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On the Dynamics of a Differential Inclusion Built upon a Nonconvex Constrained Minimization Problem[1]

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Abstract. In this paper, we study the dynamics of a differential inclusion built upon a nonsmooth, not necessarily convex, constrained minimization problem in finite-dimensional spaces. In particular, we are interested in the investigation of the asymptotic behavior of the trajectories of the dynamical system represented by the differential inclusion. Under suitable assumptions on the constraint set and the two involved functions (one defining the constraint set, the other representing the functional to be minimized), it is proved that all the trajectories converge to the set of the constrained critical points. We present also a large class of constraint sets satisfying our assumptions. As a simple consequence, in the case of a smooth convex minimization problem, we have that any trajectory converges to the set of minimizers.

Key Words. Nonconvex nonsmooth constrained minimization problems, critical points, differential inclusions, viability theory.

1. Introduction

The main goal of this paper is to investigate the asymptotic behavior of the trajectories of a suitably-defined differential inclusion associated to a nonconvex, nonsmooth minimization problem. To be specific, let

$$S = \{x \in \mathbb{R}^n : V(x) \leq 0\}$$

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be a nonempty compact set, where $V : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz map. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz map. Consider the following problem:

$$\min_S \phi. \tag{1}$$

Clearly, the set of minimizers $\mathcal{M}$ is nonempty and contained in the set of critical points of ϕ in S, which is given by

$$C = \{x \in S : 0 \in \partial\phi(x) + N_S(x)\},$$

where $\partial\phi(x)$ denotes the Clarke generalized gradient of ϕ at x and $N_S(x)$ is the normal cone at $x \in S$. Then we associate to (1) a suitable differential inclusion,

$$\dot{x}(t) \in \psi_\sigma(x(t)), \quad \text{for a.a. } t \geq 0, \tag{2}$$

depending on a parameter $\sigma > 0$; the form of the multivalued vector field $x \to \psi_\sigma(x)$ will be given later on in (4); the abbreviation a.a. stands for "almost all".

In the case where ϕ and S are convex, if the function V defining S is the indicator function of S, we have that $\mathcal{M} = C$ and the dynamics is given by the following differential variational inequality:

$$\dot{x}(t) \in -\partial\phi(x(t)) - N_S(x(t)), \quad x(t) \in S, \quad \text{for a.a. } t \geq 0. \tag{3}$$

The problem of the asymptotic behavior of the trajectories of (3) has been studied extensively in both finite-dimensional and infinite-dimensional spaces, see for instance Refs. 1–5 and the extensive references therein. In the convex case, it is well known that the solution of any Cauchy problem associated to (3) with $x_0 \in S$ is unique. Moreover, it is viable in S[i.e. $x(t) \in S$, for any $t \geq 0$]; see Refs. 1 and 6 for the study of (3) in some nonconvex cases.

Under our general assumptions on the functions V and ϕ, the set S and ϕ are not necessarily convex; thus, in particular, existence, uniqueness, and invariance with respect to S of the solutions to (3) are not anymore guaranteed. Therefore, in this paper, we introduce a different dynamics (2) defined in all of $\mathbb{R}^n$ to overcome the problem of lack of invariance of the set S with respect to the considered dynamics and the related problem of lack of uniqueness of the solution.

Having this in mind, we consider the inclusion (2), where $\psi_\sigma(x) : \mathbb{R}^n \to \mathbb{R}^n$ is defined as follows:

$$\psi_\sigma(x) = \begin{cases} -\partial\phi(x) - \sigma\partial V(x), & \text{if } x \in \mathbb{R}^n \setminus S, \\ -\partial\phi(x), & \text{if } x \in \text{int}(S), \\ -\partial\phi(x) - [0, \sigma]\partial V(x), & \text{if } x \in \partial S. \end{cases} \tag{4}$$

Here, we denote by int(S) the interior of the set S. It turns out that $x \to \psi_\sigma(x)$ is an upper semicontinuous map with nonempty compact convex values. The parameter $\sigma > 0$ can be viewed as a penalty parameter when the state x does not belong to S and it plays a crucial role throughout the paper. In particular, we will provide conditions on S and V such that, for $\sigma > 0$ sufficiently large, the set C coincides with the set of equilibria[4] of ψ_σ in a suitable neighborhood of S in $\mathbb{R}^n$. Therefore, to study the convergence of the trajectories of (2) to C, it is equivalent to study the convergence of the trajectories to the equilibria of (2). Furthermore, our assumptions will allow us to show that any trajectory $x(t)$ of (2), starting from any point of a suitable neighborhood of S, reaches C either in finite time or asymptotically. Moreover, S turns out to be invariant with respect to the dynamics (2). A similar dynamics was introduced as a gradient differential equation in Ref. 7 in the smooth case and as a gradient inclusion in Ref. 8 in the nonsmooth one. Specifically, in Ref. 7 Kennedy and Chua introduced the so-called dynamic canonical nonlinear programming circuit to implement the dynamics and to solve (1), when ϕ and V are twice continuously differentiable, via the convergence of the trajectories to the set $\mathcal{M}$.

In Ref. 8, under different assumptions on the nonsmooth functions ϕ and V, a rigorous analysis, via a Lyapunov-like approach, of the qualitative behavior of the trajectories of (2) as $t \to +\infty$ is presented. In particular, if ϕ and V are convex functions, we obtain the convergence of the trajectory of (2), starting from any point x_0 belonging to any bounded set containing S, to a point $\hat{x} \in \mathcal{M}$. Furthermore, always in the convex case, a precise study of the dynamics of (2) is presented in Ref. 9, where particular attention is devoted to the dynamics on ∂S, that is to the study of the sliding modes.

Finally, we point out that, in the most part of the above cited papers, the constraints are represented by functions $f_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \ldots, q$, and

$$S = \{x \in \mathbb{R}^n : f_j(x) \geq 0, j = 1, \ldots, q\}.$$

In order to reduce these cases to the situation considered in this paper, it is sufficient to define

$$V(x) = -\sum_{j=1}^{q} f_j^-(x), \qquad \text{where } f_j^-(x) = \min(0, f_j(x)), x \in \mathbb{R}^n.$$

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[4]A point x is an equilibrium point of a vector field $x \to F(x)$ if and only if $0 \in F(x)$.

Other choices of V are possible; as we will see in Proposition 4.1 of the Appendix (section 4), they depend on the properties that we require for the dynamics (2) with ψ_σ given as in (4).

The paper is organized as follows. In Section 2, we provide conditions on V under which the set S is invariant with respect to the dynamics (2). In Section 3, under the same assumptions, we analyze the asymptotic behavior of the trajectories of (2) and we exhibit a general class of functions ϕ and V (the class of semiconvex functions) for which we have the uniqueness of the solution of any Cauchy problem associated to (2). The convex case is also considered. Finally, in the Appendix (Section 4), we discuss a large class of sets S which satisfy the general assumptions under which we have proved our results.

2. Invariance of the Set S with Respect to the Dynamics (2)

Let $S=\{x\in\mathbb{R}^n : V(x)\leq 0\}$ where $V:\mathbb{R}^n\to\mathbb{R}$ is a locally Lipschitz map. Assume the following condition:

(H1) There exist $R>0$ and $m>0$ such that

$$\inf_{x\in(S+RB)\setminus S}\ \min_{v\in\partial V(x)}\|v\|\geq m.$$

Here, $B=B(0,1)$ is the ball centered at the origin with radius 1.

For $\sigma>0$, consider the differential inclusion

$$\dot{x}(t)\in\psi_\sigma(x(t)),\quad \text{for a.a. } t\geq 0, \tag{5}$$

where

$$\psi_\sigma(x)=\begin{cases}-\partial\phi(x)-\sigma\partial V(x), & \text{if } x\in\mathbb{R}^n\setminus S,\\ -\partial\phi(x), & \text{if } x\in\text{int }(S),\\ -\partial\phi(x)-[0,\sigma]\partial V(x), & \text{if } x\in\partial S.\end{cases} \tag{6}$$

Since ϕ and V are locally Lipschitz, we have that $x\to\partial\phi(x)$ and $x\to\partial V(x)$ are uppersemicontinuous with nonempty compact convex values (Ref. 10); thus, $\psi_\sigma:\mathbb{R}^n\to\mathbb{R}^n, \sigma>0$, is uppersemicontinuous with nonempty compact convex values and so we have the local existence of a solution of any Cauchy problem associated to (2).

The class of sets S satisfying (H1) is quite large and it will be discussed in the Appendix (Section 4). We mention here that it includes:

(a) the class of the sets S satisfying the following condition:
(H2) for any $x \in \partial S$, we have $0 \notin \partial V(x)$;
(b) the convex sets with $V(\cdot) =$ dist$(\cdot, S)$;
(c) the proximal retracts with $V(\cdot) =$ dist$(\cdot, S)$, i.e. sets S for which there exists a neighborhood $\mathcal{U} \supset S$ such that, for each $x \in \mathcal{U}$, there exists a unique $y \in S$ satisfying $\|x - y\| =$ dist(x, S);
(d) the sets S for which ∂S is a $C^{1,1}$ manifold; these are particular cases of proximal retracts;
(e) the epiLipschitz sets, i.e. sets S that locally are the epigraph of a Lipschitz function.

We can formulate now the main result of this section.

Proposition 2.1. Assume (H1). Then, there exists $\bar{\sigma} > 0$ such that, for any $\sigma > \bar{\sigma}$, the set S is invariant with respect to (2).

To prove Proposition 2.1, we need the following preliminary result.

Lemma 2.1. Assume (H1). Then, there exists $\alpha > 0$ such that any solution of (2) has the property that the function $t \to V(x(t))$ is decreasing as long as $x(t) \in S_\alpha = \{x \in (S + RB) \setminus S : 0 < V(x) < \alpha\}$.

Proof. Since $x \to \partial\phi(x)$ and $x \to \partial V(x)$ are upper-semicontinuous maps with nonempty compact convex values, we have that:

$$\text{there exists } \alpha > 0 \text{ such that, for any } x \in S_\alpha,\ \text{dist}(0, \partial V(x)) > m/2. \quad (7)$$

Let $x(\cdot)$ be a solution to (2). Since $x \to V(x)$ is a locally Lipschitz functions and since $t \to x(t)$ is almost everywhere (a.e.) differentiable, we have that, for a.a. $t \geq 0$, there exists $(d/dt)\, V(x(t))$. Furthermore, for a.a. $t \geq 0$ such that $x(t) \in S_\alpha$, we have that

$$\begin{aligned}(d/dt)\, V(x(t)) &\leq \max_{\zeta \in \partial V(x(t))} \langle \zeta, \dot{x}(t) \rangle \\ &= \max_{\zeta \in \partial V(x(t))} \langle \zeta, -\eta(t) - \sigma\mu(t) \rangle,\end{aligned}$$

for some $\eta(t) \in \partial\phi(x(t))$ and $\mu(t) \in \partial V(x(t))$. Using the Schwarz inequality, (7), and the compactness and convexity of $\partial V(x(t))$, we obtain

$$(d/dt)\, V(x(t)) \leq l^2 - \sigma m^2/4,$$

where l is an upper bound of the Lipschitz constant of both V and ϕ in the compact set $\overline{(S + RB) \setminus S}$. Let

$$\bar{\sigma} = 4l^2/m^2;$$

thus, for $\sigma > \bar{\sigma}$, it results that

$$(d/dt)\, V(x(t)) < 0,$$

for a.a. $t \geq 0$ such that $x(t) \in S_\alpha$. But $t \to V(x(t))$ is locally Lipschitz; thus, $t \to V(x(t))$ is decreasing for such t and

$$V(x(t)) \leq V(x_0) + (l^2 - \sigma m^2/4)t.$$

This concludes the proof of Lemma 2.1. □

Proof of Proposition 2.1. Assume by contradiction that S is not invariant with respect to (2); thus, there exists a solution $x(t)$ of (2) starting from $x_0 \in S$ which leaves S in finite time. Let

$$\theta = \inf\{t > 0 : x(t) \notin S\} < \infty;$$

by the continuity of $t \to x(t)$, there exists $h > 0$ such that

$$x\,((\theta, \theta + h)) \subset S_\alpha,$$

and so,

$$0 < V(x(s)) < \alpha, \quad \text{for any } s \in (\theta, \theta + h).$$

On the other hand, by Lemma 2.1, for $\sigma > \bar{\sigma}$ the function $s \to V(x(s))$ is decreasing on $(\theta, \theta + h)$ and $V(x(\theta)) = 0$; hence,

$$V(x(s)) < 0, \quad \text{for any } s \in (\theta, \theta + h),$$

which is a contradiction with

$$0 < V(x(s)) < \alpha, \quad \text{for any } s \in (\theta, \theta + h). \qquad \square$$

The following result holds true.

Proposition 2.2. Assume (H1) and let $\sigma > 4l^2/m^2$. Then, for any $r > 0$ such that $S_r = \{x \in \mathbb{R}^n : 0 < V(x) < r\} \subset (S + BR) \setminus S$, any trajectory of (2) starting from $x_0 \in S_r$ reaches S in a finite time less or equal to $r/(\sigma m^2/4 - l^2)$ and remains in S for all future times.

Proof. Let $x_0 \in S_r$ and let $x(\cdot) \in S_{\psi_\sigma}(x_0)$, the set of solutions to (2) starting from x_0. As in the proof of Lemma 2.1, for a.a. $t \geq 0$ such that $x(t) \in S_r$, we have

$$\begin{aligned}(d/dt)\, V(x(t)) &\leq l^2 - \sigma m^2/4 \\ &< 0,\end{aligned}$$

and so,

$$V(x(t)) \leq V(x_0) + (l^2 - \sigma m^2/4)t.$$

In conclusion,

$$V(x(t)) \leq 0, \quad \text{for } t \geq V(x_0)/(\sigma m^2/4 - l^2). \qquad \square$$

Remark 2.1. Proposition 2.2 can be viewed as an attainability result of a closed set by the trajectories of a dynamical system; for related control problems, see Ref. 11.

3. Reachability of the Set of Critical Points

This section is devoted to the problem of the reachability of the set C by all the trajectories of (2). We can prove the following main result.

Theorem 3.1. Assume that ϕ, V are locally Lipschitz maps and that $S = \{x \in \mathbb{R}^n : V(x) \leq 0\}$ is a nonempty compact set satisfying (H1). Then, for $\sigma > 0$ sufficiently large, there is $r > 0$ such that, for any $x_0 \in \{x \in \mathbb{R}^n : V(x) < r\}$, any trajectory $x(t)$ of (2) starting from x_0 satisfies the following properties:

(C1) $x(t)$ reaches S in finite time τ;
(C2) $x(t) \in S$ for any $t \geq \tau$;
(C3) $x(t)$ reaches C either in finite time $\bar{\tau}$ or asymptotically; i.e., $\mathrm{dist}(x(t), C) \to 0$ as $t \to +\infty$.
(C4) $\liminf_{t \to +\infty} \mathrm{dist}(x(t), C) = 0$.

Proof. (C1) and (C2) are direct consequences of Propositions 2.1 and 2.2.

Let us prove (C3): for this, consider $x(\cdot) \in \mathcal{S}_{\psi_\sigma}(x_0)$, $x_0 \in S$. Suppose that $x(t)$ does not reach C in finite time; thus, $x(t) \in S \setminus C$, for any $t \geq 0$;

in fact, S is invariant with respect to (2) for $\sigma > \bar{\sigma}$, where $\bar{\sigma}$ is given in Lemma 2.1. Arguing by contradiction, assume that

$$\limsup_{t\to+\infty} \operatorname{dist}(x(t), C) = \beta > 0. \tag{8}$$

Define

$$\bar{V}(x) = \begin{cases} V(x), & \text{if } x \notin S, \\ 0, & \text{if } x \in S. \end{cases}$$

Let

$$W_\sigma(x) = \phi(x) + \sigma \bar{V}(x).$$

W_σ is a locally Lipschitz single-valued function. Observe that, from $\sigma > \bar{\sigma}$, it results that

$$C = \{x \in S : 0 \in \partial W_\sigma(x)\}.$$

Let

$$H = \overline{S \setminus (C + (\beta/4)\,\operatorname{int}(B))};$$

from (8), there exists a sequence $t_n \to +\infty$ such that

$$\operatorname{dist}(x(t_n), C) > \beta/2 \quad \text{and} \quad \lim_{n\to\infty} \operatorname{dist}(x(t_n), C) = \beta.$$

For a.a. $t \geq 0$ such that $x(t) \notin C$, we have that

$$\begin{aligned}(d/dt)\, W_\sigma(x(t)) &\leq \max_{\zeta\in-\partial W_\sigma(x(t))} \langle \zeta, \dot{x}(t)\rangle \\ &\leq -\operatorname{dist}^2(0, \partial W_\sigma(x(t)));\end{aligned} \tag{9}$$

in fact, $0 \notin \partial W_\sigma(x(t))$ and $\partial W_\sigma(x(t))$ is a convex compact set. Let

$$\begin{aligned} m_H &= \min_{x\in H} \min_{v\in\partial W_\sigma(x)} \|v\| > 0, \\ M_H &= \max_{x\in H} \max_{v\in\partial W_\sigma(x)} \|v\| > 0, \\ M_{W_\sigma} &= \max_{x\in H} W_\sigma(x). \end{aligned}$$

For any $s, t > 0$ for which $x(s), x(t) \in H$, we have

$$x(t) - x(s) \in \int_s^t \partial W_\sigma(x(r))dr \subset l|t-s|B.$$

Thus,

$$\|x(t) - x(s)\| \leq l|t - s|.$$

Therefore, for $x(t) \in H$, it results that

$$\begin{aligned} \text{dist}(x(t), C) &\geq |\text{dist}(x(s), C) - \|x(t) - x(s)\|| \\ &\geq -l|t - s| + \beta/2 > \beta/4, \end{aligned} \tag{10}$$

for any $t \in [s, s + \beta/(4l))$. From (9) one has that $t \to W_\sigma(x(t))$ is nonincreasing for all $t \geq 0$ and that

$$(d/dt)W_\sigma(x(t)) \leq -m_H^2, \quad \text{for a.a. } t \geq 0 \text{ such that } x(t) \in H.$$

In virtue of (10), we have

$$W_\sigma(x(t)) \leq M_{W_\sigma} - m_H^2 t,$$

for any $t \in [t_n, t_n + \beta/(4l)), n \in \mathbb{N}$, and $t \to W_\sigma(x(t))$ is nonincreasing. Thus, we have that

$$\lim_{t \to +\infty} W_\sigma(x(t)) = -\infty,$$

contradicting the fact that the locally Lipschitz function W_σ assumes the minimum on S.

To conclude, let us now prove (C4). We argue once again by contradiction; hence, we assume that

$$\liminf_{t \to +\infty} \text{dist}(x(t), C) = \gamma > 0.$$

By the same argument used before in the proof of (C3), for a.a. $t \geq 0$ we have

$$(d/dt)\, W_\sigma(x(t)) \leq -m_H^2 < 0;$$

thus,

$$\lim_{t \to +\infty} W_\sigma(x(t)) = -\infty,$$

obtaining a contradiction with the fact that W_σ assumes the minimum on S. □

Remark 3.1. Observe that Theorem 3.1, for what it concerns the reachability of the set C, can be reformulated as follows: either $x(t)$ reaches C and it remains there for all the future times or $x(t)$ does not belong to C for any $t \geq 0$ sufficiently large or there exists a sequence $t_n \to +\infty$ with $x(t_n) \in C$.

If we assume more regularity conditions on ϕ and V, then we can establish a more precise result on the behavior of the trajectories of (2) as $t \to +\infty$. Indeed, we can prove the following result.

Corollary 3.1. Assume all the conditions of Theorem 3.1. Moreover, assume that ϕ and V are semiconvex[5] functions in $\mathbb{R}^n$. Then, for any $x_0 \in \mathbb{R}^n$ with $V(x_0) < r$, the unique trajectory $x(t)$ starting from x_0 is such that either there exists $\tau > 0$ for which $\mathrm{dist}(x(t), C) = 0$ for any $t \geq \tau$ or

$$\lim_{t \to +\infty} \mathrm{dist}(x(t), C) = 0.$$

Proof. One has only to remark that, under the stated assumptions, W_σ is semiconvex and the differential inclusion (2) reduces to

$$\dot{x}(t) \in -\partial W_\sigma(x(t)), \quad \text{for a.a. } t \geq 0. \tag{11}$$

Moreover, this differential inclusion has only one solution starting from x_0 with $V(x_0) < r$. In fact, let $\lambda > 0$ such that

$$x \to W_\sigma(x) + 2\lambda \|x\|^2$$

is convex on the convex hull of S_r; thus, the generalized gradient of this function is monotone and so

$$\langle \zeta - \bar{\zeta}, x - \bar{x} \rangle \geq -2\lambda \|x - \bar{x}\|^2,$$

for any $x, \bar{x}$ and for any $\zeta \in \partial W_\sigma(x)$ and $\bar{\zeta} \in \partial W_\sigma(\bar{x})$.

Assume now that $x(t)$ and $\bar{x}(t)$ are two solutions of (11) starting from x_0. Then, for a.a. $t \geq 0$, we have

$$\begin{aligned}(d/dt)\,(1/2)\,\|x(t) - \bar{x}(t)\|^2 &= \langle x(t) - \bar{x}(t), -\gamma(t) + \bar{\gamma}(t) \rangle \\ &\leq 2\lambda \|x(t) - \bar{x}(t)\|^2,\end{aligned}$$

[5]A function $u : \mathbb{R}^n \to \mathbb{R}$ is said to be semiconvex if, for any open bounded convex set $\mathcal{V} \subset \mathbb{R}^n$, there exists $\lambda > 0$ such that $x \to u(x) + 2\lambda \|x\|^2$ is convex in $\mathcal{V}$; see Ref. 12.

where $\gamma(t) \in \partial W_\sigma(x(t))$ and $\bar{\gamma}(t) \in \partial W_\sigma(\bar{x}(t))$. By the Gronwall inequality, we have

$$x(t) = \bar{x}(t), \quad \text{for any } t \geq 0.$$

We complete the proof by noticing that, if $x(\tau) \in C$ [i.e., $0 \in \partial W_\sigma(x(\tau))$], then the constant function $t \to x(\tau)$ is the unique solution of (11) starting from $x(\tau)$. □

Remark 3.2. Since both convex and $C^{1,1}$ functions are semiconvex, the previous result includes these relevant classes of functions.

We end this section by considering the convex case, i.e., the case when ϕ and V are convex and so S is. We have the following result.

Corollary 3.2. Assume all the conditions of Theorem 3.1. If ϕ and V are convex functions in $\mathbb{R}^n$, then any trajectory of (2) starting from any point $x_0 \in \mathbb{R}^n$ is such that either $x(t) \in \mathcal{M}$ for any $t \geq \tau$, for some $\tau \geq 0$, or $\lim_{t \to +\infty} \operatorname{dist}(x(t), \mathcal{M}) = 0$.

4. Appendix

Let us first note that, if S satisfies (H2) with V locally Lipschitz, then $S = \overline{\operatorname{int}(S)}$. In fact, let $x \in \partial S$; if there exists a neighborhood $\mathcal{U}$ of x in $\mathbb{R}^n$ for which $V(y) \geq 0$ for any $y \in \mathcal{U}$, then x is a local minimum for V in $\mathbb{R}^n$ and so $0 \in \partial V(x)$, contradicting (H2). Hence, any $x \in \partial S$ is a cluster point of $\operatorname{int}(S)$; thus, $\operatorname{int}(S) \neq \emptyset$.

Moreover, recall that any proximate retract satisfies (H1) with $V(\cdot) = \operatorname{dist}(\cdot, S)$; this class of sets has been studied extensively in Ref. 13.

A natural question occurs: how conditions (H2) and (H1) depend on the particular choice of V?

To answer this question, for the reader convenience, we collect in the following proposition some known results from the literature together with a result [(v) $\Rightarrow$ (ii)] for which we provide a proof.

Proposition 4.1. Let S be a compact set of $\mathbb{R}^n$ such that $S = \overline{\operatorname{int}(S)}$. We have the scheme

$$\text{(i)} \Leftarrow \text{(ii)} \Longleftrightarrow \text{(iii)} \Longleftrightarrow \text{(iv)} \Longleftrightarrow \text{(v)}$$

for the following statements:

(i) there exists a locally Lipschitz function V such that $S = \{x \in \mathbb{R}^n : V(x)\} \leq 0$ and V, S satisfy (H1);

(ii) S is epiLipschitz;
(iii) for any $x \in \partial S$, the Clarke cone $C_S(x)$ has the property that $\operatorname{int} C_S(x) \neq \emptyset$;
(iv) there exists a neighborhood $\mathcal{U}$ of S with $0 \notin \partial \operatorname{dist}(x, S)$ for any $x \in \mathcal{U} \setminus S$; i.e., S satisfies (H1) with $V(\cdot) = \operatorname{dist}(\cdot, S)$;
(v) there exists a locally Lipschitz function V such that $S = \{x \in \mathbb{R}^n : V(x) \leq 0\}$ and V satisfies (H2).

Proof. Clearly, (iv) $\Rightarrow$ (v), (iv) $\Rightarrow$ (i), and (v) $\Rightarrow$ (i). Moreover, Theorem B of Ref. 14. contains (ii) $\Rightarrow$ (v); the assertion (ii) $\Rightarrow$ (iv) can be found in Ref. 15, Proposition 2.2; and (ii) $\Leftrightarrow$ (iii) is due to Rockafellar (Ref. 16). If we prove that (v) $\Rightarrow$ (ii), then we obtain the scheme of the proposition.

Finally, we prove (v) $\Rightarrow$ (ii) for the reader convenience. First, we show that the set $\hat{S} = \overline{\mathbb{R}^n \setminus S}$ is epiLipschitz and so is S. Let $x \in \partial S$; thus, $V(x) = 0$. For $\varepsilon > 0$, define

$$A_\varepsilon = \overline{\text{co}} \left(\bigcup_{y \in B(x,\varepsilon)} \partial V(y) \right).$$

Here, $\overline{\text{co}}(E)$ denotes the closure of the convex hull of the set E. We can choose $0 < \varepsilon < m$ in such a way that $\operatorname{dist}(0, A_\varepsilon) > \varepsilon$. From the Lebourg theorem (Ref. 17), for any $t > 0, v \neq 0$, and $x' \in B(x, \varepsilon) \cap \partial S$, there exists $s \in (0, t)$ and $\zeta \in \partial V(x' + sv)$ such that

$$[V(x' + tv) - V(x')]/t = \langle \zeta, v \rangle.$$

Therefore, for any $v \in z + (\varepsilon/4)\, B$, where $z = \Pi_{A_\varepsilon}(0)$ [i.e., z is the element of A_ε of smallest norm], we have

$$[V(x' + tv) - V(x')]/t > \varepsilon^2/4.$$

Hence,

$$V(x' + tv) > 0 \quad \text{and} \quad x' + tv \in \hat{S}.$$

Thus,

$$z + (\varepsilon/4)\, B \subset T_{\hat{S}}(x').$$

Consequently, by the Cornet theorem (Ref. 18, p. 130), we obtain

$$z + (\varepsilon/8)\, B \subset z + (\varepsilon/4)\, B \subset \liminf_{x' \to x} T_{\hat{S}}(x') \subset C_{\hat{S}}(x).$$

This means that

$$\operatorname{int}(C_{\hat{S}}(x)) \neq \emptyset, \quad \text{for } x \in \partial S,$$

and so $\hat{S}$ is epiLipschitz. □

References

1. Aubin, J. P., and Cellina, A., *Differential Inclusions*, Springer Verlag, Basel, Switzerland, 1984.
2. Bolte, J., *On the Continuous Gradient Projection Method in Hilbert Spaces*, Journal of Optimization Theory and Applications, Vol. 119, pp. 235–259, 2003.
3. Bolte, J., and Teboulle, M., *Barrier Operators and Associated Gradient-Like Dynamical Systems for Constrained Minimization Problems*, SIAM Journal on Control and Optimization, Vol. 42, pp. 1266–1292, 2003.
4. Brezis, H., *Opérateurs Maximaux Monotones*, Lecture Notes, North Holland, Amsterdam, Holland, 1973.
5. Bruck, R., *Asymptotic Convergence of Nonlinear Contraction Semigroups in Hilbert Spaces*, Journal of Functional Analysis, Vol. 18, pp. 15–26, 1974.
6. Serea, O. S., *On a Reflecting Boundary Problem for Optimal Control*, SIAM Journal on Control and Optimization, Vol. 42, pp. 559–575, 2003.
7. Kennedy, M., and Chua, L., *Neural Networks for Nonlinear Programming*, IEEE Transactions on Circuits and Systems, I, Vol. 35, pp. 554–568, 1988.
8. Forti, M., Nistri, P., and Quincampoix, M., *Generalized Neural Network for Nonsmooth Nonlinear Programming Problems*, IEEE Transactions on Circuits and Systems, I, Vol. 51, pp. 1741–1754, 2004.
9. Glazos, M., Hui, S. and Zak, S., *Sliding Modes in Solving Convex Programming Problems*, SIAM Journal on Control and Optimization, Vol. 36, pp. 680–697, 1998.
10. Clarke, F., *Optimization and Nonsmooth Analysis*, Wiley, New York, NY, 1983.
11. Nistri, P., and Quincampoix, M., *On Open-Loop and Feedback Attainability of a Closed Set for Nonlinear Control Systems*, Journal of Mathematical Analysis and Applications, Vol. 270, pp. 474–487, 2002.
12. Cannarsa, P., and Sinestrari, C., *Semiconcave Functions and Optimal Control*, Progress in Nonlinear Differential Equations and Their Applications, Birkhauser, Basel, Switzerland, Vol. 58, 2004.
13. Poliquin, R., Rockafellar, T., and Thibault, L., *Local Differentiability of Distance Functions*, Transactions of the American Mathematical Society, Vol. 352, pp. 5231–5249, 2000.
14. Cornet, B., and Czarnecki, M., *Smooth Representations of EpiLipschitzian Subsets of $\mathbb{R}^n$*, Nonlinear Analysis, Vol. 37, pp. 139–160, 1999.
15. Cornet, B., and Czarnecki, M., *Existence of Generalized Equilibria*, Nonlinear Analysis, Vol. 44, pp. 555–574, 2001.

16. Rockafellar, T., *Clarke's Tangent Cones and the Boundaries of Closed Sets in* $\mathbb{R}^n$, Nonlinear Analysis, Vol. 3, pp. 145–154, 1979.
17. Lebourg, G., *Valeur Moyenne pour Gradient Généralisé*, Comptes Rendus de l'Académie des Sciences, Paris, Série I, Vol. 281, pp. 795–797, 1975.
18. Aubin, J. P., and Frankowska, H., *Set-Valued Analysis*, Birkhauser, Basel, Switzerland, 1990.