} \rightarrow \mathbb{R}$ as follows

$$F(\theta) = \int_0^T \langle z^*(\tau), \gamma(\tau - \theta, x_0(\tau)) \rangle d\tau, \quad \text{for any } \theta \in \mathbb{R}.$$

We can now formulate the following result.

Theorem 2 ([14], Theorems pp. 387 and 392) *Assume that for sufficiently small $\varepsilon > 0$ system (6) has a continuous family $\varepsilon \rightarrow x_\varepsilon(t)$ of T -periodic solutions satisfying the property*

$$x_\varepsilon(t) \rightarrow x_0(t + \theta_0) \quad \text{as } \varepsilon \rightarrow 0, \quad t \in [0, T], \quad (8)$$

then $F(\theta_0) = 0$. Moreover, if $F(\theta_0) = 0$ and $F'(\theta_0) \neq 0$ then (8) holds true and the solutions $x_\varepsilon(t)$ are asymptotically stable or unstable according to whether $F'(\theta_0)$ is negative or positive.

Proof of Theorem 1. For $\delta > 0$ let

$$\gamma_\delta(t, x) = \left(|x - y_0(t)|^2 - \min_{s \in [0, T]} \frac{1}{T} \int_0^T |x_0(\tau + s) - y_0(\tau)|^2 d\tau - \delta \right) f(x).$$

Observe that $\gamma_\delta \in C^1([0, T] \times \mathbb{R}^n, \mathbb{R}^n)$ is T -periodic with respect to time and so it can be extended from $[0, T]$ to \mathbb{R} by T -periodicity. By (7) we have now that

$$\begin{aligned} F_\delta(\theta) &= \int_0^T |x_0(\tau + \theta) - y_0(\tau)|^2 d\tau - \\ &\quad - \min_{s \in [0, T]} \int_0^T |x_0(\tau + s) - y_0(\tau)|^2 d\tau - T\delta. \end{aligned} \quad (9)$$

Under our assumptions the function $\theta \rightarrow F_\delta(\theta)$ is T -periodic and continuously differentiable. Furthermore there exists a unique $\theta_0 \in [0, T]$ such that $F_0(\theta_0) = 0$. Since the function F_0 reaches its minimum at θ_0 , we have that $F'_0(\theta_0) = 0$. Then, there exists ν_0 such that $F'_0(\theta) \neq 0$ for any $\theta \in (\theta_0 - \nu_0, \theta_0) \cup (\theta_0, \theta_0 + \nu_0)$. Thus $F'_0(\theta) < 0$ for $\theta \in (\theta_0 - \nu_0, \theta_0)$ and $F'_0(\theta) > 0$ for $\theta \in (\theta_0, \theta_0 + \nu_0)$. Therefore for any $\delta > 0$ sufficiently small there exists a unique θ_δ such that

- 1) $F_\delta(\theta_\delta) = 0$ and $F'_\delta(\theta_\delta) > 0$,
- 2) $\theta_\delta \rightarrow \theta_0$ as $\delta \rightarrow 0$.

Let $\mu > 0$, by the previous considerations there exists $\delta_\mu > 0$ such that

$$\left| \int_0^T |x_0(\tau + \theta_\delta) - y_0(\tau)|^2 d\tau - \int_0^T |x_0(\tau + \theta_0) - y_0(\tau)|^2 d\tau \right| < \frac{\mu}{2} \quad (10)$$

for any $\delta \in [0, \delta_\mu]$. Then for a given $\delta \in [0, \delta_\mu]$, by applying the Malkin's results (Theorem 2 above) and by taking into account (8), we deduce the existence of a positive number $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in [0, \varepsilon_\delta]$, system (4) possesses a unique asymptotically stable T -periodic solution $x_\mu(t)$ such that $x_\mu(t) \in \mathcal{N}_{\delta_\mu}(x_0)$, for any $t \in [0, T]$, satisfying

$$\left| \int_0^T |x_\mu(\tau) - y_0(\tau)|^2 d\tau - \int_0^T |x_0(\tau + \theta_\delta) - y_0(\tau)|^2 d\tau \right| < \frac{\mu}{2}. \quad (11)$$

Finally, (10) and (11) conclude the proof. \square

Remark 1 *In the case when equation (5) has k solutions on the interval $[0, T]$ we can reformulate Theorem 1 by simply replacing the sentence "unique asymptotically stable T -periodic solution" by " k asymptotically stable T -periodic solutions". Furthermore, we can interchange the rôle of the slave and master system in all of the previous arguments.*

3 A Static Feedback for the Synchronization of Trajectories

In this section we consider two nonlinear nonautonomous systems

$$\dot{x} = \phi(t, x) \quad (12)$$

$$\dot{y} = \psi(t, y), \quad (13)$$

where $\phi, \psi : [0, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfy the following conditions:

- (3.1) $t \rightarrow \phi(t, x)$ and $t \rightarrow \psi(t, x)$ are Lebesgue measurable functions for any $x \in \mathbb{R}^n$; $x \rightarrow \phi(t, x)$ and $x \rightarrow \psi(t, x)$ are locally Lipschitz functions for almost all (a.a.) $t \geq 0$.

(3.2) For any $\rho > 0$ there exists an integrable function $\gamma_\rho(t)$, $t \geq 0$, such that

$$|\phi(t, x)| \leq \gamma_\rho(t) \quad \text{and} \quad |\psi(t, x)| \leq \gamma_\rho(t)$$

for a.a. $t \geq 0$ and any $x \in \mathbb{R}^n$ such that $|x| \leq \rho$.

(3.3) Any local solution $x(t)$ and $y(t)$ of the Cauchy problems

$$\begin{cases} \dot{x} = \phi(t, x) \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \quad \begin{cases} \dot{y} = \psi(t, y) \\ y(0) = y_0 \in \mathbb{R}^n, \end{cases}$$

can be extended to the interval $[0, +\infty)$.

Let $y_0(t)$, $t \geq 0$, be the prescribed bounded solution of (13) to which system (12) must be synchronized by means of a suitable feedback coupling term to be added to (12). The coupling term we propose here has the form of a static feedback given by

$$B \operatorname{sgn}(x - y_0(t)), \quad (14)$$

where $B = \operatorname{diag}(b_i)$, $b_i < 0$ for any $i = 1, 2, \dots, n$, and $\operatorname{sgn}(x) = (\operatorname{sgn}(x_1), \dots, \operatorname{sgn}(x_n))$ for all $x = (x_i)_{i=1}^n \in \mathbb{R}^n$. Therefore system (12) takes the form

$$\dot{x} = \phi(t, x) + B \operatorname{sgn}(x - y_0(t)). \quad (15)$$

Since the right hand side of system (15) is discontinuous in the state variable x we must adopt a suitable notion of solution for any Cauchy problem associated to (15). Here, following [9], as a solution of (15) we intend an absolutely continuous function $x(t)$, $t \geq 0$, such that

$$\dot{x}(t) \in \phi(t, x(t)) + B \mathbf{K}[\operatorname{sgn}(x(t) - y_0(t))]$$

for a.a. $t \geq 0$, where

$$\mathbf{K}[\operatorname{sgn}(x - y)] = \bigcap_{\varepsilon > 0} \bigcap_{\mu(N)=0} \overline{\operatorname{co}}(\operatorname{sgn}(\mathcal{B}_\varepsilon(x - y) \setminus N)),$$

$N \subset \mathbb{R}^n$ is an arbitrary set of Lebesgue measure zero, $\overline{\operatorname{co}}(A)$ denotes the closure of the convex hull of the set A and $\mathcal{B}_\varepsilon(c)$ is the open ball in \mathbb{R}^n with radius ε and center c . It is immediate to see that if t_0 is a discontinuity point for some component of the vector $\operatorname{sgn}(x(t) - y_0(t))$, then the corresponding component of $\mathbf{K}[\operatorname{sgn}(x(t_0) - y_0(t_0))]$ is the interval $[-1, 1]$. The other components of $\mathbf{K}[\operatorname{sgn}(x(t_0) - y_0(t_0))]$ being $+1$ or -1 . In the sequel we will refer to such solution as a Filippov solution to system (15).

Then we associate to system (15) the Lyapunov function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$, where $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ defined as follows

$$V(e) = \sum_{i=1}^n |e_i|,$$

where $e = (e_i)_{i=1}^n$ is the tracking error given by $e_i = x_i - y_i$, $i = 1, 2, \dots, n$.

We can prove the following result.

Theorem 3 For any bounded set $I \subset \mathbb{R}^n$ there exists a positive constant $M_I > 0$ such that, if $b_i < -M_I$ for any $i = 1, 2, \dots, n$, then any Filippov solution $x(t), t \geq 0$, to system (15) starting from a point $x_0 \in I$ converges in finite time $t_0 \leq -\frac{V(e(0))}{\mu_I}$, where $\mu_I = \max\{M_I + b_i, i = 1, 2, \dots, n\} < 0$, to $y_0(t)$ and $x(t) = y_0(t)$ for any $t \geq t_0$.

Proof. Let $x(t), t \geq 0$, be a solution to (15) with $x(0) = x_0$, whenever $x_0 \in I$. Let $e(t) = (e_i(t))_{i=1}^n, e_i(t) = x_i(t) - y_{0,i}(t), t \geq 0$. From the chain rule for locally Lipschitz regular maps ([7], Theorem 2.3.9-(iii)), for a.a. $t \geq 0$, we have that

$$\frac{d}{dt}V(e(t)) = \langle \xi, \dot{e}(t) \rangle,$$

for any $\xi \in \partial V(e(t))$, where $\partial V(e(t))$ is the Clarke generalized gradient of the function V evaluated at $e(t)$ ([7], p. 27). It can be easily verified that the vector $\xi = (\xi_i)_{i=1}^n$ is given by $\xi_i = +1, (\xi_i = -1)$, for those indexes $i \in J_+(t), (J_-(t))$, for which $e_i(t) > 0, (e_i(t) < 0)$, while $\xi_i \in [-1, 1]$ for the indexes $i \in J_0(t)$ for which $e_i(t) = 0$. Therefore, for a.a. $t \geq 0$ such that $e(t) \neq 0$, we have that

$$\begin{aligned} \frac{d}{dt}V(e(t)) &= \sum_{i \in J_+(t) \cup J_-(t)} \dot{e}_i(t) \operatorname{sgn} e_i(t) \\ &= \sum_{i \in J_+(t) \cup J_-(t)} [\phi_i(t, x(t)) - \psi_i(t, y_0(t)) + b_i \operatorname{sgn} e_i(t)] \operatorname{sgn} e_i(t) \\ &= \sum_{i \in J_+(t) \cup J_-(t)} [\phi_i(t, x(t)) - \psi_i(t, y_0(t))] \operatorname{sgn} e_i(t) + b_i \\ &\leq \max\{M_I + b_i, i = 1, 2, \dots, n\} = \mu_I, \end{aligned} \tag{16}$$

where $M_I \geq |\phi_i(t, x(t)) - \psi_i(t, y_0(t))|$ for a.a. $t \geq 0$ and for any $i = 1, 2, \dots, n$. Observe that, under our assumptions on the vector fields $\phi(t, x)$ and $\psi(t, x)$, such a constant $M_I > 0$ does exist. In fact, $y_0(t), t \geq 0$, is a bounded trajectory in \mathbb{R}^n and it is not hard to see that any solution $x(t), t \geq 0$, to (15) starting from the bounded set $I \subset \mathbb{R}^n$ is also bounded in \mathbb{R}^n . By our assumption $\mu_I < 0$, therefore by integrating (16) between 0 and $t > 0$ we obtain

$$V(e(t)) - V(e(0)) \leq \mu_I t.$$

In conclusion, for $t \geq -\frac{V(e(0))}{\mu_I} > 0$ we have that $V(e(t)) = 0$ and thus $e(t) = 0$ for any $t \geq -\frac{V(e(0))}{\mu_I}$. □

4 A Dynamic Feedback for the Synchronization of Trajectories

In practice, since the switching is not instantaneous and the tracking error is not perfectly calculated, we have a serious drawback by using the previous approach, that is the so-called chattering phenomenon as shown in Example 2 of Section 3. To avoid this phenomenon we propose in the sequel a different coupling term to add to (12) in order to solve our synchronization problem, namely a dynamic feedback coupling term defined by means of the singular perturbation theory. This method to eliminate the chattering and to make a controlled dynamical system insensitive with respect to external perturbations has been introduced in [4], where applications to specific tracking problems have been also presented. Specifically, given the reference trajectory $y_0(t)$, $t \geq 0$, which is a solution of system (13), we introduce the function $s : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ as follows

$$s(t, \xi_0, x) = e^{Ct}(\xi_0 - y_0(0)) - (x - y_0(t)),$$

where C is a given real symmetric matrix such that $\max \lambda(C) \leq -\alpha < 0$, here $\lambda(C)$ denotes the set of the eigenvalues of the matrix C . Then, under conditions (3.1) – (3.3), for $\varepsilon > 0$ small we consider the system

$$\begin{cases} \dot{x} &= \phi(t, x) - Bu \\ \varepsilon \dot{u} &= \frac{\partial s}{\partial t}(t, \xi_0, x) + \frac{\partial s}{\partial x}(t, \xi_0, x)[\phi(t, x) - Bu] := g(t, \xi_0, x, u). \end{cases} \quad (17)$$

We are now in the position to prove the following result.

Theorem 4 *Assume, that B is a real negative defined $n \times n$ matrix. Then for any $\delta > 0$ there exists $\varepsilon > 0$ such that*

$$\lim_{t \rightarrow \infty} |x_\varepsilon(t) - y_0(t)| < \delta,$$

where $x_\varepsilon(t)$, $t \geq 0$, is the solution of (17) such that $x_\varepsilon(0) = \xi_0$.

Proof. Let $\varepsilon = 0$ and consider the corresponding system

$$\begin{cases} \dot{x} = \phi(t, x) - Bu \\ 0 = Ce^{Ct}(\xi_0 - y_0(0)) + \dot{y}_0(t) - \phi(t, x) + Bu, \end{cases}$$

Resolving the second algebraic equation with respect to u and substituting in the first equation we obtain

$$\dot{x}_0(t) - \dot{y}_0(t) = Ce^{Ct}(\xi_0 - y_0(0)), \quad t \geq 0$$

or equivalently

$$x_0(t) - y_0(t) = e^{Ct}(\xi_0 - y_0(0)), \quad t \geq 0,$$

with $x_0(t)$ satisfying $x_0(0) = \xi_0$. By our assumption on the matrix C we have that $\lim_{t \rightarrow \infty} (x_0(t) - y_0(t)) = 0$. Now we show that for given $\delta > 0$ there exists $\varepsilon > 0$ such that

$$|x_\varepsilon(t) - x_0(t)| \leq \delta$$

for any $t \geq 0$. For this, we use the singular perturbation theory on unbounded time intervals. Specifically, in the sequel we verify that all the assumptions of the Theorem at p. 523 of [16] are satisfied for system (17) in order to conclude that

$$\lim_{\varepsilon \rightarrow 0} x_\varepsilon(t) = x_0(t)$$

uniformly in $[0, +\infty)$. First of all, for any $(\hat{t}, \hat{x}) \in \mathbb{R}_+ \times \mathbb{R}^n$ the equilibrium point

$$u(\hat{t}, \hat{x}) = -B^{-1} [Ce^{C\hat{t}}(\xi_0 - y_0(0)) + \psi(\hat{t}, y_0(\hat{t})) - \phi(\hat{t}, \hat{x})]$$

of the equation $\varepsilon du/d\tau = g(\hat{t}, \xi_0, \hat{x}, u)$ is asymptotically stable. In other words, the solution $z(\tau)$, $\tau \geq 0$, of the Cauchy problem (the boundary layer)

$$\begin{cases} \dot{z} = g(\hat{t}, \xi_0, \hat{x}, z) \\ z(0) = z_0. \end{cases}$$

converges asymptotically to $u(\hat{t}, \hat{x})$. In fact, consider the Lyapunov function $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ given by

$$V(\hat{t}, \hat{x}, z) = \frac{|z - u(\hat{t}, \hat{x})|^2}{2}.$$

We have, that

$$\begin{aligned} \frac{d}{d\tau} V(\hat{t}, \hat{x}, z(\tau)) &= \left\langle \frac{\partial}{\partial z} V(\hat{t}, \hat{x}, z(\tau)), g(\hat{t}, \xi_0, \hat{x}, z(\tau)) \right\rangle \\ &= \langle z(\tau) - u(\hat{t}, \hat{x}), g(\hat{t}, \xi_0, \hat{x}, z(\tau)) - g(\hat{t}, \xi_0, \hat{x}, u(\hat{t}, \hat{x})) \rangle \\ &= \langle z(\tau) - u(\hat{t}, \hat{x}), B(z(\tau) - u(\hat{t}, \hat{x})) \rangle \\ &\leq -\nu |z(\tau) - u(\hat{t}, \hat{x})|^2 \end{aligned}$$

for some $\nu > 0$, since B is negative defined. Moreover, observe that the asymptotic stability is exponential and uniform with respect to (\hat{t}, \hat{x}) when \hat{x} belongs to bounded sets.

Furthermore, the origin $x = 0$ is a uniformly stable equilibrium point of (17) when $\varepsilon = 0$. To show this it is sufficient to observe that for $\varepsilon = 0$ we have

$$\dot{x}(t) - \dot{y}_0(t) = Ce^{Ct}(\xi_0 - y_0(0)), \quad t \geq 0$$

and the change of variable $e = x - y_0$ makes the origin $e = 0$ of this dynamics (exponentially) asymptotically stable, in fact

$$\langle e, \dot{e} \rangle = \langle e, Ce \rangle \leq -\alpha |e|^2.$$

At this point all the assumptions of the Theorem at p. 523 of [16] are satisfied, thus

$$\lim_{\varepsilon \rightarrow 0} x_\varepsilon(t) = x_0(t)$$

uniformly in $[0, +\infty)$ and the conclusion of the theorem easily follows. \square

Remark 2 *The convergence theorem of [16] employed in the proof of Theorem 4 also establishes that the absolutely continuous function $u_\varepsilon(t)$, $t \geq 0$, given by (17) is such that*

$$\lim_{\varepsilon \rightarrow 0} u_\varepsilon(t) = u_0(t)$$

uniformly on any interval $[t_1, +\infty)$, $t_1 > 0$, where

$$u_0(t) = -B^{-1} [Ce^{Ct}(\xi_0 - y_0(0)) + \psi(t, y_0(t)) - \phi(t, x_0(t))], \quad t \geq 0,$$

is the so-called equivalent control of the theory of variable structure systems (see [23] and [24]). In other words, it is the ideal control which realizes exactly the condition $s(t, x_0(t)) = 0$, $t \geq 0$. This convergence property together with the absolute continuity of the functions $u_\varepsilon(t)$ and $u_0(t)$, $t \geq 0$, prevents the chattering phenomenon.

Finally, we would like to point out that the proposed dynamical feedback coupling term $-Bu$, designed by means of (17), makes the dynamical system $\dot{x} = \phi(t, x)$ insensitive with respect to (possibly unknown) perturbations. In fact, consider any bounded perturbation $p : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ of system $\dot{x} = \phi(t, x)$ satisfying (3.1) – (3.2). Thus system (17) takes the form

$$\begin{cases} \dot{x} &= \phi(t, x) + p(t, x) - Bu \\ \varepsilon \dot{u} &= \frac{\partial s}{\partial t}(t, \xi_0, x) + \frac{\partial s}{\partial x}(t, \xi_0, x)[\phi(t, x) + p(t, x) - Bu]. \end{cases} \quad (18)$$

It is easy to verify that for this system we can repeat the arguments used in the proof of Theorem 3. In fact, for $\varepsilon = 0$, the dynamics of the error $e = x - y_0$ for system (18) is the same of that of system (17), namely

$$\dot{x}(t) - \dot{y}_0(t) = Ce^{Ct}(\xi_0 - y_0(0)), \quad t \geq 0,$$

Moreover, $u(\hat{t}, \hat{x})$ is still an asymptotically stable equilibrium point of the second equation of system (18).

5 Examples

In the following examples we have chosen to show the behavior of only the first component of the solutions of the slave and master systems, the other components having a quite similar behavior.

Example 1 At first we illustrate the result of Section 2 on the phase synchronization in \mathbb{R}^2 . We take the same system as slave and master systems, namely

$$\begin{cases} \dot{x}_1 = 2 \left(x_1 - \frac{1}{3}x_1^3 + x_2 - \frac{9}{20} \right) \\ \dot{x}_2 = -\frac{1}{2} \left(x_1 + \frac{4}{5}x_2 - \frac{7}{10} \right) \end{cases} \quad (19)$$

which has an asymptotically stable limit cycle with period $T \simeq 9.83$. In particular, let $y_0(t)$ be the T -periodic solution of (19) which starts at $t = 0$ from the point $(-0.7481, 1.5164)$ of the limit cycle. In this way, since

$$\min_{s \in [0, T]} \frac{1}{T} \int_0^T |y_0(\tau + s) - y_0(\tau)|^2 d\tau = 0,$$

system (4) becomes

$$\begin{cases} \dot{x}_1 = 2 \left(x_1 - \frac{1}{3}x_1^3 + x_2 - \frac{9}{20} \right) (1 + \varepsilon(|x - y_0(t)|^2 - \delta)) \\ \dot{x}_2 = -\frac{1}{2} \left(x_1 + \frac{4}{5}x_2 - \frac{7}{10} \right) (1 + \varepsilon(|x - y_0(t)|^2 - \delta)) \end{cases} \quad (20)$$

and we consider the solution $x(t)$ of (20) which starts from $x(0) = (5, -5)$, with the choice $\varepsilon = 0.01$ and $\delta = 0.05$. In Figure 1 the solid line is the graph of the first component $x_1(t)$ of $x(t)$ in the intervals $[T, 2T]$ (left picture) and $[50T, 51T]$ (right picture), while the dashed line is the graph of the first component of the T -periodic solution $y_0(t)$. Due to the asymptotical stability of the limit cycle of (19), we see that $x_1(t)$ already has the shape of the first component of $y_0(t)$ after only one period, but the phases of these trajectories are still considerably different. The picture on the right shows that the phase difference between the first components of $x(t)$ and $y_0(t)$ is significantly reduced after 50 periods thanks to the coupling term added in (20).

Example 2 We exploit the result of Section 3 by considering the static feedback synchronization of the following neural network introduced in [29]

$$\dot{x} = - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1.25 & -3.2 & -3.2 \\ -3.2 & 1.1 & -4.4 \\ -3.2 & 4.4 & 1 \end{bmatrix} \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix}, \quad (21)$$

where $f(s) = (|s + 1| - |s - 1|)/2$. In [29] it is shown that system (21) has a chaotic attractor. As master system we use another neural network

$$\dot{x} = - \begin{bmatrix} \frac{10}{7} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} x + \begin{bmatrix} -\frac{20}{7} & 10 & 0 \\ 1 & -30 & 1 \\ 0 & \frac{100}{7} & -1.9 \end{bmatrix} \begin{bmatrix} x_1^3 \\ x_2 \\ x_3 \end{bmatrix} + I(t), \quad (22)$$

where $I(t)$ is the periodic input such that (22) has the 2π -periodic solution $y_0(t) = (\cos t, \sin t, -\cos t)$. We take the gains $b_1 = b_2 = b_3 = -3.5$ in the coupling term (14) and consider the solution x of (15) such that $x(0) = (-1, 1, 1)$; then in Figure 2 the graph of the first component $x_1(t)$ of $x(t)$ is plotted. In particular, the picture on the left shows the convergence in finite time to the first component of the reference solution $y_0(t)$; on the other hand, the picture on the right is a zoom of the left one around the hitting zone and is a clear evidence of the chattering phenomenon.

Example 3 To illustrate the dynamic feedback synchronization considered in Section 4 and to compare the issues with those of the previous example, we consider again the two neural networks (21) and (22) of Example 2 with the same forcing term $I(t)$. We take $C = B = \text{diag}(-1, -1, -1)$ and $\varepsilon = 0.001$ and consider the solution x of (17) such that $x(0) = (-1, 1, 1)$. In Figure 3 the graphs of the first components $x_1(t)$ of $x(t)$ (solid line) and $\cos t$ of $y_0(t)$ (dotted line) are plotted. In particular, the picture on the left shows how $x_1(t)$ approaches $\cos t$, while the picture on the right is a zoom of the previous one around $t = 2\pi$ and shows that $x_1(t)$ remains in a neighborhood of $\cos t$ without any chattering.

Remark 3 Observe that in Examples 2 and 3 we have taken the initial conditions for the master system on the reference trajectory $y_0(t)$. On the other hand, in [18] it is shown that $y_0(t)$ is globally exponentially stable, therefore these initial conditions could be also chosen as far as we like from $y_0(t)$.

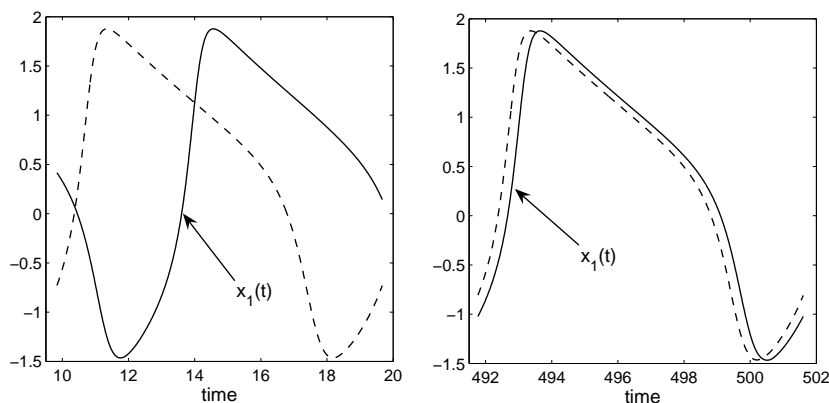


Figure 1: Time behavior of $x_1(t)$ (solid line) in Example 1 in the intervals $[T, 2T]$ (left picture) and $[50T, 51T]$ (right picture). The dashed line is the graph of the first component of the T -periodic solution $y_0(t)$ of the master system.

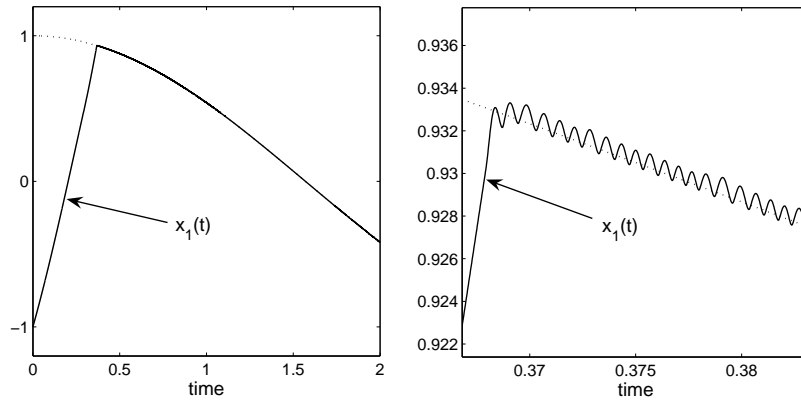


Figure 2: Time behavior of $x_1(t)$ (solid line) of Example 2. The dotted line is the graph of $\cos t$, that is the first component of the periodic solution of the master system (22).

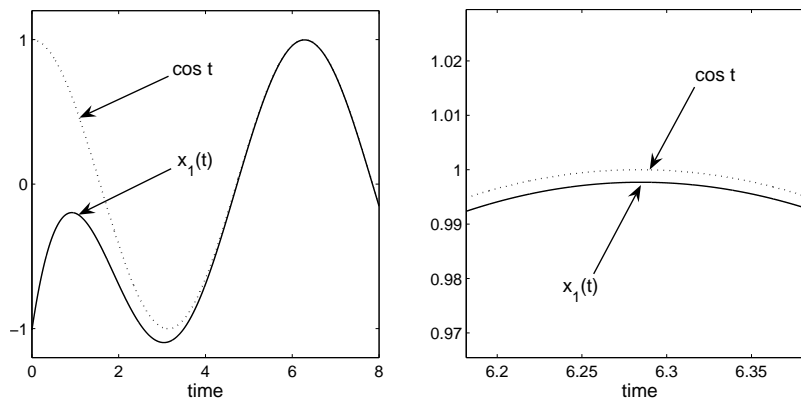


Figure 3: Time behavior of $x_1(t)$ (solid line) of Example 3. The dotted line is the graph of $\cos t$, that is the first component of the periodic solution of the master system.

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