Generalized Neural Network for Nonsmooth Nonlinear Programming Problems

Mauro Forti, Paolo Nistri, and Marc Quincampoix

***Abstract*—In 1988 Kennedy and Chua introduced the dynamical canonical nonlinear programming circuit (NPC) to solve in real time nonlinear programming problems where the objective function and the constraints are *smooth* (twice continuously differentiable) functions. In this paper, a generalized circuit is introduced (G-NPC), which is aimed at solving in real time a much wider class of *nonsmooth* nonlinear programming problems where the objective function and the constraints are assumed to satisfy only the weak condition of being regular functions. G-NPC, which derives from a natural extension of NPC, has a neural-like architecture and also features the presence of constraint neurons modeled by ideal diodes with *infinite* slope in the conducting region. By using the Clarke's generalized gradient of the involved functions, G-NPC is shown to obey a gradient system of differential inclusions, and its dynamical behavior and optimization capabilities, both for convex and nonconvex problems, are rigorously analyzed in the framework of nonsmooth analysis and the theory of differential inclusions. In the special important case of linear and quadratic programming problems, salient dynamical features of G-NPC, namely the presence of *sliding modes*, trajectory convergence in *finite time*, and the ability to compute the *exact* optimal solution of the problem being modeled, are uncovered and explained in the developed analytical framework.**

***Index Terms*—Convergence in finite time, gradient inclusions, neural networks, nonlinear programming, nonsmooth nonconvex optimization, sliding modes.**

Notation

$\mathbb{R}^n$	Real n space.
A	$= [A_{ij}] \in \mathbb{R}^{n\times n}$, square matrix.
A'	Transpose of A.
x	$= (x_1, \cdots, x_n)' \in \mathbb{R}^n$, column vector.
$\langle x, y\rangle$	$= \sum_{i=1}^n x_i y_i$, scalar product of $x, y \in \mathbb{R}^n$.
$\|x\|_2$	$= [\sum_{i=1}^n x_i^2]^{1/2}$ Euclidean norm of $x \in \mathbb{R}^n$.
$\overline{E}$	Closure of set $E \subset \mathbb{R}^n$.
$\mathrm{int}(E)$	Interior of E.
∂E	Boundary of E.
$\mathrm{dist}(x, E)$	$= \inf_{y\in E} \|x - y\|_2$, distance of $x \in \mathbb{R}^n$ from E.
$\mathrm{co}(E)$	Convex hull of E.
$b(0, R)$	$= \{y \in \mathbb{R}^n : \|y\|_2 < R\}$, ball with center 0 and radius R.
$\emptyset$	Empty set.
$\nabla V(x)$	Gradient of $V(x) : \mathbb{R}^n \to \mathbb{R}$.
$\partial V(x)$	Clarke's generalized gradient of $V(x) : \mathbb{R}^n \to \mathbb{R}$.

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I. Introduction

The analog circuit approach for solving nonlinear programming problems has received a great deal of attention in the last two decades, see [1]–[8], and references therein. The idea is that of using a dedicated electrical circuit that simulates both the objective and constraint functions, and is able to compute the optimal solution during the analog transient motion toward an equilibrium point which coincides with such an optimal solution. The approach is especially desirable for all those on-line applications where computing the optimum in real time is of fundamental importance, as in some signal processing and robotic problems.

Beyond doubt, the dynamical canonical nonlinear programming circuit (NPC) introduced by Kennedy and Chua in 1988 [1], is a cornerstone in the analog circuit approach to optimization. Indeed, NPC has synthesized the research on this topic in the circuit theory community, see [9]–[11] and their references, with that in the emerging field of neural computation [12], [13]. Among the most desirable features of NPC is the simple and realistic electronic implementation for the most common applications, as for example linear and quadratic programming problems.

NPC has a neural-like architecture and is conceived for solving *smooth* programming problems where the objective and constraint functions are supposed to be twice continuously differentiable functions. As in a typical penalty method, NPC implements the gradient of a smooth energy function that is the sum of the objective function and a barrier function defined by the constraints. It is of importance to observe that to guarantee constraint satisfaction, and to approximately reconcile the solution of the problem with that computed by the circuit, the nonlinearities of the constraint neurons are required to be diode-like functions with a *very-high slope* in the conducting region, the slope playing the role of a penalty parameter [1, Sec. IV].

There is the experimental evidence, from computer simulations and measurements on actual prototypes, that NPC displays salient dynamical features, i.e., *sliding modes* on the surfaces defined by the constraints, and convergence of trajectories toward the optimal solution in *finite time*. These phenomena, which are related to the simultaneous presence of low and very high slopes in the diode-like nonlinearities involved in NPC, cannot be satisfactorily explained by a standard analysis of smooth dynamical systems of differential equations as those considered in [1]. Rather, they are expected to find a natural and rigorous explanation in the analysis of the limiting case where the diode-like functions are modeled by ideal diodes with *infinite* slope in the conducting region. Indeed, in this way a circuit is obtained whose dynamics is no longer described by a standard differential equations, but instead by a differential inclusion. Furthermore, there are well-established mathematical tools, namely the theory of differential equations with discontinuous right-hand side and differential inclusions [14]–[16], which enable to rigourously analyze such an ideal case and analytically predict sliding modes or the phenomenon of trajectory convergence in finite time.

It is also worth remarking that within the framework of analog optimization circuits modeled by differential inclusions it is natural to deal with more general nonlinear programming problems where not only the constraint functions, but also the objective function, are allowed to be *nonsmooth*. In fact, this leads to considering circuits whose dynamics is described by the (set-valued) Clarke's generalized gradient of such functions. Nonsmooth optimization is certainly of interest for the engineering applications, consider as an example the widely employed class of continuous piecewise affine functions [17], [18].

Motivated by the previous discussion, in this paper we introduce a generalized circuit for nonsmooth programming problems [generalized NPC (G-NPC)], which derives from a natural extension of NPC. The case is considered where the diodes are ideal and possess infinite slope in the conducting region, which corresponds to using a nonsmooth penalty approach. G-NPC has a neural-like architecture and is aimed at solving in real time, the really general class of nonsmooth programming problems where the objective and constraint functions are assumed to satisfy only the weak condition of being *regular* (see Section I-A-2 for the definition of regular functions).

G-NPC is designed to implement the Clarke's generalized gradient of the nonsmooth energy given by the objective plus the barrier function defined by the constrains, and is analyzed in the framework of nonsmooth analysis and the theory of inclusions [14]–[16], [19]. The paper addresses the definition, existence, uniqueness, boundedness and convergence toward equilibrium points of the trajectories of G-NPC. Moreover, it studies the optimization capabilities of G-NPG, both for convex and nonconvex problems. In particular, the issues are addressed of how to exactly reconcile the solution of the programming problem with that computed by G-NPC, and to guarantee convergence in finite time by suitably exploiting the presence of sliding modes, where the convergence time can be quantitatively estimated on the basis of the relevant quantities defining the programming problem. Special attention is devoted to the application to the important case of linear and quadratic programming problems.

A. *Preliminaries*

Here, we present definitions and results concerning set-valued maps and nonsmooth analysis, which are needed in the remainder of the paper.

1) Set-Valued Maps: Suppose that to each point x of a set $E \subset \mathbb{R}^n$ there corresponds a nonempty set $F(x) \subset \mathbb{R}^n$. Then, $x \multimap F(x)$ is a set-valued map from E to $\mathbb{R}^n$. A set-valued map $F : E \multimap \mathbb{R}^n$ with nonempty values is said to be upper semicontinuous at $x_0 \in E$ if for any open set $\mathcal{V}$ containing $F(x_0)$, there exists a neighborhood $\mathcal{U}$ of x_0 such that $F(\mathcal{U}) \subset \mathcal{V}$. If E is closed, F has nonempty closed values, and it is bounded in a neighborhood of each point $x \in E$, then, F is upper semicontinuous on E if and only if its graph $\{(x,y) \in E \times \mathbb{R}^n : y \in F(x)\}$ is closed.

2) Locally Lipschitz and Regular Functions: Function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be Lipschitz near $x \in \mathbb{R}^n$ if there exist positive numbers κ and ϵ such that we obtain $|f(x_2) - f(x_1)| \leq \kappa \|x_2 - x_1\|_2$ for all $x_1, x_2 \in x + \epsilon b(0,1)$. If f is Lipschitz near any point of its domain, then it is said to be locally Lipschitz.

We suppose henceforth that f is Lipschitz near x. The generalized directional derivative of f at x in the direction $v \in \mathbb{R}^n$ is given by

$$f^0(x;v) = \limsup_{\substack{y \to x \\ t \to 0^+}} \frac{f(y+tv) - f(y)}{t}.$$

The quantity $f^0(x;v)$ is well defined and finite. Furthermore, the Clarke's generalized gradient of f at x is defined as

$$\partial f(x) = \left\{ \xi \in \mathbb{R}^n : f^0(x;v) \geq \langle v, \xi \rangle \text{ for all } v \in \mathbb{R}^n \right\}.$$

By accounting for the properties of f^0, it is possible to show that $\partial f(x)$ is a nonempty convex compact subset of $\mathbb{R}^n$. One has, for any $v \in \mathbb{R}^n$, $f^0(x;v) = \max\{\langle \xi, v \rangle : \xi \in \partial f(x)\}$, hence, knowing f^0 is equivalent to knowing $\partial f(x)$.

When f is a continuously differentiable (smooth) function at x, $\partial f(x)$ reduces to the singleton $\nabla f(x)$, and when f is convex $\partial f(x)$ coincides with the classical subdifferential of convex analysis, that is, $\partial f(x)$ is the set of vectors $\xi \in \mathbb{R}^n$ satisfying $f(x+u) - f(x) \geq \langle u, \xi \rangle$, for all $u \in \mathbb{R}^n$.

There are several equivalent definitions of the generalized gradient. One of them, which is useful for the practical computation of $\partial f(x)$, is based on the property that a locally Lipschitz function is differentiable almost everywhere (in the sense of Lebesgue measure). Indeed, it can be proved that

$$\partial f(x) = \overline{\text{co}} \left\{ \lim_{n \to \infty} \nabla f(x_n) : x_n \to x,\ x_n \notin \mathcal{N},\ x_n \notin \Omega_f \right\}$$

where Ω_f is the set of points in $x + \epsilon b(0,1)$ where f fails to be differentiable, $\mathcal{N}$ is an arbitrary set of measure zero, and $\text{co}\{\cdot\}$ denotes the convex hull.

Now, the concept of a *regular* function, which is relevant throughout the paper, is defined.

Definition 1: A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ which is Lipschitz near x is said to be regular at x provided the following conditions hold.

1) For all $v \in \mathbb{R}^n$, the usual one-sided directional derivative $f'(x;v) = \lim_{t \to 0^+} [f(x+tv) - f(x)]/t$ exists.
2) For all $v \in \mathbb{R}^n$, $f'(x;v) = f^0(x;v)$. ■

Regular functions form a rather wide set, and several classes of them are presented in [20, Prop. 2.3.6] and [21, App. A]. In particular, a nonsmooth convex function on $\mathbb{R}^n$ is regular at any $x \in \mathbb{R}^n$. It is also worth remarking that the (possibly) nonconvex locally Lipschitz functions involved in the most common engineering applications are also regular. For example, this is true of locally Lipschitz functions whose domain can be decomposed in a finite number of closed sets such that the restriction of the given function to each of them is smooth (C^1).

3) Differential Calculus for Regular Functions: In this paper, a Lyapunov-like approach is employed to study trajectory convergence for a general class of analog optimization circuits whose dynamics is described by a differential inclusion. At the core is the use of differential calculus for generalized gradients, which enables to show among other facts that an energy function is decreasing along the circuit trajectories. The crucial point is that the rules of calculus assume a simple and useful form for regular functions, namely, they are expressed in terms of equalities, instead of inclusions as in the case where f is only locally Lipschitz. Thus, in the regular case, the rules appear in a form similar to that of the classical formulas for differentiable functions.

For instance, for a finite family of functions $f_j, j = 1, \cdots, n$, which are regular at x, we have $\partial(\sum_{j=1}^{n} f_j)(x) = \sum_{j=1}^{n} \partial f_j(x)$. If f_j were only Lipschitz near x, then the previous equality must be replaced by the inclusion $\partial(\sum_{j=1}^{n} f_j)(x) \subseteq \sum_{j=1}^{n} \partial f_j(x)$. An analogous situation occurs in the calculus of the derivative of a composed function (chain rule), which is of key importance in the Lyapunov-like approach used in the paper. The following holds, see [20, Th. 2.3.9-(iii), Prop. 2.2.4].

Property 1 (Chain Rule): If $W : \mathbb{R}^n \to \mathbb{R}$ is regular at $x(t)$ and $x(\cdot) : \mathbb{R} \to \mathbb{R}^n$ is differentiable at t and Lipschitz near t, then

$$\frac{d}{dt} W(x(t)) = \langle \xi, \dot{x}(t) \rangle \qquad \forall \xi \in \partial W(x(t)). \tag{1}$$

■

We stress the fact that if W is Lipschitz at $x(t)$, but not necessarily regular at $x(t)$, then, the time derivative would be expressed by the inclusion $dW(x(t))/dt \subseteq \{\langle \xi, \dot{x}(t) \rangle : \xi \in \partial W(x(t))\}$, which is not convenient in the applications of this paper.

II. Generalized Circuit for Nonlinear Programming

Here, we introduce the nonlinear programming problem which is dealt with in the paper. Moreover, in Section II-A we briefly review the circuit proposed in [1] to solve the problem under suitable smoothness assumptions on the nonlinear functions involved. Then, in Section II-B we present the generalized circuit introduced in this paper to solve the same problem under less restrictive smoothness assumptions.

Let us consider the nonlinear programming problem

$$\begin{aligned} &\text{minimize } \phi(x) \\ &\text{subject to } f_j(x) \geq 0, \qquad j = 1, \cdots, p \end{aligned} \tag{2}$$

where $\phi : \mathbb{R}^q \to \mathbb{R}$, $f_j : \mathbb{R}^q \to \mathbb{R}$, $j = 1, \cdots, p$, and p, q are positive integers.

The region where the constraints are satisfied (feasibility region) is given by

$$\mathcal{S} = \{x \in \mathbb{R}^q : f_j(x) \geq 0, \ j = 1, \cdots, p\}.$$

Henceforth, we always assume that $\mathcal{S}$ is a nonempty, bounded set, with nonempty interior. For a given $x \in \mathbb{R}^q$, let $I^-(x) = \{j \in \{1, \cdots, p\} : f_j(x) < 0\}$ be the set of constraints that are violated at x. Moreover, let $I^0(x) = \{j \in \{1, \cdots, p\} : f_j(x) = 0\}$, and $I^+(x) = \{j \in \{1, \cdots, p\} : f_j(x) > 0\}$.

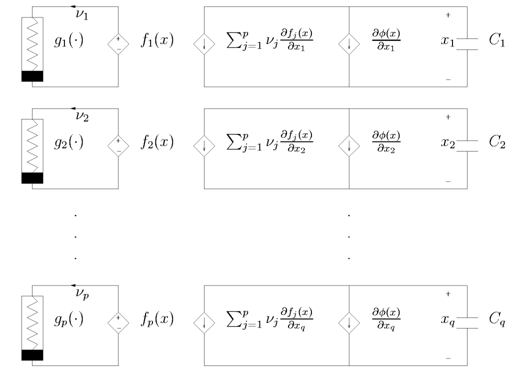

Fig. 1. NPC proposed by Kennedy and Chua in [1].

A. NPC

Kennedy and Chua [1] have proposed a dedicated electrical circuit (NPC), which is aimed at solving problem (2), under the following smoothness assumptions on ϕ and f_j.

Assumption 1: The objective function $\phi \in C^2(\mathbb{R}^q)$. ■

Assumption 2: The constraint functions $f_j \in C^2(\mathbb{R}^q)$, $j = 1, \cdots, p$. ■

NPC, which is represented in Fig. 1, satisfies the system of ordinary differential equations

$$\dot{x}_i = -\frac{\partial \phi(x)}{\partial x_i} - \sum_{j=1}^{p} \nu_j \frac{\partial f_j(x)}{\partial x_i}, \qquad i = 1, \cdots, q \tag{3}$$

where

$$\nu_j = g_j(f_j(x)), \qquad j = 1, \cdots, p.$$

In writing these equations, we have assumed, without loss of generality, normalized values of 1 F for the circuit capacitances C_i.

The nonlinear resistors $g_j, j = 1, \cdots, p$, are characterized by the piecewise-linear function [1]

$$g_j(\rho) = g(\rho) = \begin{cases} 0, & \rho \geq 0 \\ \frac{1}{r}\rho, & \rho < 0. \end{cases} \tag{4}$$

This corresponds to a transposed diode, where $1/r > 0$ is the slope in the conducting region $\rho < 0$. Note that g_j is continuous on $\mathbb{R}$. Moreover, it is a passive nonlinearity, i.e., $\rho g_j(\rho) \geq 0$ for all $\rho \in \mathbb{R}$.

Consider the energy function

$$E^r(x) = \phi(x) + \sum_{j=1}^{p} \int_{0}^{f_j(x)} g(\rho) d\rho. \tag{5}$$

Under the stated assumptions it follows that $E^r \in C^1(\mathbb{R}^q)$. Moreover, accounting for (4), we have

$$E^r(x) = \phi(x) + \frac{1}{r} B(x) \tag{6}$$

where

$$B(x) = \begin{cases} 0, & x \in \mathcal{S} \\ \frac{1}{2}\sum_{j\in I^-(x)} f_j^2(x), & x \in \mathbb{R}^q \setminus \mathcal{S}. \end{cases} \tag{7}$$

Clearly, $E^r(x)$ reduces to $\phi(x)$ when $x \in \mathcal{S}$, and B is seen to be a *smooth* (C^1) barrier function with respect to the feasibility region $\mathcal{S}$.

It is shown in [1] that NPC is a gradient system with respect to E^r, i.e., we obtain

$$\dot{x}_i = -\frac{\partial\phi(x)}{\partial x_i} - \sum_{j=1}^{p} \nu_j \frac{\partial f_j(x)}{\partial x_i} = -\frac{\partial E^r(x)}{\partial x_i}, \qquad i = 1, \cdots, q. \tag{8}$$

If $x(t)$ is a solution of (3), then, $dE^r(x(t))/dt = -\|\dot{x}(t)\|_2^2$, i.e., E^r is strictly decreasing along nonstationary solutions of (3), while it is constant on stationary (equilibrium) solutions. This means that (3) can be employed to search for (local) minima of function E^r.

On the basis of (5)–(8), it is clear that NPC exploits a standard *smooth* penalty approach to reduce the constrained optimization problem (2), i.e., minimize ϕ over the set $\mathcal{S}$, to the solution of an unconstrained optimization problem, i.e., minimize E^r over the whole space $\mathbb{R}^q$. The latter problem is then solved through a gradient method on the smooth energy E^r. The barrier function B is employed to prevent trajectories of (3) from staying outside the feasibility region $\mathcal{S}$. Note that the slope $1/r$ of the diode-like function g represents a penalty parameter: The larger is $1/r$, the stronger is the penalty $(1/r)B(x)$ for violating the constraints. In fact, it has been found through specific examples that large values of $1/r$ are needed to approximatively reconcile the solution computed by NPC with that of the original constrained optimization problem (2) [1, Sec. IV].

B. Generalized NPC

Let us consider the nonlinear programming problem (2) under the following assumptions.

Assumption 3: The objective function ϕ is regular on $\mathbb{R}^q$. ■

Assumption 4: The constraint functions $f_j, j = 1, \cdots, p$, are regular on $\mathbb{R}^q$. ■

The smoothness requirements in Assumption 3 and Assumption 4 are much less restrictive than the corresponding ones made in [1] (cf. Assumption 1 and Assumption 2, respectively). To solve this more general problem, we introduce the circuit in Fig. 2 (the G-NPC). The circuit contains linear capacitors (normalized to $C_i = 1$ F), controlled current and voltage sources, and nonlinear resistors d_j. The current waveforms injected by the controlled current sources are given by $\sum_{j=1}^{p} \nu_j \partial_i f_j(x)$ and $\partial_i\phi(x)$, where ∂_i denotes the ith component of the Clarke's generalized gradient. Finally, the nonlinear resistors d_j, $j = 1, \cdots, q$, are chosen in G-NPC as

$$d_j(\rho) = d(\rho) = \begin{cases} 0, & \rho > 0 \\ [-\sigma, 0], & \rho = 0 \\ -\sigma, & \rho < 0 \end{cases} \tag{9}$$

where $\sigma > 0$ is a given parameter, see Fig. 3(a) and (b).

G-NPC represents a natural extension of NPC in [1]. Indeed, in the special case where ϕ and f_j are smooth, as in

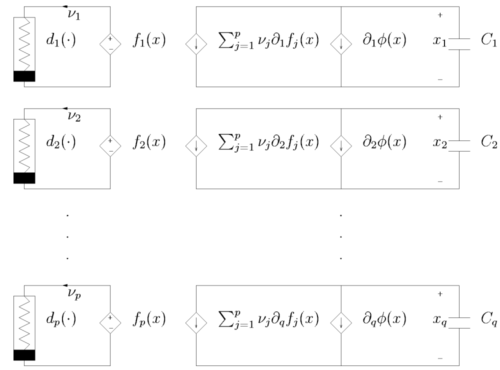

Fig. 2. G-NPC.

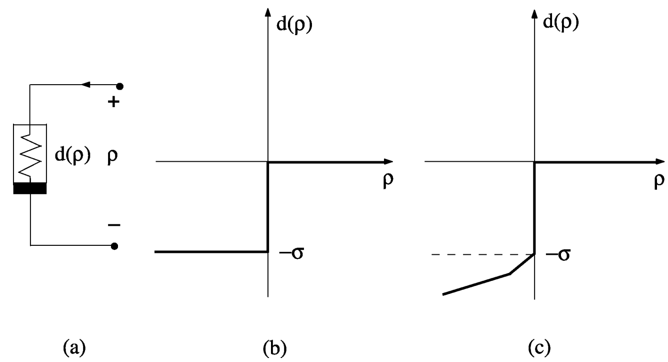

Fig. 3. (a) Nonlinear resistor d used in G-NPC and its characteristics. (b) Hard comparator function, or ideal diode with a negative saturation level $-\sigma$. (c) Diode with nonideal behavior after negative saturation.

Assumption 1 and Assumption 2, respectively, it turns out that $\partial_i\phi(x) = \partial\phi(x)/\partial x_i$ and $\partial_i f_j(x) = \partial f_j(x)/\partial x_i$, hence the current sources in G-NPC reduce exactly to the corresponding ones in NPC. Moreover, apart from the negative saturation level $-\sigma$, nonlinear resistor d in (9) may be thought of as the limit for the slope $1/r \to +\infty$ of the diode-like function g of NPC in (4).

Let us now discuss the dynamical circuit equations satisfied by G-NPC. Consider the nonlinear resistor d in (9), see Fig. 3(b). When $\rho \neq 0$ (suppose ρ is a voltage), the current flowing into the resistor, $d(\rho)$, is single-valued. However, when $\rho = 0$, $d(0)$ can assume all values between 0 and the negative saturation level $-\sigma$. From a mathematical point of view, $d : \mathbb{R} \multimap \mathbb{R}$ is thus a *set-valued* map. Furthermore, when the objective function ϕ is smooth, $\partial_i\phi(x) = \partial_i\phi(x)/\partial x_i$ is single-valued. However, for nonsmooth ϕ, i.e., ϕ which are only regular, the generalized gradient $\partial_i\phi(x)$ is a set-valued map. For example, if $\phi(x) : \mathbb{R} \to \mathbb{R}$ is given by $\phi(x) = |x|$, we have

$$\partial\phi(x) = \overline{\text{co}}\,[\text{sgn}(x)] = \begin{cases} 1, & \rho > 0 \\ [-1, 1], & \rho = 0 \\ -1, & \rho < 0. \end{cases} \tag{10}$$

Analogous considerations hold for the constraint functions f_j.

The previous discussion makes it clear that the dynamics of G-NPC is no longer described by a standard differential equation where the velocity vector field $\dot{x}$ is a single-valued map. Rather, the dynamics obeys a differential inclusion where $\dot{x}$ turns out to be a set-valued map. Indeed, by explicitly writing the circuit equations of G-NPC, we get from Fig. 2 that it satisfies the system of *differential inclusions*

$$\dot{x}_i \in -\partial_i \phi(x) - \sum_{j=1}^{p} \nu_j \partial_i f_j(x), \qquad i = 1, \cdots, q$$

where

$$\nu_j \in d\left(f_j(x)\right), \qquad j = 1, \cdots, p.$$

In a compact form, we obtain

$$\dot{x}_i \in F_i^\sigma(x) = -\partial_i \phi(x) - \sum_{j=1}^{p} d\left(f_j(x)\right) \partial_i f_j(x), \qquad i = 1, \cdots, q \quad (11)$$

where $F^\sigma(x) = (F_1^\sigma(x), \cdots, F_n^\sigma(x))' : \mathbb{R}^n \multimap \mathbb{R}^n$ is the set-valued map defining G-NPC.

The nonlinear resistor d in (9) corresponds to the familiar input-output characteristic of a passive hard comparator with a negative threshold $-\sigma$ [Fig. 3(b)], which can be realized using the combination of an operational amplifier and a battery.[1] Otherwise, function d may be thought of as being the characteristic of an ideal passive transposed diode with a negative saturation level $-\sigma$. Such an ideal diode can be implemented by an operational amplifier with a p-n junction diode in its feedback path [22].[2] [3] For an elementary nonsmooth ϕ, there exist obvious implementations of $\partial_i \phi(x)$. As an example, if $\phi(x) : \mathbb{R} \to \mathbb{R}$ is given by $\phi(x) = |x|$, we have $\partial \phi(x) = \overline{\text{co}}[\text{sgn}(x)]$ (see (10)), which again corresponds to a hard comparator. For more complex ϕ, we need to resort to techniques to synthesize general nonlinear resistive networks, see, e.g., [22] and the papers cited in [1].

We conclude this section by presenting some results on the dynamics of G-NPC, and the nonlinear programming problem (2), which are needed for the development.

First of all, we need to explain what is meant by a solution of a Cauchy problem associated to the system of differential inclusions (11). Since F^σ is a set-valued map with nonempty compact convex values, a possible definition, which we shall adopt in this paper, is that of Filippov [14].

Definition 2 ([14]): A solution of (11) on an interval $[t_1, t_2]$ $(t_2 > t_1)$, which satisfies the initial condition $x(t_1) = x_0$, is an absolutely continuous function $x(t)$ defined in $[t_1, t_2]$, such that $x(t_1) = x_0$, and for almost all (a.a.) $t \in [t_1, t_2]$, we have $\dot{x}(t) \in F^\sigma(x(t))$. ■

The importance for the engineering applications of the concept of solutions in the sense of Filippov is due to the fact that they are good approximations of solutions of actual systems with very high slope nonlinearities [15], [23], [24].

Property 2: Suppose that ϕ satisfies Assumption 3, and f_j, $j = 1, \cdots, p$, satisfy Assumption 4. Then, for any $x_0 \in \mathbb{R}^q$ there is at least a local solution of (11) with initial condition $x(0) = x_0$. ■

Proof: Since F^σ is an upper semicontinuous set-valued map with nonempty compact convex values, the local existence of a solution $x(t)$ of (11) on [0, t_1] ($t_1 > 0$), with $x(0) = x_0$, is a straightforward consequence of [14, Th. 1, p. 77]. ■

An equilibrium point $x^e \in \mathbb{R}^q$ of G-NPC is a stationary solution of (11). Clearly, x^e is an equilibrium point of (11) if and only if we obtain $0 \in F^\sigma(x^e)$. The set of equilibrium points of G-NPC is thus given by

$$\mathcal{E}_\sigma = \left\{x \in \mathbb{R}^q : 0 \in F^\sigma(x)\right\}. \quad (12)$$

Consider the nonlinear programming problem (2), and suppose that ϕ satisfies Assumption 3 and $f_j(x)$, $j = 1, \cdots, p$, satisfy Assumption 4. Under the stated hypotheses, $\mathcal{S}$ is a compact set. Let

$$\mathcal{M} = \arg\min_{x \in \mathcal{S}} \phi(x) \neq \emptyset \quad (13)$$

be the set of global minimizers of problem (2). Consider the set of critical points of ϕ in $\mathcal{S}$

$$\mathcal{C} = \{x \in \mathcal{S} : 0 \in \partial\phi(x) + N_\mathcal{S}(x)\} \quad (14)$$

where $N_\mathcal{S}(x)$ is the normal cone to $\mathcal{S}$ at x [16]. From [20, Corollary p. 52], $\mathcal{C}$ is nonempty. Moreover, from standard results on nonsmooth optimization we have $\mathcal{M} \subset \mathcal{C}$, and for a convex programming problem, i.e., a problem where ϕ is convex on $\mathbb{R}^q$, and $-f_j$, $j = 1, \cdots, n$, are convex on $\mathbb{R}^q$, we obtain $\mathcal{M} = \mathcal{C}$, see Section III-C.

III. Main Results

Here, the main results on trajectory convergence and optimization capabilities of G-NPC are presented. Section III-A considers the most general case where the objective function ϕ and the constraint functions $-f_j$ are nonconvex. Then, Section III-B establishes additional results in the case where $-f_j$ are assumed to be convex. Finally, Section III-C gives stronger results for G-NPC when both ϕ and $-f_j$ are convex, i.e., in the case of nonsmooth convex programming problems. Section IV will cover the application of G-NPC to two convex programming problems of special interest, i.e., linear and quadratic programming problems.

A. General Nonconvex Programming Problems

Here, we suppose that the objective function ϕ satisfies Assumption 3, and the constraints f_j, $j = 1, \cdots, p$, satisfy Assumption 4, but we allow ϕ and $-f_j$ to be nonconvex functions. Our goal is to show that G-NPC has optimization capabilities analogous to those established for NPC in [1].

Let us introduce for G-NPC the energy function

$$W^\sigma(x) = \phi(x) + \sum_{j=1}^{p} \int_0^{f_j(x)} \hat{d}(\rho) d\rho \quad (15)$$

where

$$\hat{d}(\rho) = \begin{cases} 0, & \rho \geq 0 \\ -\sigma, & \rho < 0. \end{cases}$$

[1]The threshold $-\sigma$ will play an important role in the design of G-NPC to solve specific optimization problems, see Section III.

[2]This corresponds to the realization proposed in [1, Fig. 5(b)] for the constraint neurons. Measurements we have conducted on the circuit in [1, Fig. 5(b)] show that such a diode in fact closely approximates the considered function d in Fig. 3(b) and (c).

[3]It is of interest to note that the theory developed in the paper continues to be valid, with minor changes, if we substitute the third equation defining d in (9), with the following one: $d(\rho) \leq -\sigma$ for $\sigma < 0$, see Fig. 3(c). This modification can account for nonideal effects in the actual realization of the diode.

Function W^σ is *nonsmooth*, indeed it is only regular on $\mathbb{R}^q$ [20, Proposition 2.3.6-c]. Moreover

$$W^\sigma(x) = \phi(x) + \sigma\mathcal{B}(x)$$

where

$$\mathcal{B}(x) = \begin{cases} 0, & x \in \mathcal{S} \\ \sum_{j\in I^-(x)} (-f_j(x)), & x \in \mathbb{R}^q \setminus \mathcal{S} \end{cases} \tag{16}$$

is a *nonsmooth* barrier function.

The next basic result holds.

Property 3: Suppose that ϕ satisfies Assumption 3, and f_j, $j = 1, \cdots, p$, satisfy Assumption 4. Then, $\partial\mathcal{B}(x)$ exists for any $x \in \mathbb{R}^q$ and we obtain for (11)

$$\dot{x} \in F^\sigma(x) = -\partial W^\sigma(x) = -\partial\phi(x) - \sigma\partial\mathcal{B}(x) \tag{17}$$

i.e., G-NPC is a circuit that implements the generalized gradient of the energy function W^σ. ■

Proof: The barrier function $\mathcal{B}$ in (16) can be rewritten as $\mathcal{B}(x) = \sum_{j=1}^p |h_j(x)|$, for $x \in \mathbb{R}^q$, where $h_j(x) = (|f_j(x)| - f_j(x))/2$.

For any $x \in \mathbb{R}^q \setminus \mathcal{S}$ we have $I^-(x) \neq \emptyset$. Hence, taking into account that $\partial \sum = \sum \partial$ [20, Corollary p. 40], we obtain from (16)

$$\partial\mathcal{B}(x) = \sum_{j\in I^-(x)} \partial\left(-f_j(x)\right).$$

Let $x \in \partial\mathcal{S}$, thus $I^-(x) = \emptyset$ and $I^0(x) \neq \emptyset$. Hence

$$\partial\mathcal{B}(x) = \sum_{j\in I^0(x)} \partial|h_j|(x)$$

where $h_j(x) = 0$. Since $|\cdot|$ is convex, then it is regular at any point $h_j(x)$, see [20, Proposition 2.3.6]. Therefore, by [20, Theorem 2.3.9-(i)] we obtain

$$\partial|h_j|(x) = [0,1]\cdot\partial(-f_j)(x).$$

Finally, if $x \in \mathrm{int}(\mathcal{S})$, then $\partial\mathcal{B}(x) = 0$. ■

It follows from (17) that the equilibrium points $\mathcal{E}_\sigma$ of G-NPC, as given in (12), coincide with the critical points of the energy W^σ over $\mathbb{R}^q$, i.e.,

$$\mathcal{E}_\sigma = \{x \in \mathbb{R}^q : 0 \in F^\sigma(x)\} = \{x \in \mathbb{R}^q : 0 \in -\partial W^\sigma(x)\}.$$

The next result is a consequence of Property 3.

Property 4: Suppose that ϕ satisfies Assumption 3, and f_j, $j = 1, \cdots, p$, satisfy Assumption 4. Let $x(t)$, $t \in [t_1, t_2]$, where $t_1 < t_2$, be a solution of (11). Then, $W^\sigma(x(t))$ is differentiable for a.a. $t \in [t_1, t_2]$ and

$$\frac{d}{dt}W^\sigma\left(x(t)\right) = -\left\|\dot{x}(t)\right\|_2^2 \leq 0. \tag{18}$$

In particular, if $x(t) \notin \mathcal{E}_\sigma$, then

$$\frac{d}{dt}W^\sigma\left(x(t)\right) \leq -m_\sigma^2\left(x(t)\right) < 0 \tag{19}$$

where $m_\sigma(x(t)) = \min_{\xi\in-\partial W^\sigma(x(t))} \|\xi\|_2 > 0$. If instead $x(t) \in \mathcal{E}_\sigma$, then

$$\frac{d}{dt}W^\sigma\left(x(t)\right) = 0. \tag{20}$$

■

Proof: Since $x(t)$, $t \in [t_1, t_2]$, is absolutely continuous and W^σ is regular, then $x(t)$ and $W^\sigma(x(t))$ are differentiable for a.a. $t \in [t_1, t_2]$. Moreover, from (1)

$$\frac{d}{dt}W^\sigma\left(x(t)\right) = \langle\xi, \dot{x}(t)\rangle \qquad \forall\xi \in \partial W^\sigma\left(x(t)\right). \tag{21}$$

From Property 3, $\dot{x}(t) \in -\partial W^\sigma(x(t))$, hence by choosing $\xi = -\dot{x}(t) \in \partial W^\sigma(x(t))$, we obtain (18).

Suppose that $x(t) \notin \mathcal{E}_\sigma$, thus $0 \notin -\partial W^\sigma(x(t))$. Since the set $\partial W^\sigma(x(t))$ is nonempty and compact, we have $m_\sigma^2(x(t)) = \min_{\xi\in-\partial W^\sigma(x(t))} \|\xi\|_2 > 0$ and $\langle a, a\rangle = \|a\|_2^2 \geq m_\sigma^2(x(t))$ for any $a \in -\partial W^\sigma(x(t))$. Therefore, since $\dot{x}(t) \in -\partial W^\sigma(x(t))$, we have

$$\frac{d}{dt}W^\sigma\left(x(t)\right) = \|\dot{x}(t)\|_2^2 \leq -m_\sigma^2\left(x(t)\right) < 0.$$

Finally, if $x(t) \in \mathcal{E}_\sigma$, then $0 \in \partial W^\sigma(x(t))$. By choosing $\xi = 0$ in (21), we obtain (20). ■

Property 4 simply states that the energy function W^σ is strictly decreasing along nonstationary solutions of (11), while it remains constant on stationary solutions. Property 3 and Property 4 generalize to G-NPC the corresponding results of NPC proved in [1], see Section II-A. In particular, it is seen that G-NPC implements a *nonsmooth* penalty method to solve the nonlinear programming problem (2), and it is suitable to search for (local) minima of the nonsmooth energy function W^σ.

B. Convex Constraints

In Section III-A, the most general setting where the objective and constraint functions are nonconvex has been studied. The aim of this section is to establish basic results on trajectory convergence of G-NPC under the next assumption.

Assumption 5: The constraint functions $-f_j$, $j = 1, \cdots, p$, are convex on $\mathbb{R}^q$. Furthermore, there exists $\tilde{x} \in \mathrm{int}(\mathcal{S})$ such that $f_j(\tilde{x}) > 0$, for $j = 1, \cdots, p$. ■

This assumption is not overly restrictive, since a large part of all relevant applications involve programming problems where the constraint functions $-f_j$ are convex and satisfy Assumption 5. It is also noted that Assumption 5 implies Assumption 4 (cf. [20, Proposition 2.3.6]). Under Assumption 5, the feasibility region $\mathcal{S}$ is convex. Furthermore, the barrier function $\mathcal{B}$ defined in (16) is convex on $\mathbb{R}^q$.

Without loss of generality, we suppose henceforth that $\tilde{x} = 0 \in \mathrm{int}(\mathcal{S})$. If not, we are brought back to this case by the change of variables $x \to x - \tilde{x}$. Thus, we have $f_j(0) > 0$ for $j = 1, \cdots, p$.

Let us consider a sufficiently large sphere centered at the origin, $b(0, R)$, such that

$$\mathcal{S} \subset b(0, R).$$

Our goal is to address convergence to the set $\mathcal{C}$ of the critical points of ϕ in $\mathcal{S}$, as given in (14), of the trajectories of G-NPC starting within $b(0,R)$. Consider the nonempty compact set $\overline{b(0,R)\setminus\mathcal{S}}$. Since the map $x \multimap \partial\phi(x)$ is upper semi-continuous with nonempty, compact and convex values, we can introduce the quantity

$$M_\phi(R) = \max_{x\in\overline{b(0,R)\setminus\mathcal{S}}} \left\{ \max_{v\in\partial\phi(x)} \|v\|_2 \right\} < +\infty. \quad (22)$$

Moreover, let

$$f_m = \min_{j\in\{1,\cdots,p\}} f_j(0) > 0 \quad (23)$$

and

$$M_{\mathcal{B}}(R) = \max_{x\in\overline{b(0,R)\setminus\mathcal{S}}} \frac{\|x\|_2}{f_m + \mathcal{B}(x)} < +\infty. \quad (24)$$

The next theorem is the main result in this section.

Theorem 1: Suppose that ϕ satisfies Assumption 3, and f_j, $j = 1,\cdots,p$, satisfy Assumption 5. If the negative saturation level $-\sigma$ of the nonlinear function d in (9) is such that

$$\sigma > \Gamma(R) = M_\phi(R)M_{\mathcal{B}}(R) \quad (25)$$

then, the following are true.

a) The set $\mathcal{C}$ of critical points of ϕ in $\mathcal{S}$ coincides with the set of equilibrium points of (11) in $b(0,R)$, i.e., we have

$$\mathcal{C} = \mathcal{E}_\sigma \cap b(0,R) \neq \emptyset.$$

b) Given any $x_0 \in b(0,R)$, any solution $x(t)$ of (11) starting from x_0 at $t=0$ is bounded and hence defined for all $t \geq 0$, and in particular $x(t) \in b(0,R)$ for $t \geq 0$. Moreover, $x(t)$ converges to $\mathcal{C}$ as $t \to +\infty$, i.e.,

$$\lim_{t\to+\infty} \mathrm{dist}\,(x(t),\mathcal{C}) = 0. \qquad \blacksquare$$

Proof:

(a) Let us show that $\mathcal{C} = \{x\in\mathcal{S} : 0 \in \partial\phi(x) + N_{\mathcal{S}}(x)\} = \mathcal{E}_\sigma \cap b(0,R) = \{x \in b(0,R) : 0 \in \partial\phi(x) + \sigma\partial\mathcal{B}(x)\}$.

From (38) of Lemma 2 in Appendix I, it is known that $\|\xi\|_2 > 0$ for any $\xi \in -\partial W^\sigma(x) = -\partial\phi(x) - \sigma\partial\mathcal{B}(x)$ and any $x \in b(0,R)\setminus\mathcal{S}$.

Furthermore, if $x \in \partial\mathcal{S}$, then for $\sigma > \Gamma(R)$ it can be proved that $0 \in \partial\phi(x) + \sigma\partial\mathcal{B}(x)$ if and only if $0 \in \partial\phi(x) + N_{\mathcal{S}}(x)$. To this end, we begin by noting the following facts. For $x \in \partial\mathcal{S}$ we have $\partial\mathcal{B}(x) = \sum_{j\in I^0(x)}[0,1]\cdot\partial(-f_j)(x)$. Therefore, by following an argument as in the proof of Property 6 in Appendix I, we obtain

$$\langle x,z\rangle \geq \sum_{j\in I^0(x)} \lambda_j f_j(0) \geq 0$$

for any $z = \sum_{j\in I^0(x)}\lambda_j z_j \in \partial\mathcal{B}(x)$, with $z_j \in \partial(-f_j)(x)$, and $\lambda_j \in [0,1]$ for any $j \in I^0(x)$. If $\sigma > \Gamma(R)$, and $\lambda_j \geq 1$ for some $j \in I^0(x)$, then we obtain $\|\xi\|_2 > M_\phi(R)$ for any $\xi \in -\sigma\partial\mathcal{B}(x)$ (see the proof of Lemma 2 of Appendix I). Moreover, $N_{\mathcal{S}}(x) = \lim_{\sigma\to+\infty}\sigma\partial\mathcal{B}(x) = \sum_{j\in I^0(x)}[0,+\infty)\cdot\partial(-f_j)(x)$ [20, Theorem 2.4.7 and Corollary 1, pp. 55–56].

Now, suppose that x is a critical point belonging to $\partial\mathcal{S}$, i.e., $x \in \mathcal{C}\cap\partial\mathcal{S}$, hence $x \in b(0,R)$ and $0 \in \partial\phi(x) + N_{\mathcal{S}}(x)$. This implies that there exists $u \in N_{\mathcal{S}}(x)$ such that $u \in -\partial\phi(x)$ and $\|u\|_2 \leq M_\phi(R)$. Moreover, since as seen before we obtain $\|\xi\|_2 > M_\phi(R)$, $\xi \in -\sigma\partial\mathcal{B}(x)$, for $\sigma > \Gamma(R)$ and $\lambda_j \geq 1$ for $j \in I^0(x)$, we obtain

$$u = \sum_{j\in I^0(x)} \lambda_j^u z_j^u$$

for some $0 \leq \lambda_j^u < 1$, and $z_j^u \in \partial(-f_j)(x)$, $j \in I^0(x)$. Therefore, $u \in \sigma\partial\mathcal{B}(x)$, i.e., $0 \in \partial\phi(x) + \sigma\partial\mathcal{B}(x)$ and hence, $x \in \mathcal{E}^\sigma \cap b(0,R)$. This implies that $\mathcal{C}\cap\partial\mathcal{S} \subset b(0,R)\cap\mathcal{E}_\sigma$. The converse, $\mathcal{E}_\sigma\cap b(0,R) \subset \mathcal{C}\cap\partial\mathcal{S}$ is obvious.

Finally, if x is a critical point belonging to $\mathrm{int}(\mathcal{S})$, i.e., $x \in \mathcal{C}\cap\mathrm{int}(\mathcal{S})$, then $\partial\mathcal{B}(x) = N_{\mathcal{S}}(x) = \{0\}$.

(b) Consider a solution $x(t)$ of (11) starting from $x_0 \in b(0,R)$ at $t = 0$. For a.a. $t > 0$ such that $x(t) \in b(0,R)\setminus\mathcal{S}$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|_2^2 = \langle x(t),\dot{x}(t)\rangle = \langle x(t), -v(t) - \sigma z(t)\rangle$$

with $v(t) \in \partial\phi(x(t))$ and $z(t) \in \partial\mathcal{B}(x(t))$. Thus, from (39) of Lemma 2 in Appendix I we obtain

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|_2^2 < 0.$$

This implies that $x(t) \in b(0,R)$ for all $t \geq 0$.

It remains to prove that $\lim_{t\to+\infty}\mathrm{dist}(x(t),\mathcal{C}) = 0$. First of all, observe that the energy W^σ is continuous on $\mathbb{R}^q$ and hence bounded from below on the compact set $\overline{b(0,R)}$. Hence, $W^\sigma(x(t))$ is bounded from below for all $t \geq 0$.

We argue by contradiction, i.e., we assume that $\limsup_{t\to+\infty}\mathrm{dist}(x(t),\mathcal{C}) = 2\alpha > 0$. Therefore, there exists a sequence of instants $\{t_n\}$ such that $t_n \to +\infty$ as $n \to +\infty$ and $\lim_{n\to+\infty}\mathrm{dist}(x(t_n),\mathcal{C}) = 2\alpha$. Consider the compact set

$$H = \overline{b(0,R)\setminus\left(\mathcal{C} + \frac{\alpha}{\kappa_0}b(0,1)\right)}$$

where $\kappa_0 > 2$ can be chosen such that $[\mathcal{C} + (\alpha/\kappa_0)b(0,1)] \subset b(0,R)$. The map $x \multimap \{\|w\|_2 : w \in -\partial W^\sigma(x)\}$ assumes the minimum $m > 0$ and the maximum $M > 0$ on H. In fact, the map $x \multimap -\partial W^\sigma(x)$ is upper semicontinuous with nonempty compact convex values, thus the set $-\partial W^\sigma(H)$ is compact [15, Proposition 3, p. 42] and it does not contain the zero vector. Then, the assertion follows from the continuity of the map $y \to \|y\|_2$ on $-\partial W^\sigma(H)$. Then, from (19) we obtain

$$\frac{d}{dt}W^\sigma(x(t)) \leq -m^2 < 0$$

for a.a. $t \geq 0$ such that $x(t) \in H$. Since, for sufficiently large n, say $n > \overline{n}$, we have $\mathrm{dist}(x(t_n),\mathcal{C}) \geq \alpha(2\kappa_0 - 1)/\kappa_0$, it follows that $x(t) \in H$ for any $t \in (t_n, t_n + \delta)$, and for all $n > \overline{n}$, provided $0 < \delta \leq 2\alpha(\kappa_0 - 1)/(\kappa_0 M)$. Hence, on any of the infinitely many time intervals $(t_n, t_n + \delta)$, where $n > \overline{n}$, the energy W^σ undergoes a negative jump which is lower than $-\delta m^2$. By noting that, from Property 4, W^σ is nonincreasing along $x(t)$ for all $t \geq 0$, we easily reach a contradiction to the fact that $W^\sigma(x(t))$ is bounded from below for all $t \geq 0$.■

Remarks:

1) In many programming problems, the lower bound $\Gamma(R)$ in (25) can be rather easily estimated. If needed, a simpler estimate is obtained by noting that

$$\Gamma(R) < \tilde{\Gamma}(R) = \frac{RM_\phi(R)}{f_m}.$$

2) The next example shows that if $\sigma < \Gamma(R)$, then (a) and (b) of Theorem 1 may no longer hold.

Example 1. We want to minimize the scalar function $\phi(x) = -x$ subject to the convex (regular) constraint $f(x) = 1 - |x| \geq 0$, hence, $\mathcal{S} = [-1, 1]$. If we let $R = 2$, we have $\mathcal{S} \subset b(0, 2)$ and, from (25), $\Gamma(2) = 1$. Suppose to choose $\sigma = 1/2 < \Gamma(2)$. For this choice, it is easily seen that the differential inclusion (11) reduces to

$$\dot{x} = F^{\frac{1}{2}}(x) = \begin{cases} 1, & x \in (-1, 1) \\ \frac{1}{2}, & x > 1 \\ \frac{3}{2}, & x < -1 \\ \left[\frac{1}{2}, 1\right], & x = 1 \\ \left[1, \frac{3}{2}\right], & x = -1. \end{cases}$$

Then, it turns out that (11) has no equilibrium points, i.e., $\mathcal{E}_\sigma = \emptyset$. Moreover, since $\dot{x} = F^{1/2}(x) > \gamma^2 > 0$ for all $x \in \mathbb{R}$, all solutions $x(t)$ of (11) are unbounded and tend to $+\infty$ as $t \to +\infty$. ■

3) Theorem 1 guarantees convergence to $\mathcal{C}$ of each trajectory $x(t)$ of (11) starting in $b(0, R)$. When function ϕ is nonconvex, the set of global minimizers $\mathcal{M}$ is in general a strict subset of the set $\mathcal{C}$ of critical points. Therefore, it is not always possible to guarantee convergence to $\mathcal{M}$, as shown in the next examples.

Example 2(a). Consider the problem of minimizing the smooth nonconvex scalar function $\phi(x) = x^3$ subject to the convex (regular) constraint $f(x) = 1 - |x| \geq 0$. We have $\mathcal{S} = [-1, 1]$, $\mathcal{M} = \{-1\}$, and $\mathcal{C} = \{-1, 0\}$. If we let $R = 2$, for sufficiently large σ we have, for G-NPC in (11), $\mathcal{E}_\sigma \cap b(0, 2) = \{-1, 0\} = \mathcal{C}$. It is easy to verify that the solution $x(t)$ of (11) starting from $x_0 = 1/2$ converges from the right to the unstable equilibrium point 0, as $t \to +\infty$. Hence, $x(t)$ converges to $\mathcal{C}$, in accordance with Theorem 1, but it does not converge to $\mathcal{M}$. ■

Example 2(b). Consider the same problem as in Example 2(a) but with $\phi(x) = -(x - 1/2)^2$. We have $\mathcal{S} = [-1, 1]$, $\mathcal{M} = \{-1\}$, and $\mathcal{C} = \{-1, 1/2, 1\}$. By letting $R = 2$, for sufficiently large σ we have, for G-NPC in (11), $\mathcal{E}_\sigma \cap b(0, 2) = \{-1, 1/2, -1\} = \mathcal{C}$. If we choose $x_0 = 2/3$, it is seen that the corresponding trajectory $x(t)$ converges to 1as $t \to +\infty$. Again, we have convergence to $\mathcal{C}$ but not to $\mathcal{M}$. Note that $x = 1$ is a stable equilibrium point corresponding to a local minimum of ϕ in $\mathcal{S}$. ■

4) Since (11) is a system of differential inclusions, the uniqueness of the solution of a Cauchy problem associated to G-NPC in (11) is in general not guaranteed [15], see the next example.

Example 3. We wish to minimize $\phi(x) = |x_1| - |x_2| : \mathbb{R}^2 \to \mathbb{R}$, subject to the constraints $f_1(x) = 1 - |x_1| \geq 0$, $f_2(x) = 1 - |x_2| \geq 0$. Function ϕ is regular but nonconvex, and the constraints are convex and therefore regular. We have $\mathcal{S} = [-1, 1] \times [-1, 1]$, $\mathcal{M} = \{(0, 1)', (0, -1)'\}$, and $\mathcal{C} = \{(0, 0)', (0, 1)', (0, -1)'\}$. If we let $R = 2$, for σ sufficiently large we obtain, for G-NPC in (11), $\mathcal{E}_\sigma \cap b(0, 2) = \{(0, 0)', (0, 1)', (0, -1)'\} = \mathcal{C}$.

Consider the absolutely continuous function $x^a(t) = (x_1^a(t), x_2^a(t))' : [0, +\infty) \to \mathbb{R}^2$ defined as follows

$$x_1^a(t) = \begin{cases} \frac{1}{2} - t, & 0 \leq t \leq \frac{1}{2} \\ 0, & t > \frac{1}{2} \end{cases}$$

$$x_2^a(t) = \begin{cases} 0, & 0 \leq t \leq \frac{1}{2} \\ \frac{1}{2} - t, & \frac{1}{2} < t \leq \frac{3}{2} \\ -1, & t > \frac{3}{2} \end{cases}$$

and the absolutely continuous function $x^b(t) = (x_1^b(t), x_2^b(t))' : [0, +\infty) \to \mathbb{R}^2$ defined as

$$x_1^b(t) = x_1^a(t)$$
$$x_2^b(t) = -x_2^a(t).$$

It can be easily verified that both functions $x^a(t)$ and $x^b(t)$ are solutions of (11) on $[0, +\infty)$ starting from the same initial condition $x_0 = (1/2, 0)'$ at $t = 0$. The first one reaches $(0, 0)'$ and then converges to $(0, -1)' \in \mathcal{M}$ as $t \to +\infty$, while the second reaches $(0, 0)'$ but then converges to $(0, 1)' \in \mathcal{M}$ as $t \to +\infty$. Thus, the property of uniqueness of solutions does not hold in this case. ■

5) We observe that the proof of (b) in Theorem 1 states that the positive ω-limit set, $\omega(x(\cdot))$, of any trajectory $x(t)$ of (11) starting from $x_0 \in b(0, R)$, is contained in the set $\mathcal{C}$.

C. Convex Programming Problems

Here, we address convergence of the trajectories of G-NPC under the following assumption.

Assumption 6: The objective function ϕ is convex on $\mathbb{R}^q$. ■

The next theorem gives a condition under which the trajectories of G-NPC starting in $b(0, R)$ converge to the set of global minimizers $\mathcal{M}$, as in (13), of the nonlinear programming problem (2).

Theorem 2: Suppose that ϕ satisfies Assumption 6, and f_j, $j = 1, \cdots, p$, satisfy Assumption 5. If $\sigma > \Gamma(R)$, where $\Gamma(R)$ is given in (25), then the following are true.

a) The set $\mathcal{M}$ of global minimizers of problem (2) coincides with the set of equilibrium points of (11) in $b(0, R)$, i.e., we have

$$\mathcal{M} = \mathcal{E}_\sigma \cap b(0, R) \neq \emptyset.$$

b) Given any $x_0 \in b(0, R)$, there is a unique solution $x(t)$ of (11) starting from x_0 at $t = 0$, which is defined for all $t \geq 0$, and is such that $x(t) \in b(0, R)$ for $t \geq 0$. Moreover, $x(t)$ converges to $\mathcal{M}$ as $t \to +\infty$, i.e.,

$$\lim_{t \to +\infty} \operatorname{dist}(x(t), \mathcal{M}) = 0. \quad ■$$

Proof: Let us show that for a convex programming problem we obtain $\mathcal{C} = \mathcal{M}$. For smooth functions, this is the classical result of Kuhn–Tucker. In the following, we simply adapt the argument employed in the proof of that result to the present nonsmooth case.

From Theorem 1, we have $\mathcal{C} = \mathcal{E}_\sigma \cap b(0, R)$. Let $\tilde{x} \in \mathcal{C}$, hence $0 \in -\partial W^\sigma(\tilde{x})$. If $\tilde{x} \in \mathrm{int}(S)$, then $\tilde{x}$ is a minimizer of ϕ in $\mathcal{S}$, see [15, Corollary 5, p. 33]. If $\tilde{x} \in \partial\mathcal{S}$, let $\lambda_j \in [0, 1]$, $j \in I^0(\tilde{x})$, be such that

$$0 \in \partial\phi(\tilde{x}) + \sigma \sum_{j \in I^0(\tilde{x})} \lambda_j \partial(-f_j)(\tilde{x}).$$

Since the function

$$W_0^\sigma(x) = \phi(x) + \sigma \sum_{j \in I^0(\tilde{x})} \lambda_j(-f_j)(x)$$

is convex on $\mathbb{R}^q$, we have that $\tilde{x}$ is a global minimizer of W_0^σ in $\mathbb{R}^q$. Therefore, $W_0^\sigma(\tilde{x}) \le W_0^\sigma(x)$, for any $x \in \mathbb{R}^q$. On the other hand, for $j \in I^0(\tilde{x})$ and $x \in \mathcal{S}$ we have $-f_j(x) \le 0$, hence $W_0^\sigma(x) \le \phi(x)$, for any $x \in \mathcal{S}$ and $W_0^\sigma(\tilde{x}) = \phi(\tilde{x})$. In conclusion, $\phi(\tilde{x}) = W_0^\sigma(\tilde{x}) \le W^\sigma(\tilde{x}) \le \phi(x)$, for any $x \in \mathcal{S}$.

On the basis of Theorem 1, to complete the proof, it suffices to show the uniqueness of solutions. Under the stated assumptions on ϕ and $f_j, j = 1, \cdots, p$, W^σ is a convex function on $\mathbb{R}^q$, while from Property 3, (11) is a gradient system on W^σ. Therefore, the uniqueness of the solution of (11) starting from x_0 at $t = 0$ follows from standard results on gradient differential inclusions [15, Theorem 1, p. 159]. ■

We emphasize the fact that, in addition to the result in Theorem 1, in Theorem 2 , we have convergence of each solution of (11) to the set of global minimizers $\mathcal{M}$ (cf. Remark 3 in Section III-B). Furthermore, for each initial condition $x_0 \in b(0, R)$, there exists a *unique* solution of (11) starting from x_0 (cf. Remark 4 in Section III-B).

In the case where ϕ is convex, we are also able to give a condition ensuring convergence to $\mathcal{M}$ in *finite time*, as it is stated in the next result.

Theorem 3: Suppose that, in addition to the hypotheses of Theorem 2, there exists $\eta_1 > 0$ such that

$$\inf_{x \in \mathcal{S} \setminus \mathcal{M}} \left\{ \min_{\xi \in -\partial W^\sigma(x)} \|\xi\|_2 \right\} > \eta_1. \tag{26}$$

Then, given any $x_0 \in b(0, R) \setminus \mathcal{S}$, there is a unique solution $x(t)$ of (11) starting from x_0 at $t = 0$, which converges to $\mathcal{M}$ in finite time, i.e., we obtain $x(t) \in \mathcal{M}$ for all $t \ge \tilde{t}_c$, for some $\tilde{t}_c \ge 0$. Furthermore, the convergence time of $x(t)$ to $\mathcal{M}$, which is defined as $\mathcal{T}_c = \inf\{\tilde{t}_c \ge 0 : x(t) \in \mathcal{M}, t \ge \tilde{t}_c\}$, can be estimated by

$$\mathcal{T}_c \le t_e = (W^\sigma(x_0) - \phi_m) \frac{1}{\min\{\eta_1^2, \eta_2^2\}} \tag{27}$$

where $\phi_m = \phi(x^\star)$, with $x^\star \in \mathcal{M}$, is the global minimum of ϕ in $\mathcal{S}$; moreover

$$\eta_2 = M_\phi(R) \frac{\sigma - \Gamma(R)}{\Gamma(R)} > 0$$

where $M_\phi(R)$ is defined in (22) and $\Gamma(R)$ in (25). ■

Proof: Let $x(t)$ be a solution of (11) starting from $x_0 \in b(0, R)$ at $t = 0$. From Theorem 2, it is known that $x(t) \in b(0, R)$ for $t \ge 0$, and $x(t) \to \mathcal{M}$ as $t \to +\infty$. Hence, due to the continuity of W^σ, $\lim_{t \to +\infty} W^\sigma(x(t)) = \min_{x \in \overline{b(0,R)}} W^\sigma(x) = \phi_m$.

From (38) of Lemma 2 in Appendix I and (19), we obtain

$$\frac{d}{dt} W^\sigma(x(t)) < -\eta_2^2$$

for a.a. $t \ge 0$ such that $x(t) \in b(0, R) \setminus \mathcal{S}$. Furthermore, for a.a. $t \ge 0$ such that $x(t) \in \mathcal{S} \setminus \mathcal{C}$ we obtain from (19) and (26)

$$\frac{d}{dt} W^\sigma(x(t)) < -\eta_1^2.$$

Therefore, we easily conclude that $W^\sigma(x(t)) = \phi_m$ for $t \ge t_e$, where t_e is given in (27). This in turn implies that $x(t) \in \mathcal{M}$ for $t \ge t_e$. ■

Remarks:

1) Condition (26) of Theorem 3 is the crucial assumption ensuring convergence in finite time to $\mathcal{M}$ of the solutions of (11) starting in $b(0, R)$. It is stressed that (26) is in general not necessarily met, and there is not convergence in finite time of the solutions of (11), see the next example. In Section IV we will single out the important case of linear programming (LP) problems, where condition (26) is instead certainly satisfied and convergence in finite time to $\mathcal{M}$ is always guaranteed.

   Example 4. Consider the problem of minimizing the convex scalar function $\phi(x) = x^2$, subject to $f(x) = 1 - |x| \ge 0$. We have $\mathcal{S} = [-1, 1]$ and $\mathcal{M} = \{0\}$. It can be verified that the solution $x(t)$ of (11) starting from $x_0 = 1/2$ at $t = 0$ is such that $|x(t)| \le 1/2$ for $t \ge 0$, i.e., it satisfies the differential equations $\dot{x}(t) = -2x(t)$ for $t \ge 0$. As a consequence, $x(t)$ converges asymptotically to 0 as $t \to +\infty$, i.e., there is not convergence in finite time. It is also easily seen that in this case hypothesis (26) of Theorem 3 is not satisfied. ■

2) Suppose, in addition to the assumptions of Theorem 2, that ϕ is strictly convex on $\mathbb{R}^q$. In this case, there is a unique global minimizer of ϕ in $\mathcal{S}$, i.e., $\mathcal{M} = \{x^\star\}$ is a singleton. It follows from Theorem 2 that, given $x_0 \in b(0, R)$, there is a unique solution of (11) starting from x_0 at $t = 0$, which converges to $x^\star$ as $t \to +\infty$. Under conditions analogous to those of Theorem 3 we can also guarantee global convergence in finite time to $x^\star$.

IV. Linear and Quadratic Programming Problems

Let us consider the following convex programming problem:

$$\begin{aligned} &\text{minimize } \phi(x) = a'x + \frac{1}{2}x'Qx \\ &\text{subject to } f_j(x) = \langle A_j, x\rangle - c_j \\ &\qquad = \sum_{k=1}^{q} A_{j,k}x_k - c_j \ge 0, \qquad j = 1, \cdots, p. \end{aligned} \tag{28}$$

where $a \in \mathbb{R}^q$, $Q \in \mathbb{R}^{q \times q}$ and $A_j \in \mathbb{R}^q$, $c_j \in \mathbb{R}$, $j = 1, \cdots, p$. Without loss of generality, suppose that Assumption 5 is satisfied with $\tilde{x} = 0$. This implies that $-c_j > 0$, $j = 1, \cdots, p$. If Q is a symmetric positive definite matrix, (28) is a quadratic programming (QP) problem, while in the special case $Q = 0$, (28) reduces to the LP problem

$$\begin{aligned} &\text{minimize } \phi(x) = a'x \\ &\text{subject to } f_j(x) = \langle A_j, x\rangle - c_j \\ &\qquad = \sum_{k=1}^{q} A_{j,k} x_k - c_j \geq 0, \qquad j = 1, \cdots, p. \end{aligned} \tag{29}$$

A. Quadratic Programming Problems

For QP problems the dynamical equations (11) of G-NPC can be written as follows:

$$\begin{aligned} \dot{x}_i \in -a_i - &\sum_{k=1}^{q} Q_{ik} x_k \\ &- \sum_{j=1}^{p} A_{i,j} d\left(\sum_{k=1}^{q} A_{j,k} x_k - c_j\right), \qquad i = 1, \cdots, q. \end{aligned}$$

From the definition of the nonlinearity d in (9), we have $d(\sum_{k=1}^{q} A_{j,k}x_k - c_j) = -\sigma$ for $j \in I^-(x)$, $d(\sum_{k=1}^{q} A_{j,k}x_k - c_j) = -\sigma[0,1]$ for $j \in I^0(x)$, and $d(\sum_{k=1}^{q} A_{j,k}x_k - c_j) = 0$ for $j \in I^+(x)$. Hence, for $i = 1, \cdots, q$, we obtain

$$\dot{x}_i \in \begin{cases} -a_i - \sum_{k=1}^{q} Q_{ik}x_k + \sigma \sum_{j\in I^-(x)} A_{i,j} \\ \quad +\sigma \sum_{j \in I^0(x)} [0,1] \cdot A_{i,j}, & x \in \mathbb{R}^q \setminus \mathcal{S} \\ -a_i - \sum_{k=1}^{q} Q_{ik}x_k \\ \quad +\sigma \sum_{j\in I^0(x)} [0,1] \cdot A_{i,j}, & x \in \mathcal{S} \end{cases}$$

and in vector form

$$\begin{aligned} &\dot{x} \in F_{\mathrm{QP}}^{\sigma}(x) \\ &= \begin{cases} -a - Qx + \sigma \sum_{j \in I^-(x)} A_j \\ \quad +\sigma \sum_{j\in I^0(x)} [0,1] \cdot A_j, & x \in \mathbb{R}^q \setminus \mathcal{S} \\ -a - Qx + \sigma \sum_{j \in I^0(x)} [0,1] \cdot A_j, & x \in \mathcal{S}. \end{cases} \end{aligned} \tag{30}$$

The next result holds.

Theorem 4: Suppose that $\sigma > \Gamma(R)$, where $\Gamma(R)$ is given in (25). Then, given any $x_0 \in b(0, R)$, there is a unique solution of the QP circuit (30) starting from x_0 at $t = 0$, which converges to $x^\star$ as $t \to +\infty$, where $x^\star$ is the unique global minimizer of ϕ in $\mathcal{S}$. ■

Proof: For the QP problem (28), $\phi(x) = a'x + (1/2)x'Qx$ is strictly convex, hence there is a unique global minimizer, $x^\star$, of ϕ in $\mathcal{S}$, i.e., $\mathcal{M} = \{x^\star\}$. The result in the theorem is thus a direct consequence of Theorem 2. ■

B. Linear Programming Problems

Let us now consider the LP problem (29). The dynamical equations reduce to

$$\begin{aligned} &\dot{x} \in F_{\mathrm{LP}}^{\sigma}(x) \\ &= \begin{cases} -a + \sigma \sum_{j\in I^-(x)} A_j \\ \quad +\sigma \sum_{j\in I^0(x)} [0,1] \cdot A_j, & x \in \mathbb{R}^q \setminus \mathcal{S} \\ -a + \sigma \sum_{j \in I^0(x)} [0,1] \cdot A_j, & x \in \mathcal{S}. \end{cases} \end{aligned} \tag{31}$$

Moreover, the energy function (15) becomes

$$W_{\mathrm{LP}}^{\sigma}(x) = a'x - \sigma \sum_{j\in I^-(x)} (\langle A_j, x\rangle - c_j)$$

and from Property 3 we obtain $\dot{x} \in F_{\mathrm{LP}}^{\sigma}(x) = -\partial W_{\mathrm{LP}}^{\sigma}(x)$, for $x \in \mathbb{R}^q$.

The next property shows that the threshold $\Gamma(R)$ as defined in (25) assumes a particularly simple form for LP problems.

Property 5: For the LP problem (29), we obtain

$$\Gamma(R) = \|a\|_2 M_{\mathcal{B}}(R) \leq \Gamma_{\mathrm{LP}} = \frac{\tilde{R}\|a\|_2}{f_m} \tag{32}$$

for any $R > \tilde{R}$, where $\tilde{R} = \max_{y\in\partial\mathcal{S}} \|y\|_2$, and $f_m = \min_{j\in\{1,\ldots,p\}}(-c_j) > 0$. ■

Proof: See Appendix II. ■

Now, we prove the lemma which is crucial to study trajectory convergence for the LP circuit (31).

Lemma 1: Consider the set-valued map F_{LP}^{σ} defining the LP circuit (31). If $a \neq 0$ and $\sigma > \Gamma_{\mathrm{LP}}$, where Γ_{LP} is given in (32), we obtain

$$\inf_{x\in\partial\mathcal{S}\setminus\mathcal{M}} \left\{ \min_{\xi \in -\partial W_{\mathrm{LP}}^{\sigma}(x)} \|\xi\|_2 \right\} = \mu_{\mathrm{LP}} > 0. \tag{33}$$

■

Proof: Let $\sigma > \Gamma_{\mathrm{LP}}$. From Property 5 and (a) of Theorem 2, we have $\mathcal{M} = \mathcal{E}_\sigma$ on $\mathbb{R}^q$. Furthermore, since $a \neq 0$ it follows from standard results on LP problems that $\mathcal{M} \subset \partial\mathcal{S}$. Let $x \in \partial\mathcal{S} \setminus \mathcal{M}$, hence $I^-(x) = \emptyset$, $I^0(x) \neq \emptyset$ and consider the set

$$F_{\mathrm{LP}}^{\sigma}(x) = -\partial W_{\mathrm{LP}}^{\sigma}(x) = -a + \sigma \sum_{j\in I^0(x)} [0,1] \cdot A_j.$$

Since $x \in \partial\mathcal{S} \setminus \mathcal{M} = \partial\mathcal{S} \setminus \mathcal{E}_\sigma$, then $0 \notin F_{\mathrm{LP}}^{\sigma}(x)$. Being $F_{\mathrm{LP}}^{\sigma}(x)$ a compact set, and $0 \notin F_{\mathrm{LP}}^{\sigma}(x)$, we obtain $\min_{\xi\in-\partial W_{\mathrm{LP}}^{\sigma}(x)} \|\xi\|_2 = m(I^0(x)) > 0$. The family Φ_0 of distinct sets of indexes $I^0(x)$ obtained by varying x in $\partial\mathcal{S}\setminus\mathcal{M}$ is finite, hence we obtain

$$\begin{aligned} \inf_{x\in\partial\mathcal{S}\setminus\mathcal{M}} \left\{ \min_{\xi\in-\partial W_{\mathrm{LP}}^{\sigma}(x)} \|\xi\|_2 \right\} &= \min_{I^0(x)\in\Phi_0} \left\{ m\left(I^0(x)\right) \right\} \\ &= \mu_{\mathrm{LP}} > 0. \end{aligned}$$

■

The next theorem is the main result on the application of G-NPC to solve LP problems.

Theorem 5: Suppose that $\sigma > \Gamma_{\mathrm{LP}}$, where Γ_{LP} is given in (32). Then, given any $x_0 \in \mathbb{R}^q$, there is a unique solution $x(t)$ of the LP circuit (31) starting from x_0 at $t = 0$, which converges to $\mathcal{M}$ in finite time, i.e., we obtain $x(t) \in \mathcal{M}$ for all $t \geq \tilde{t}_c$, for some $\tilde{t}_c \geq 0$.

If $a \neq 0$ the convergence time of $x(t)$, which is defined as $\mathcal{T}_c = \inf\{\tilde{t}_c \geq 0 : x(t) \in \mathcal{M}, t \geq \tilde{t}_c\}$, can be estimated by

$$\mathcal{T}_c \leq t_e = \left(W_{\mathrm{LP}}^{\sigma}(x_0) - \phi_m\right) \frac{1}{\min\{\eta_1^2, \eta_2^2\}} \tag{34}$$

where $\phi_m = \phi(x^\star)$, with $x^\star \in \mathcal{M}$, is the global minimum of $\phi(x) = a'x$ in $\mathcal{S}$; moreover

$$\eta_1 = \min\{\|a\|_2, \mu_{\mathrm{LP}}\} > 0 \tag{35}$$

where μ_{LP} is defined in (33) and

$$\eta_2 = \|a\|_2 \frac{\sigma - \Gamma_{\mathrm{LP}}}{\Gamma_{\mathrm{LP}}} = \frac{\sigma f_m}{\tilde{R}} - \|a\|_2 > 0. \tag{36}$$

If instead $a = 0$, we have

$$\mathcal{T}_c \leq t_e = \frac{W_{\rm LP}^{\sigma}(x_0)}{\eta_2^2}. \qquad \blacksquare$$

Proof: Suppose that $a \neq 0$. To prove the result stated in the theorem it suffices to note that, on the basis of Lemma 1, the hypotheses of Theorem 3 are satisfied with η_1 as in (35).

Then, suppose that $a = 0$. By an argument as in the proof of Theorem 3, we have $dW_{\rm LP}^{\sigma}(x(t))/dt < -\eta_2^2 < 0$ for a.a. $t \geq 0$ such that $x(t) \in b(0, R) \setminus \mathcal{S}$, where η_2 is defined in (36). For $x(t) \in b(0, R) \setminus \mathcal{S}$ it also results $W_{\rm LP}^{\sigma}(x(t)) > 0$, while for $x(t) \in \mathcal{S}$, $W_{\rm LP}^{\sigma}(x(t)) = 0$. Therefore, it easily follows that $x(t) \in \mathcal{S}$ for $t \geq t_e = W_{\rm LP}^{\sigma}(x_0)/\eta_2^2$. Finally, note that $\dot{x}(t) = 0$ for a.a. $t \geq t_e$ such that $x(t) \in \mathcal{S}$. Otherwise, $W_{\rm LP}^{\sigma}$ would undergo a negative jump for $t \geq t_e$, while we obtain $W_{\rm LP}^{\sigma}(x(t)) = 0$ for $t \geq t_e$. Hence, $x(t) = \text{constant}$ for $t \geq t_e$. $\blacksquare$

Remarks:

1) Theorem 5 actually ensures that in the case of LP problems there is *always* convergence in finite time to $\mathcal{M}$ of the trajectories of G-NPC starting everywhere in $\mathbb{R}^q$. This is due to the fact that condition (26) in Theorem 3 is certainly satisfied for $a \neq 0$, see the proof of Theorem 5 and Lemma 1, and compare Remark 1 in Section III-C.
2) As it was pointed out in Section II-A, the neural network proposed by Kennedy and Chua [1], see NPC in (3), is actually employed in the case where the slope $1/r$ of the diode-like function g in (4) is very high. As a consequence, the result in Theorem 5 can be applied with good approximation also to study the behavior of NPC. In other words, the present study has justified from the viewpoint of nonsmooth optimization the phenomena of sliding modes, and convergence in finite time, which have been observed by means of computer simulations in the use of NPC for solving LP problems.
3) A previous paper [6] has introduced a different class of neural networks to solve in finite time, via the use of a sliding mode approach, LP problems which are given in standard form. The neural network in [6] has significant architectural differences with respect to NPC and G-NPC, compare the block diagram in [6, Fig. 1] with Fig. 1 and Fig. 2, and has not been investigated from the viewpoint of implementation by an actual circuit. In fact, from [6, Fig. 1] that neural network seems to need a more complex electronic implementation with respect to NPC or G-NPC. We refer the reader to [6] also for a comprehensive review of previous approaches to LP problems based on analog solvers.
4) The problem considered for the LP circuit (31) can be also formulated as that of reaching a closed set of a finite dimensional space by the trajectories of a suitably defined controlled system. Here, the closed set is the set $\mathcal{M}$ of the global minimizers of the LP problem, and the control is the penalty parameter σ of the adopted penalty method (cf. (9)). Other approaches, based on viability and invariance theory, to solve the attainability problem of closed sets are presented in [25], [26].

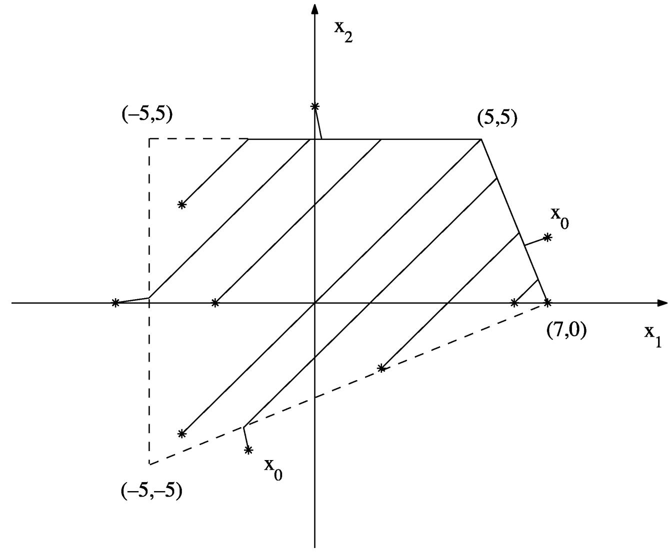

Fig. 4. Phase-space trajectories starting at ten different initial conditions for the LP circuit of Example 5 ($\sigma = 6$). Each initial conditions x_0 is represented by an asterisk. The feasibility region $\mathcal{S}$ is delimited by dashed lines.

Example 5*:* To illustrate the result in Theorem 5, consider the following LP problem studied in [12]. Minimize

$$\phi(x) = -x_1 - x_2$$

subject to the affine constraints

$$\begin{aligned} -\frac{5}{12}x_1 + x_2 + \frac{35}{12} &\geq 0 \\ x_1 + 5 &\geq 0 \\ -x_2 + 5 &\geq 0 \\ -\frac{5}{2}x_1 - x_2 + \frac{35}{2} &\geq 0. \end{aligned}$$

The feasibility region $\mathcal{S}$ is shown in Fig. 4, and it is seen that the global minimum of ϕ in $\mathcal{S}$ is reached at $x^\star = (5,5)'$, and $\phi_m = \phi(x^\star) = -10$.

Consider the application of the LP circuit (31) to solve this problem. If we let $\hat{R} = 7.1$, then $\mathcal{S} \subset b(0, \hat{R})$, hence from (32)

$$\Gamma_{\rm LP} \leq \frac{\hat{R}\|a\|_2}{f_m} = \frac{7.1 \times 12 \times \sqrt{2}}{5} = 3.45.$$

Therefore, the hypotheses of Theorem 5 are certainly satisfied if we choose, for example, $\sigma = 6$, and the trajectories of (31) are globally convergent to $(5,5)'$ in finite time.

Let us evaluate $\mu_{\rm LP}$ as defined in (33) for $\sigma = 6$. The boundary of $\mathcal{S}$ is made up of four segments and on each open segment the quantity $\min_{\xi \in F_{\rm LP}^{\sigma}(x)} \|\xi\|_2$ is constant. On the basis of this observation, a straightforward though tedious computation shows that $\mu_{\rm LP} = 0.557$, hence $\eta_1 = 0.557$ and $\eta_2 = 0.743$. From (34) we conclude that for any $x_0 \in \mathbb{R}^2$ the solution of (31) starting from x_0 at $t = 0$ converges in finite time, $\mathcal{T}_c$, to $x^\star = (5,5)'$, and $\mathcal{T}_c \leq t_e$ where, on the basis of Theorem 5, the estimated convergence time is

$$t_e \leq \left(W_{\rm LP}^{\sigma}(x_0) - \phi_m\right) \frac{1}{\min\{\eta_1^2, \eta_2^2\}} = \frac{W_{\rm LP}^{\sigma}(x_0) + 10}{0.31}. \tag{37}$$

Fig. 4 shows the phase-space trajectories of (31), starting from ten different initial conditions x_0, as obtained with MATLAB simulations ($\sigma = 6$). It is seen that all trajectories converge toward the optimal solution $(5,5)'$. In particular

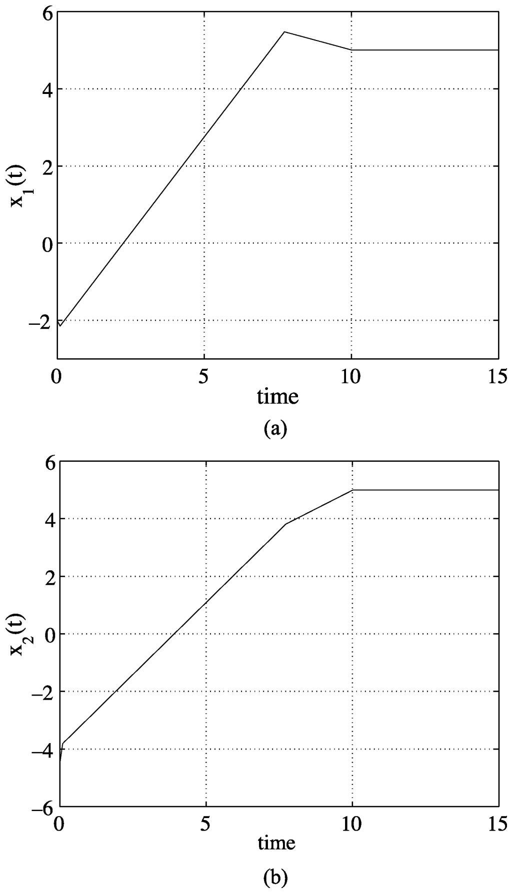

Fig. 5. (a) Time-domain evolution of the state variable $x_1(t)$. (b) Evolution of $x_2(t)$, for the LP circuit of Example 5 ($x_0 = (-2, -4.5)'$, $\sigma = 6$).

TABLE I
COMPARISON BETWEEN ESTIMATED (t_e) AND TRUE ($\mathcal{T}_c$) CONVERGENCE TIME FOR TEN DIFFERENT INITIAL CONDITIONS

x_0	t_e	$\mathcal{T}_c$
$(-6,0)'$	70.97	10.1
$(0,6)'$	32.26	5.0
$(7,0)'$	9.67	9.67
$(6,0)'$	12.9	9.0
$(2,-2)'$	32.26	9.67
$(-3,0)'$	41.93	8.0
$(-4,-4)'$	58.06	9.0
$(-2,-4.5)'$	67.74	10.02
$(-4,3)'$	35.48	9.0
$(7,2)'$	41.93	6.33

Fig. 5 depicts the time evolution of the state variables $x_1(t)$ and $x_2(t)$ for the trajectory starting from $x_0 = (-2, -4.5)'$. From the figure it clearly emerges that $x(t)$ converges to the exact optimal solution in finite time. An analogous result has been verified in all other simulations.

For the ten initial conditions considered in Fig. 4, Table I compares the estimated convergence time t_e in (37), and the true convergence time $\mathcal{T}_c$ as obtained with MATLAB. It is seen that the estimate (37) is always satisfied. It is also noted that in some cases (e.g., $x_0 = (7,0)'$) the estimate is accurate, while in others situations the estimate is quite conservative.[4]

V. CONCLUSION

The paper has proposed a programming circuit which generalizes that introduced by Kennedy and Chua in [1], and is aimed at solving in real time a large class of *nonsmooth* programming problems of interest for the engineering applications. The proposed circuit has a neural-like architecture and in particular it possesses constraint neurons modeled by ideal diodes with infinite slope in the conducting region, which corresponds to the use of a nonsmooth penalty approach.

The generalized circuit has been rigorously analyzed in the framework of the theory of differential inclusions, and thorough results on trajectory convergence toward equilibrium points and optimization capabilities have been obtained. Moreover, the paper has singled out classes of problems of special interest, such as linear programming problems, where the circuit is able to compute the *exact* global optimal solution in *finite time*, by suitably exploiting the presence of sliding modes. Essentially, the results here obtained mean that the NPC originally conceived by Kennedy and Chua, once it has been extended in the way discussed in the paper, can be rigorously justified also from the viewpoint of nonsmooth optimization theory.

It is believed that the techniques developed in the paper, in particular the Lyapunov-like approach used to analyze trajectory convergence and optimization capabilities, are potentially useful also to study other classes of nonlinear circuits containing familiar nonsmooth nonlinearities, as ideal diodes or ideal operational amplifiers used as hard comparators. Up to now, such circuits have been mainly investigated from the static viewpoint of existence and uniqueness of the equilibrium point, while the relevant dynamical features seem to have received not much attention.

The property of convergence in finite time, which is peculiar to some classes of nonsmooth circuits considered in this paper, is of interest in view of the applications requiring that the programming problems be solved on-line. Future work aims at obtaining less conservative estimates of the (finite) convergence time, with respect to those established in Theorem 3 and Theorem 5. Hopefully, this can be done by exploiting other approaches based on viability and invariance theory (cf. Remark 4 in Section IV), which are expected to enable to gain further insight on the geometrical structure of the energy surface (objective function and constraints) of each specific class of programming problems which is dealt with.

[4]The conservativeness of the estimate (34) can be explained by considering that in the proof of Theorem 5 the worst-case situation is considered where a trajectory is supposed to always move with the lowest possible velocity $\min\{\eta_1, \eta_2\}$, which corresponds to the minimum derivative (in absolute value) of the energy W^{σ}_{LP}. It is believed that a more accurate estimate of $\mathcal{T}_c$ could be obtained by exploiting more knowledge on the geometry of the energy function W^{σ}_{LP}. This, however, goes beyond the scope of the present work, and represents an interesting topic for future research.

APPENDIX I

Property 6: Suppose that f_j, $j = 1, \cdots, p$, satisfy Assumption 5. Then, we have

$$\langle x, z\rangle \geq \sum_{j\in I^-(x)} f_j(0) + \sum_{j\in I^-(x)} |f_j(x)| > 0$$

for any $z \in \partial\mathcal{B}(x)$, and any $x \in \mathbb{R}^q \setminus \mathcal{S}$. ■

Proof: Since $-f_j$, $j = 1, \cdots, p$, are convex on $\mathbb{R}^q$ and $f_j(0) > 0$ for any $j = 1, \cdots, p$, we have for $x \in \mathbb{R}^q \setminus \mathcal{S}$

$$0 < f_j(0) \leq -f_j(x) + f_j(0) \leq \langle x, z\rangle$$

for $j \in I^-(x)$, and $z \in -\partial f_j(x)$. Thus, if $z \in \partial\mathcal{B}(x)$, then $z = \sum_{j\in I^-(x)} z_j$, with $z_j \in -\partial f_j(x)$, and

$$\begin{aligned}\langle x, z\rangle &= \sum_{j\in I^-(x)} \langle x, z_j\rangle \\ &\geq \sum_{j\in I^-(x)} (-f_j(x) + f_j(0)) \\ &= \sum_{j\in I^-(x)} f_j(0) + \sum_{j\in I^-(x)} |f_j(x)| > 0.\end{aligned}$$ ■

Lemma 2: Suppose that ϕ satisfies Assumption 3, and f_j, $j = 1, \cdots, p$, satisfy Assumption 5. If $\sigma > \Gamma(R)$, where $\Gamma(R)$ is defined in (25), the following properties hold.

a) We have, for any $\xi \in -\partial W^\sigma(x)$ and any $x \in b(0, R) \setminus \mathcal{S}$

$$\|\xi\|_2 \geq M_\phi(r)\frac{\sigma - \Gamma(r)}{\Gamma(r)} > 0 \tag{38}$$

.

b) We have

$$\langle x, v\rangle + \sigma\langle x, z\rangle > 0 \tag{39}$$

for any $v \in \partial\phi(x)$, any $z \in \partial\mathcal{B}(x)$, and any $x \in b(0, R)\setminus \mathcal{S}$. ■

Proof:

a) Taking into account (16) and (23), from Property 6 we obtain $\|x\|_2\|z\|_2 \geq \langle x, z\rangle \geq f_m + \mathcal{B}(x) > 0$, for any $z \in \partial\mathcal{B}(x)$ and any $x \in \mathbb{R}^q \setminus \mathcal{S}$. Then, being $\sigma > \Gamma(R)$, we get from (24) and (25)

$$\begin{aligned}\|u\|_2 &\geq \sigma\frac{f_m + \mathcal{B}(x)}{\|x\|_2} \\ &\geq \frac{\sigma}{\max_{x\in\overline{b(0,R)\setminus\mathcal{S}}} \frac{\|x\|_2}{f_m+\mathcal{B}(x)}} \\ &= M_\phi(R)\frac{\sigma}{\Gamma(R)}.\end{aligned}$$

Since $\sigma/\Gamma(R) > 1$ we can put $\epsilon = \sigma/\Gamma(R) - 1 > 0$ and obtain that $\|u\|_2 \geq M_\phi(R)(1 + \epsilon)$, for any $u \in \sigma\partial\mathcal{B}(x)$ and any $x \in \mathbb{R}^q \setminus \mathcal{S}$. Finally, since $\partial W^\sigma(x) = \partial\phi(x) + \sigma\partial\mathcal{B}(x)$, we obtain from (22), $\|\xi\|_2 \geq \epsilon M_\phi(R) > 0$, for any $\xi \in -\partial W^\sigma(x)$ and any $x \in b(0, R) \setminus \mathcal{S}$, i.e., (38).

b) Equation (22) yields $\langle x, v\rangle \leq \|x\|_2\|v\|_2 \leq M_\phi(R)\|x\|_2$, for any $v \in \partial\phi(x)$, and any $x \in b(0, R) \setminus \mathcal{S}$. Moreover, from Property 6 we have $\langle x, z\rangle \geq f_m + \mathcal{B}(x)$, for any $z \in \partial\mathcal{B}(x)$ and any $x \in \mathbb{R}^q \setminus \mathcal{S}$. Thus, being $\sigma > \Gamma(R)$, we obtain $\sigma\langle x, z\rangle > M_\phi(R)\|x\|_2$, for any $z \in \partial\mathcal{B}(x)$ and any $x \in \mathbb{R}^q \setminus \mathcal{S}$, i.e., (39). ■

APPENDIX II

Proof of Property 5

Let us evaluate, for $x \in \overline{b(0, R) \setminus \mathcal{S}}$, the quantity $\|x\|_2/(f_m+\mathcal{B}(x))$.

If $x \in \partial\mathcal{S}$, then $\|x\|_2 \leq \tilde{R}$ and $\mathcal{B}(x) = 0$. Hence, $\|x\|_2/(f_m + \mathcal{B}(x)) \leq \tilde{R}/f_m$ and from (24)

$$\|a\|_2 M_\mathcal{B}(R) = \|a\|_2 \max_{x\in\partial\mathcal{S}} \frac{\|x\|_2}{f_m + \mathcal{B}(x)} \leq \frac{\tilde{R}\|a\|_2}{f_m}.$$

Consider now the set of points $x \in [\overline{b(0, R) \setminus \mathcal{S}}] \setminus \partial\mathcal{S}$. Recall that $\mathcal{S}$ is a compact convex set and $0 \in \mathrm{int}(\mathcal{S})$. Let $\tilde{x} \in C_R$, where $C_R = \{x \in \mathbb{R}^q : \|x\|_2 = R\}$, and consider the half line $r = \alpha\tilde{x}$, with $\alpha \geq 0$, and the segment $r \cap [\overline{b(0, R) \setminus \mathcal{S}}] \setminus \partial\mathcal{S} = \{y = \alpha\tilde{x} : \alpha_m(\tilde{x}) < \alpha \leq 1\}$, where $\alpha_m(\tilde{x}) > 0$ is defined by $\alpha_m(\tilde{x})\tilde{x} \in \partial\mathcal{S}$. It is clear all points $x \in [\overline{b(0, R) \setminus \mathcal{S}}] \setminus \partial\mathcal{S}$ can be obtained by varying $\tilde{x}$ on C_R and α on $(\alpha_m(\tilde{x}), 1]$.

We want to show that

$$\Xi(\alpha) = \frac{\|\alpha\tilde{x}\|_2}{f_m + \mathcal{B}(\alpha\tilde{x})} = \frac{\|\alpha\tilde{x}\|_2}{f_m - \sum_{j\in I^-(\alpha\tilde{x})} (\langle A_j, \alpha\tilde{x}\rangle - c_j)}$$

is a nonincreasing function of α for $0 < \alpha_m(\tilde{x}) < \alpha \leq 1$. Note that if $j \in I^-(\alpha\tilde{x})$, then $-\langle A_j, \alpha\tilde{x}\rangle > -c_j > 0$.

Choose α_1 and α_2 such that $0 < \alpha_m(\tilde{x}) < \alpha_1 < \alpha_2 \leq 1$. Then, we obtain $I^-(\alpha_1\tilde{x}) \subseteq I^-(\alpha_2\tilde{x})$. Indeed, if $j \in I^-(\alpha_1\tilde{x})$, then $-\langle A_j, \alpha_1\tilde{x}\rangle > -c_j > 0$, hence $-\langle A_j, \alpha_2\tilde{x}\rangle > -\langle A_j, \alpha_1\tilde{x}\rangle > -c_j > 0$, i.e., $j \in I^-(\alpha_2\tilde{x})$. Now, we have

$$\begin{aligned}\Xi(\alpha_2) &= \frac{\|\alpha_2\tilde{x}\|_2}{f_m - \sum_{j\in I^-(\alpha_2\tilde{x})} (\langle A_j, \alpha_2\tilde{x}\rangle - c_j)} \\ &\leq \frac{\|\alpha_2\tilde{x}\|_2}{f_m - \sum_{j\in I^-(\alpha_1\tilde{x})} (\langle A_j, \alpha_2\tilde{x}\rangle - c_j)} \\ &= \frac{\|\tilde{x}\|_2}{\frac{1}{\alpha_2}\left(f_m + \sum_{j\in I^-(\alpha_1\tilde{x})} c_j\right) - \sum_{j\in I^-(\alpha_1\tilde{x})}\langle A_j, \tilde{x}\rangle}\end{aligned}$$

while

$$\begin{aligned}\Xi(\alpha_1) &= \frac{\|\alpha_1\tilde{x}\|_2}{f_m - \sum_{j\in I^-(\alpha_1\tilde{x})} (\langle A_j, \alpha_1\tilde{x}\rangle - c_j)} \\ &= \frac{\|\tilde{x}\|_2}{\frac{1}{\alpha_1}\left(f_m + \sum_{j\in I^-(\alpha_1\tilde{x})} c_j\right) - \sum_{j\in I^-(\alpha_1\tilde{x})}\langle A_j, \tilde{x}\rangle}.\end{aligned}$$

Since $(1/\alpha_1)(f_m + \sum_{j\in I^-(\alpha_1\tilde{x})} c_j) \leq (1/\alpha_2)(f_m + \sum_{j\in I^-(\alpha_1\tilde{x})} c_j) \leq 0$, and the denominator of $\Xi(\alpha_1)$ is positive, it is easily seen that $\Xi(\alpha_1) \geq \Xi(\alpha_2) > 0$. Furthermore, as $\alpha_1 \to \alpha_m(\tilde{x})^+$, then $\alpha_1\tilde{x} \to \partial\mathcal{S}$ and $\mathcal{B}(\alpha_1\tilde{x}) \to 0$, whenever $\tilde{x} \in C_R$. Therefore, for any $\alpha_m(\tilde{x}) < \alpha \leq 1$ we obtain

$$\Xi(\alpha) \leq \lim_{\alpha_1\to\alpha_m^+(\tilde{x})} \Xi(\alpha_1) \leq \frac{\tilde{R}}{f_m}.$$

Hence

$$\|a\|_2 \sup_{x\in[\overline{b(0,R)\setminus\mathcal{S}}]\setminus\partial\mathcal{S}} \frac{\|x\|_2}{f_m + \mathcal{B}(x)} = \|a\|_2 M_\mathcal{B}(R) \leq \frac{\tilde{R}\|a\|_2}{f_m}.$$ ■

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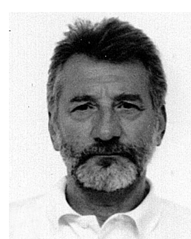

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