Bollettino U. M. I.
(5) **17**-A (1980), 523-530

A Result on the Existence of Infinitely Many Solutions of a Nonlinear Elliptic Boundary Value Problem at Resonance.

I. Massabò - P. Nistri (Cosenza)
M. P. Pera (Firenze)

Sunto. – *Si dimostra che un problema ellittico non lineare in risonanza rappresentato da una equazione del tipo* $Lu = h(u)$, *dove* $L: E \to F$ (E, F *spazi di Banach reali) è un operatore di Fredholm di indice* $i_L \geqslant 0$ *ed* $h: E \to F$ *è un operatore continuo non lineare uniformemente limitato, ha, se* $i_L > 0$, *una componente connessa non limitata di soluzioni che interseca la varietà affine* $\xi + (W \oplus E_1)$, *ove* $E_0 \oplus E_1 = \operatorname{Ker} L$, $\dim E_0 = i_L$, $\xi \in E_0$ *e* W *è un qualsiasi complemento topologico di* Ker L *in* E.

Introduction.

In this paper we deal with an equation of the form

$$(*) \qquad Lu = h(u)$$

where $L: E \to F$ (E, F real Banach spaces) is a Fredholm operator with nonnegative index i_L and $h: E \to F$ is an uniformly bounded continuous nonlinearity. In our case, equation (*) corresponds to an elliptic boundary value problem at resonance.

Our purpose is to show that (*) has an unbounded connected set of solutions if $i_L > 0$. More precisely we prove that, under suitable assumptions, there exists a subspace E_1 of the kernel of L(Ker L) and an i_L-dimensional complement E_0 of E_1 in Ker L such that, for any $\xi \in E_0$ and any topological complement W of Ker L in E, equation (*) has an unbounded connected component of solutions which meets the affine manifold $\xi + (W \oplus E_1)$.

Our results are in the same direction of those of Amann-Ambrosetti-Mancini in [2] and the proofs are based on the abstract Theorem 1.1 which is related with the theory of 0-regularizable maps introduced by Furi-Pera in [6, 7] (see also [5]).

1. – Definitions and preliminary results.

Let E, F be real Banach spaces. A continuous map $f\colon E \to F$ is said to be *compact* it it maps bounded subsets of E into relatively compact subsets of F; f is said to be *proper* if $f^{-1}(K)$ is compact for every compact set $K \subset F$. Observe that if $f\colon E \to F$ is proper on bounded closed sets of E and $h\colon E \to F$ is compact, then $f + h$ is proper on bounded closed sets. Moreover, let $f, g\colon E \to F$ be continuous; we say that a continuous map $H\colon E \times [0, 1] \to F$ is a *0-homotopy* joining f with g if

i) $H(\cdot, 0) = f$, $H(\cdot, 1) = g$;

ii) $H(\cdot, 0) - H(\cdot, \cdot)\colon E \times [0, 1] \to F$ is compact;

iii) the set $\{u \in E\colon H(u, \lambda) = 0 \text{ for some } \lambda \in [0, 1]\}$ is bounded.

Furthermore, we recall (see [6]) that a continuous map $f\colon E \to F$ is said to be *0-regular* if $a)$ $f^{-1}(0)$ is bounded and the equation $f(u) = h(u)$ has a solution in E for any compact map $h\colon E \to F$ with bounded support; $b)$ f is proper on bounded closed subsets of E.

Let $F\colon E \to F$ be continuous and assume that there exists a continuous map $\varphi\colon E \to G$, where G is a real Banach space, such that φ maps bounded sets into bounded sets and the map $(f, \varphi)\colon E \to F \times G$ defined by $(f, \varphi)(u) = (f(u), \varphi(u))$ is 0-regular. Then we say (cfr. [6]) that f is *0-regularizable* by φ (or, equivalently, that φ 0-regularizes f).

Finally, recall that a linear operator $L\colon E \to F$ is called a *Fredholm* operator, if $\dim \operatorname{Ker} L < +\infty$, $\operatorname{Im} L$ is closed and $\operatorname{codim} \operatorname{Im} L < +\infty$. The *index* of L is the integer $i_L = \dim \operatorname{Ker} L - \operatorname{codim} \operatorname{Im} L$. It is not hard to see that a bounded Fredholm operator is proper on bounded closed sets. Moreover it is known that if $L\colon E \to F$ is a bounded Fredholm operator with k-codimensional image, there exists a bounded operator $Q\colon F \to \boldsymbol{R}^k$ (unique up to a linear isomorphism of $\boldsymbol{R}^k$ into itself) such that $\operatorname{Ker} Q = \operatorname{Im} L$. In the sequel we always denote by Q such an operator.

In the next section we give an application of the abstract Theorem 1.1 below, to the solvability of a boundary value problem at resonance. The proof of Th. 1.1 is a direct consequence of [6, Th. 2.1] and of the results contained in [7]. We would like to remark that the proofs of such results are essentially based on some homotopy arguments avoiding topological degree.

THEOREM 1.1. – *Let $L\colon E \to F$ be a bounded Fredholm operator* *of index* $i_L \geqslant 0$ *and* $h\colon E \to F$ *be compact. Let* $S^+ = \{u \in E\colon Lu = \lambda h(u)$ *for some* $\lambda \in]0, 1]\}$ *and let* $P\colon E \to E$ *be a bounded linear projection onto* $\operatorname{Ker} L$. *Assume that*

i) *there exists a* k-*dimensional subspace* E_1 *of* $\operatorname{Ker} L$ $(k = \operatorname{codim} \operatorname{Im} L)$ *such that* $Qh|E_1\colon E_1 \to \boldsymbol{R}^k$ *is* 0-*homotopic to a homeomorphism* $\sigma\colon E_1 \to \boldsymbol{R}^k$;

ii) *the set* $S^+ \cap P^{-1}(E_1)$ *is bounded;*

iii) *if* $P_1\colon \operatorname{Ker} L \to \operatorname{Ker} L$ *is any linear projection such that* $\operatorname{Ker} P_1 = E_1$ *and* $\xi \in \operatorname{Im} P_1$, *the set* $\{u \in E\colon Lu = h(u),\ Pu \in E_1 + \mu\xi$ *for some* $\mu \in [0, 1]\}$ *is bounded.*

Then $L - h$ *is* 0-*regularizable by the affine map* $\varphi\colon E \to \operatorname{Im} P_1$ *defined by* $\varphi(u) = P_1 Pu - \xi$. *Hence, if* $i_L > 0$, *the equation* $Lu = h(u)$ *has an unbounded connected component* Σ_ξ *of solutions such that* $\Sigma_\xi \cap (\xi + P^{-1}(E_1)) \neq \emptyset$. (*In particular, if* $i_L = 0$, *then* $L - h$ *is* 0-*regularizable by* $\varphi = 0$, i.e. $L - h$ *is* 0-*regular*).

2. – Application to elliptic equations.

Let Ω be a bounded domain in $\boldsymbol{R}^n$ with boundary $\partial\Omega$. For semplicity we assume that $\partial\Omega$ is an $(n-1)$-dimensional smooth manifold such that Ω lies locally on one side of $\partial\Omega$. We denote by $\mathfrak{L}$ a linear uniformly elliptic partial differential operator of order $2m$, $m \geqslant 1$, with smooth real valued coefficients and $\mathcal{B}$ a normal family of m smooth boundary operators of order $\leqslant 2m-1$ which covers $\mathfrak{L}$ on $\partial\Omega$ (cfr. [8]). Finally, let $f\colon \bar{\Omega} \times \boldsymbol{R} \to \boldsymbol{R}$ be a function satisfying the following hypotheses:

$$(f_1)\qquad \left|\begin{array}{ll} f(\cdot, t)\colon \bar{\Omega} \to \boldsymbol{R} & \text{is measurable for every } t \in \boldsymbol{R}\,; \\ f(x, \cdot)\colon \boldsymbol{R} \to \boldsymbol{R} & \text{is continuous for almost any } x \in \Omega\,; \\ \multicolumn{2}{l}{\text{there exists a constant } c \geqslant 0 \text{ such that } |f(t, x)| \leqslant c} \\ \multicolumn{2}{l}{\text{for all } (x, t) \in \Omega \times \boldsymbol{R}\,.} \end{array}\right.$$

We are looking for solutions of the nonlinear elliptic boundary value problem

$$(2.1)\qquad \begin{cases} \mathfrak{L}u = f(x, u) & \text{in } \Omega \\ \mathcal{B}u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

By a solution of (2.1) we mean a function $u \in W^{2m,p}(\Omega)$ for some $p > 1$, which satisfies the differential equation pointwise a.e. and the boundary conditions in the sense of the traces (cfr. [1]).

To restate problem (2.1) in the abstract form of the previous section, let $p > 1$ be arbitrary and let $F = L^p(\Omega)$. Then it is known (cfr. [1] and [8] for the case $p = 2$) that the pair $(\mathcal{L}, \mathcal{B})$ induces a densely defined closed Fredholm operator L with domain $E = D(L) = \{u \in W^{2m,p}(\Omega): \mathcal{B}u = 0\}$. E and F endowed with the usual norms of $W^{2m,p}(\Omega)$ and $L^p(\Omega)$ are Banach spaces; $\|\cdot\|_E$ and $\|\cdot\|_F$ will denote the norms in E and F respectively. Furthermore, $L: E \to F$ is a bounded Fredholm operator with a right inverse $L^+: \operatorname{Im} L \to E$ (i.e. LL^+ is the inclusion $\operatorname{Im} L \hookrightarrow F$) which is bounded by the Agmon-Douglis-Nirenberg's estimates [1]. So, there exists a k-dimensional subspace F_0 of F, where $k = \operatorname{codim} \operatorname{Im} L$, such that $F = \operatorname{Im} L \oplus F_0$. Observe that, since L is a Fredholm operator in $L^p(\Omega)$, $\operatorname{Im} L$ is equal to the anhilator of the kernel of the dual operator L' in $L^{p'}(\Omega)$ with $1/p + 1/p' = 1$; moreover, our regularity assumptions imply that $\operatorname{Ker} L'$ is contained in $C^\infty(\Omega)$. Hence, we can identify F_0 with the subspace $\operatorname{Ker} L'$ of $L^{p'}(\Omega)$. On the other hand, by (f_1), we get that the Nemytskii operator h defined by $h(u)(x) = f(x, u(x))$ for almost all $x \in \Omega$, maps continuously F into itself and is uniformly bounded (i.e. there exists $c > 0$ such that $\|h(u)\|_F \leqslant c$ for all $u \in F$). Since E is compactly imbedded in F, h can and will be regarded as a compact map from E into F.

Then problem (2.1) takes the more precise form

$$Lu = h(u) \qquad u \in E . \tag{2.2}$$

Furthermore, assume that

(f_2) the limits $f_\pm(x) = \lim_{u \to \pm\infty} f(x, u)$ exist for almost every $x \in \Omega$;

(uc) $(\mathcal{L}, \mathcal{B})$ has the « unique continuation property », i.e. if $v \in \operatorname{Ker} L$ vanishes on a set of positive measure, then $v = 0$;

(f_3) there exists a linear operator $T: F_0 \to \operatorname{Ker} L$ such that $M_T(z) > 0$ for all $z \in F_0 \setminus \{0\}$, where, following [9],

$$M_T(z) = \int_{Tz>0} f_+ z + \int_{Tz<0} f_- z .$$

Observe that condition (f_3) implies that T is injective; hence, $i_L \geqslant 0$.

After these preparations, we are now in a position to state our main result.

THEOREM 2.1. - *Let* $L: E \to F$ *and* $h: E \to F$ *be as specified above. Assume that* (f_1), (f_2), (f_3), (uc) *are satisfied. Then* $L - h$ *is*

0-regularizable. Moreover, if $i_L > 0$*, there exists an* i_L*-dimensional subspace* E_0 *of* Ker L *such that for every* $\xi \in E_0$ *and for every topological complement* W *of* Ker L *in* E*, the equation* $Lu = h(u)$ *has an unbounded connected component of solutions* Σ_ξ *which intersects the affine manifold* $\xi + (W \oplus TF_0)$*. If* $i_L = 0$*, then* $L - h$ *is 0-regular.*

PROOF. – Let $\psi_1, \psi_2, \ldots, \psi_k$ be a basis for F_0 and let $Q: F \to \boldsymbol{R}^k$ be the bounded linear operator defined by

$$Qz = \left(\int_\Omega z\psi_1, \ldots \int_\Omega z\psi_k\right).$$

Clearly, from the definition of F_0, Ker Q = Im L. Moreover, Ker L can be regarded as the direct sum of TF_0 with an i_L-dimensional subspace E_0 of E. We want to show that $Qh|TF_0$ is 0-homotopic to a homeomorphism. Let us prove first that there exists $r_0 > 0$ such that for any $r \geq r_0$, the following inequality

$$(Qh(rv), a) > 0 \tag{2.3}$$

holds for all $v \in TF_0$ such that $v = a^1 T\psi_1 + a^2 T\psi_2 + \ldots + a^k T\psi_k$, $a = (a^1, a^2, \ldots, a^k)$, $|a| = 1$. ($(\cdot,\cdot)$ denotes the usual inner-product in $\boldsymbol{R}^k$).

Assume the contrary; then there exist a sequence $(r_n)_{n\in\boldsymbol{N}}$, $r_n \to +\infty$ and a sequence $(a_n)_{n\in\boldsymbol{N}}$ in $D = \{a \in \boldsymbol{R}^k: |a| = 1\}$ such that

$$(Qh(r_n v_n), a_n) \leq 0,$$

that is

$$\int_\Omega h(r_n v_n) z_n \leq 0$$

where

$$z_n = \sum_{i=1}^k a_n^i \psi_i.$$

Hence,

$$\limsup_{n\to\infty} \int_\Omega h(r_n T z_n) z_n \leq 0. \tag{2.4}$$

By the same argument used in the proof of ([2], Theorem 2.5), we can prove that $M_T(\bar{z}) \leq 0$, where $\bar{z} \in F_0$ is a cluster point of the bounded sequence $(z_n)_{n\in\boldsymbol{N}}$, contradicting the assumption (f_3). Therefore (2.3) is true. Consequently, let $H: TF_0 \times [0, 1] \to \boldsymbol{R}^k$ be the homotopy defined by $H(v, \lambda) = \lambda\sigma(v) + (1 - \lambda)Qh(v)$ where $\sigma: TF_0 \to \boldsymbol{R}^k$ is the isomorphism given by

$$\sigma(v) = (a^1, a^2, \ldots, a^k), \qquad v = a^1 T\psi_1 + a^2 T\psi_2 + \ldots + a^k T\psi_k.$$

Then by (2.3) there exists $c > 0$ such that for every $v \in TF_0$ with $\|v\|_E > c$, we have

$$(H(v, \lambda), \sigma(v)) = \lambda|\sigma(v)|^2 + (1-\lambda)(Qh(v), \sigma(v)) > 0;$$

therefore, $Qh|TF_0$ is 0-homotopic to the homeomorphism $\sigma\colon TF_0 \to \mathbf{R}^k$. Let us now prove the boundedness of the set $S^+ \cap P^{-1}(TF_0)$ where

$$S^+ = \{u \in E\colon Lu = \lambda h(u) \text{ for some } \lambda \in]0, 1]\}$$

and $P\colon E \to E$ is any bounded linear projection onto Ker L. Denote by

$$W_{S^+} = \{w \in \operatorname{Ker} P\colon u = v + w \in S^+ \text{ for some } v \in \operatorname{Ker} L\}$$

and observe that the Agmon-Douglis-Nirenberg's estimates (see [1]) imply the existence of a costant $c_1 > 0$ such that $\|w\|_E \leq c_1 \|Lw\|_F = c_1 \lambda \|h(u)\|_F \leq c_2$ for all $w \in W_{S^+}$.

Furthermore, the set $\{v \in TF_0\colon u = v + w \in S^+ \text{ for some } w \in W_{S^+}\}$ is bounded. In fact, assume that there exist a sequence $(r_n v_n)_{n \in N}$, $r_n \to +\infty$, $v_n \in TF_0$, $v_n = \sum_{i=1}^{n} a_n^i T\psi_i$ with $a_n \in D$ for all $n \in N$ and a sequence $(w_n)_{n \in N}$, $w_n \in W_{S^+}$ such that $Qh(r_n v_n + w_n) = 0$. Hence,

$$0 = \limsup_{n \to \infty} \left(a_n, Qh\left(r_n\left(v_n + \frac{w_n}{r_n}\right)\right)\right).$$

Now, as above, we may assume that $(v_n)_{n \in N}$ converges a.e. in Ω to some $\bar{v} = T\bar{z} \in TF_0 \setminus \{0\}$. Thus, since W_{S^+} is bounded, we get $M_T(\bar{z}) = 0$; contradicting the assumption (f_3). Therefore $S^+ \cap P^{-1}(TF_0)$ is bounded, so that (ii) of Theorem 1.1 is verified.

Finally, to prove (iii) of Theorem 1.1 let $P_1\colon \operatorname{Ker} L \to \operatorname{Ker} L$ be a linear projection such that $\operatorname{Ker} P_1 = TF_0$ and denote by E_0 the subspace Im P_1. As above, we can show that, for any $\xi \in E_0$, the set $\{u \in E\colon Lu = h(u), Pu \in TF_0 + \mu\xi \text{ for some } \mu \in [0, 1]\}$ is bounded. Therefore all the hypotheses of Theorem 1.1 are verified so that the affine map $P_1 P - \xi$ 0-regularizes $L - h$. Thus, the equation $Lu = h(u)$ has an unbounded connected component Σ_ξ of solutions such that

$$\Sigma_\xi \cap (\xi + P^{-1}(TF_0)) \neq \emptyset$$

Now, if W is any topological complement of Ker L, let $P\colon E \to E$ be the bounded linear projection onto Ker L such that Ker $P = W$;

so, for any $\xi \in E_0$ there exists a component Σ_ξ intersecting the affine manifold $\xi + (W \oplus TF_0)$. Lastly, if $i_L = 0$, Theorem 1.1 implies that $L - h$ is 0-regular. Q.E.D.

REMARK 2.1. – Let L and h be as in Theorem 2.1. If L is onto, Theorem 2.1 remains true under the only assumption (f_1). In fact, in this case it is easy to see that all the hypotheses of Theorem 1.1 are satisfied. Thus, for any $\xi \in \operatorname{Ker} L$ and any bounded linear projection $P\colon E \to E$ onto $\operatorname{Ker} L$, the map $P - \xi$ 0-regularizes $L - h$. Consequently, if $i_L > 0$ and $W = \operatorname{Ker} P$, the equation $Lu = h(u)$ has an unbounded connected component Σ_ξ of solutions intersecting $\xi + W$; in particular, if $i_L = 0$, $L - h$ is 0-regular.

REMARK 2.2. – Notice that, in Theorem 2.1, we have proved a more precise result. Namely, for any topological complement W of $\operatorname{Ker} L$ in E and any $\xi \in \operatorname{Ker} L$, the equation $Lu = h(u)$ has an unbounded connected component which meets $\xi + (W \oplus TF_0)$.

REMARK 2.3. – Observe that, by our regularity hypotheses, $\operatorname{Ker} L \subset C^\infty(\Omega)$. It is easily seen that the set

$$\left\{u \in E\colon \int_\Omega uv = 0 \text{ for all } v \in \operatorname{Ker} L\right\}$$

is a topological complement of $\operatorname{Ker} L$ in E. So, we can identify W in Theorem 2.1 with this subspace. Moreover assume $p = 2$ and denote by E_0 the L^2-orthogonal complement of TF_0 in $\operatorname{Ker} L$. Then, by Theorem 2.1, for every $\xi \in E_0$, the equation $Lu = h(u)$ has a solution of the form $u = \xi + v$ where $\int_\Omega \xi v = 0$ and $v = Tz + w$ with $z \in TF_0$ and $w \in W$.

REMARK 2.4. – We would like to point out that, with the above technique, giving an appropriate form to (f_2) and (f_3) (cfr. [3,4]) one could handle nonlinearities satisfying a growth restriction of the type

$$|f(x, Du(x))| \leq a(x) + b|Du(x)|^s$$

where

$$Du(x) = [D^\alpha u(x), x \in \Omega, 0 \leq |\alpha| \leq 2m - 1]$$

$$a \in L^2(\Omega)\,, \quad b > 0 \text{ and } 0 \leq s < 1\,.$$

REFERENCES

[1] S. Agmon - A. Douglis - L. Nirenberg, *Estimates near the boundary of elliptic partial differential equations satisfying general boundary conditions*, Comm. Pure Appl. Math., **12** (1959), 623-727.

[2] H. Amann - A. Ambrosetti - G. Mancini, *Elliptic equations with non invertible Fredholm part and bounded nonlinearities*, Math. Z., **158** (1978), 179-194.

[3] L. Cesari, *Functional analysis, nonlinear differential equations and the alternative method*, Nonlinear Functional Analysis and Differential Equations, edited by L. Cesari, R. Kannan and J. D. Schuur, Marcel Dekker, Inc., New York-Basel, 1-197.

[4] D. G. De Figueiredo, *The Dirichlet problem for nonlinear elliptic equations. A Hilbert space approach*, Partial Differential Equations and Related Topics (Dold and Eckmann ed.), Springer-Verlag, Lecture Notes Math., **446** (1975), 144-165.

[5] M. Furi - M. Martelli - A. Vignoli, *On the solvability of nonlinear equations in normed spaces*, to appear in Ann. Mat. Pura Appl.

[6] M. Furi - M. P. Pera, *On the existence of an unbounded connected set of solutions for nonlinear equations in Banach spaces*, to appear in Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.

[7] M. Furi - M. P. Pera, *An elementary approach to boundary value problems at resonance*, to appear in Nonlinear Analysis T.M.A.

[8] J. Lions - E. Magenes, *Nonhomogeneous boundary value problems and applications*, vol. I, Springer-Verlag, Berlin, 1977.

[9] M. Schechter, *A nonlinear elliptic boundary value problem*, Ann. Scuola Norm. Sup. Pisa, **27** (1973), 707-716.

Pervenuta alla Segreteria dell' U. M. I.
il 10 dicembre 1979