Sliding Mode Techniques with Robust Observers

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Abstract

In this paper we address the problem of robust tracking for systems whose state is not available. Since the control strategy used, based on a sliding manifold approach, foresees a state feedback, an observer must be employed. The LTR procedure allows to design a robust observer, but its main drawback is the presence of a fast initial transient, which results in a large overshoot in the control law. To solve this problem we propose a modification of the classic observer design through the insertion of a decaying exponential term whose role is to slow down the initial state acquisition. The estimated state enters a controller designed as to keep the state in a prescribed neighborhood of a given manifold at any time instant. The width of the neighborhood is determined by the value of a "small" positive parameter ϵ. Finally, a simulation example is presented to show the effectiveness of the proposed procedure.

1. Introduction

The tracking problem for linear time invariant perturbed dynamical systems is one of the most discussed in the Control Theory literature. Shortly, the objective is to find a (feedback) control law to steer the plant state (or output) along a prescribed reference trajectory. Two have been the most known approaches to the problem: the VSS strategy and the LQ design. Roughly speaking, the first method (see [1] for an exhaustive presentation of the strategy) defines a sliding surface on which the system state is forced to lay by a discontinuous control action, while the second (see [2]) defines a control law as to minimize the integral square error between reference and actual state over different time intervals.

Recently, for linear time invariant systems and a class of nonlinear systems, the authors have proposed a strategy based on a sliding manifold approach [3, 4, 5], resulting in a linear full state feedback control law. More specifically, the control law turns out to be the solution of a differential equation containing a "small" parameter ϵ. According to the Singular Perturbation Theory, the smaller the parameter is, the closer the state of the system to a suitably defined manifold will be. Moreover the feedback system exhibits robustness properties typical of high–gain systems, also avoiding the peaking phenomenon that afflicts these systems. In fact, the control law does not induce a time scale separation between sets of state variables, but only between the control and the state. Then the fast variable is the control, while the state is "slow". In particular, the control approaches quickly a well–defined "equivalent control" where the corresponding state approaches the manifold uniformly with respect to time.

The methods mentioned above are based on state feedback, then for systems whose state is not available an observer is required. It is well known that the insertion of an observer in the control loop decreases the robustness of the control law, but the formulation of the LTR Theory [6] has overcome this inconvenient. The technique is based on a parameterization of the set of observer gains G with a tuning parameter q so that when q increases the observer becomes more and more robust. In its original formulation, the LTR strategy was devoted to recover the phase and gain margins of the LQ regulator. Subsequently, the structure of the LTR observer has been widely explored (see [7] and references therein for an exhaustive analysis of the topic). The location of the observer poles has also been investigated. In particular, when q tends to infinity a set of controller poles goes to infinity. When the observer output is used in the control scheme as an estimate of the actual state to feed the state feedback controller, the faraway poles of the observer cause a large overshoot in the plant input until the actual state has been adequately estimated by the observer. Moreover, the quicker the estimate is, the larger the overshoot. To solve this problem, one can notice that, although the readiness of the observer is necessary for robustness, it can be relaxed in the first stage, during the error transient. This consideration leads us to a possible solution to the initial overshoot problem, namely to modify the structure of the observer with an exponential decaying term, by using the only available initial information, the system output to slow down the initial estimate acquisition.

2. Observer design

In this section we recall the results on the asymptotic pole location of the full order LTR observer. We consider the linear time invariant perturbed system

$$\begin{aligned} \dot{x} &= Ax + Bu + w(x,t), \quad x(0) = x_0 && (1) \\ y &= Cx && (2) \end{aligned}$$

where $x \in R^n$, $u \in R^m$ and $y \in R^m$ are the state vector, the input vector and the output vector respectively and $w(x,t)$ takes into account error models, disturbances and nonlinearities. Throughout the paper we assume that the system (1)–(2) is controllable, observable, minimum phase and $\text{rank}B = \text{rank}C = m$.

Then we design the identity full order observer

$$\dot{\hat{x}} = (A - GC)\hat{x} + Bu + Gy - Ke^{St}y_0 \qquad (3)$$

where $\hat{x}$ is the state estimate and $y_0 = y(0)$, S is a $m \times m$ stability matrix and K is a $n \times m$ matrix to be selected later. Let us neglect in a first stage the last term and consider the classical LTR observer. According to the LTR technique, the estimator gain matrix G is

$$G = G(q) = P(q)C^T R^{-1}, \quad q > 0 \qquad (4)$$

where R is a positive definite symmetric matrix and the matrix $P = P(q)$ is the unique positive definite solution of the ARE

$$AP + PA^T + Q(q) - PC^T R^{-1} CP = 0. \qquad (5)$$

The matrix $Q(q)$ plays the key role in the whole procedure. Its expression is

$$Q(q) = Q_0 + q^2 BVB^T \qquad (6)$$

where Q_0 is a positive semidefinite symmetric matrix. The following limit is the well known crucial result of the LTR Theory

$$\lim_{q\to\infty} \frac{G(q)}{q} = BW \qquad (7)$$

where $W = V^{1/2}R^{-1/2}$. Observe that (7) implies

$$\begin{aligned} \lim_{q\to\infty} \quad & G(q)[I + C(sI - A)^{-1}G(q)]^{-1} && (8) \\ & = B[C(sI - A)^{-1}B]^{-1}. \end{aligned}$$

Then it is easy to show that

$$\lim_{q\to\infty} [sI - A + G(q)C]^{-1}B = 0. \qquad (9)$$

In particular, this means that by increasing the value of the real scalar parameter $q > 0$ the observer becomes more and more robust with respect to input disturbances $w = w(x,t)$ satisfying the matching condition

$$w(x,t) = Bv(x,t). \qquad (10)$$

For any $q > 0$ we consider now the error dynamics

$$\dot{e} = (A - G(q)C)e \qquad (11)$$

where $e = \hat{x} - x$. For any $q > 0$ the poles of $(sI - A + G(q)C)^{-1}$ are the zeros of the polynomial

$$\gamma_q(s) = \det(sI - A)\det[I + C(sI - A)^{-1}G(q)] \quad (12)$$

where $G(q) = qBW + D(q)$ and $\lim_{q\to\infty} D(q)/q = 0$.

As $q \to \infty$ some of the roots go to infinity, while those that stay finite approaches the zeros of

$$\det(sI - A)\det[C(sI - A)^{-1}BW] \qquad (13)$$

provided that this expression is not identically zero. Since $\det W \neq 0$, these are the $p \leq n - m$ transmission zeros of the plant, that by assumption belong to the left–half plane.

An estimate of the distance of the faraway poles from the origin can be derived by using the argument in [8, Chapter 3.8], resulting in the following approximation of the faraway roots of this polynomial for large values of q

$$s = (-\alpha)^{\frac{1}{n-p}} q^{\frac{m}{n-p}} \qquad (14)$$

Therefore this first approximation indicates a Butterworth configuration of order $n - p$ with the following estimate

$$\alpha q^{\frac{m}{n-p}} \qquad (15)$$

of the distance of the faraway poles from the origin.

We can now summarize the previous considerations in the following Theorem.

Theorem 1. Consider the error dynamics

$$\dot{e} = (A - G(q)C)e, \quad e(0) = e_0 \qquad (16)$$

where $G(q) = qBW + D(q)$ and $D(q)/q \to 0$ as $q \to \infty$ is designed by means of the LTR technique as outlined before. Then, as $q \to \infty$, p of the poles of $(sI - A + G(q)C)^{-1}$ approach the p zeros s_i of $\det(sI - A)\det[C(sI - A)^{-1}B]$. The remaining $n - p$ poles go to infinity. A first approximation of the distance of the faraway poles from the origin is $\alpha q^{\frac{m}{n-p}}$. ■

As a consequence of Theorem 1, up to a (non singular) change of variables, if necessary, we have that

$$e(t,q) - \exp\begin{pmatrix} J_p t & 0 \\ 0 & q^{\frac{m}{n-p}} J_\infty t \end{pmatrix} e_0 = O(1) \qquad (17)$$

where $O(1) \to 0$ as $q \to \infty$ and $e(t,q)$ is the solution of (16). Furthermore $J_p = \text{blockdiag}(s_1, s_2, \dots, s_p)$, with $Re\, s_i < 0$ for any $i = 1, 2, \dots, p$ and J_∞ is the $(n-p) \times (n-p)$ matrix whose eigenvalues are the unit directions in the left–half plane of the $n-p$ poles of (12) which tend to infinity as $q \to \infty$.

Remark 1. If $\text{Ker}BWC \oplus \text{Im}BWC = R^n$ then we can use in (11) the Singular Perturbation Theory to study the slow and the fast dynamics of the error (see [9]).

3. Control design

In this section we discuss the design of a controller to track a given trajectory using only output measurements. To simplify the exposition we will refer to the regulation problem, i.e. the reference trajectory is zero. The extension to the case of non zero trajectories is straightforward and will be sketched in Remark 2 at the end of the section.

Consider now the perturbed system

$$\begin{aligned} \dot{x} &= Ax + Bu + Bv(x,t) && (18) \\ y &= Cx && (19) \end{aligned}$$

We assume that v is a continuous function in $R^n \times [0,\infty)$, even if we can assume less regularity.

Let $q > 0$ and consider the following error dynamics

$$\dot{e} = (A - G(q)C)e - K(q)e^{S(q)t}y_0 - Bv(x,t) \qquad (20)$$

where $S(q)$ is a $m \times m$ stability matrix for any $q > 0$ and $K(q)$ is a $n \times m$ matrix; both of them will be selected later. We point out that the term $K(q)e^{S(q)t}y_0$ will enable us to slow down the initial state estimate acquisition.

Now we define a function $s : R^n \to R^m$, $m \le n$, as follows

$$s(z) = -Hz \qquad (21)$$

where H is a $m \times n$ matrix such that HB has all of its eigenvalues in the right half plane and will be chosen later. Consider now

$$s(\hat{x}) = s(x+e) = -H(x+e). \qquad (22)$$

For any $q > 0$, let us define the function $h_q : R^n \times R^m \times [0,\infty) \to R^m$ as follows

$$h_q(x,e,t) = \frac{\partial s}{\partial e}[(A - G(q)C)e - K(q)e^{S(q)t}y_0$$
$$-Bv(x,t)] + \frac{\partial s}{\partial x}[Ax + Bu + Bv(x,t)]. \qquad (23)$$

For any $q > 0$ and $\epsilon > 0$ we form the following system of differential equations

$$\begin{aligned} \dot{x} &= Ax + Bu + Bv(x,t),\ x(0) = x_0 && (24) \\ \epsilon\dot{u} &= g_q(x,u,t),\ u(0) = u_0 && (25) \end{aligned}$$

where

$$\begin{aligned} g_q(x,u,t) &= H[-(A - G(q)C)e(t,q) \\ &\quad + K(q)e^{S(q)t}y_0 - Ax - Bu] && (26) \end{aligned}$$

and

$$\begin{aligned} e(t,q) &= e^{(A-G(q)C)t}e_0 \\ &\quad - \int_0^t e^{(A-G(q)C)(t-\tau)} K e^{S\tau} y_0 d\tau \\ &\quad - \int_0^t e^{(A-G(q)C)(t-\tau)} Bv(x,\tau)d\tau && (27) \end{aligned}$$

We now select the parameters of the exponential term. We let $K(q) = G(q)$ and $S(q) = \rho(q)S_0$, where S_0 is a constant $m \times m$ stability matrix and the continuous function $\rho : R^+ \to R^+$ is such that $\lim_{q\to\infty} \rho(q)/q = k > 0$. A possible selection of ρ is

$$\rho(q) = \begin{cases} 1 & \text{if } 0 < q < q_0 \\ \frac{q}{q_0} & \text{if } q \ge q_0 \end{cases} \qquad (28)$$

for some $q_0 > 0$.

Note that with this choice of the exponential term we have $g_q(x(0), 0, 0) = 0$ for any $q > 0$, where we have assumed $\hat{x}(0) = 0$ and $u(0) = 0$. This implies that $\epsilon\dot{u}(0) = 0$ for any $\epsilon > 0$, while in the classic LTR approach the initial slope of the control is $\dot{u}(0) = -HG(q)Cx_0/\epsilon$.

Now denote by $(x(t,\epsilon,q), u(t,\epsilon,q))$ the solution of (24)–(25). For any $\epsilon > 0$, by the continuous dependance of this solution on the parameter q, we have that

$$\lim_{q\to\infty} (x(t,\epsilon,q), u(t,\epsilon,q)) = (x(t,\epsilon), u(t,\epsilon)) \qquad (29)$$

uniformly in the time interval $[0,\infty)$, where $(x(t,\epsilon), u(t,\epsilon))$ is the solution of the limit problem

$$\begin{aligned} \dot{x} &= Ax + Bu + Bv(x,t),\ x(0) = x_0 && (30) \\ \epsilon\dot{u} &= g_\infty(x,u,t),\ u(0) = u_0. && (31) \end{aligned}$$

Here

$$g_\infty(x,u,t) = -H\Bigg[\begin{pmatrix} J_p e^{J_p t} & 0 \\ 0 & 0 \end{pmatrix} e_0$$
$$+ \frac{1}{k} BWS_0^{-1} y_0 + Ax + Bu + Bv(x,t)\Bigg] \qquad (32)$$

In fact the following result holds.

Lemma 1. Let $f : [0, \infty) \to R^m$ be a function for which there exists the Laplace transform $F(s)$, $Re\, s > \sigma > 0$. Then

$$\lim_{q \to \infty} (A - G(q)C) \int_0^t e^{(A - G(q)C)(t-\tau)} B f(\tau) d\tau = -Bf(t) \quad (33)$$

Proof. For any $q > 0$ let

$$A - G(q)C = qN(q) \quad (34)$$

where $N(q) = A/q - BWC + D(q)/q\, C$, with $D(q)/q \to 0$ as $q \to \infty$.

By assumption, $N(q)$ is nonsingular for any $q > 0$, and

$$\lim_{q \to \infty} N(q) = -BWC. \quad (35)$$

Then the left hand side of (33) can be written as follows

$$qN(q) \int_0^t e^{qN(q)(t-\tau)} B f(\tau) d\tau. \quad (36)$$

For any $q > 0$ this function represents the output of the system

$$\begin{aligned} \dot{\xi} &= qN(q)\xi + Bf,\ \xi(0) = 0 & (37) \\ \eta &= qN(q)\xi & (38) \end{aligned}$$

whose transfer function is

$$N(q) \left(\frac{sI}{q} - N(q) \right)^{-1} B. \quad (39)$$

For any $s \in \mathbf{C}$ we have that

$$\lim_{q \to \infty} qN(q)(sI - qN(q))^{-1} B = -BWC(BWC)^{+} B \quad (40)$$

Here the right inverse $(BWC)^{+}$ is defined by $(BWC)^{+} T(s) = Z(s) + \ker C$, with $BWC\, Z(s) = T(s)$. Since $\mathrm{rank} B = \mathrm{rank} C = \mathrm{rank} BWC = m$ and $T(s) = BF(s) \neq 0$ for any $s \in \mathbf{C}$ for which $F(s) \neq 0$, we have the assertion. ∎

Replacing the last term in the left hand side of (40) with $G(q)/q(sI/q + \rho(q)/qS_0)^{-1} y_0$ the same argument shows that

$$\lim_{q \to \infty} (A - G(q)C) \int_0^t e^{(A - G(q)C)(t-\tau)} G(q) e^{\rho(q) S_0 \tau} d\tau = -\frac{1}{k} BW S_0^{-1} y_0. \quad (41)$$

Let us return now to system (30)–(31). Since $\mathrm{rank} B = m$ in (30), changing variable if necessary, we may assume without loss of generality that

$$B = \begin{pmatrix} 0 \\ B_2 \end{pmatrix} \quad (42)$$

where B_2 is a $m \times m$ nonsingular matrix. Moreover the transformed system turns out to be still controllable.

Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad (43)$$

and $H = (H_1 \ \ H_2)$, and denote by $Re\lambda_{\min}(M)$ ($Re\lambda_{\max}(M)$) the minimum (the maximum) of the real part of all the eigenvalues of the matrix M.

Following the steps in [5], we can prove the following result

Theorem 2. Let $\delta, \beta, \gamma > 0$. Assume that

$$\begin{aligned} Re\lambda_{\min}(H_2 B_2) &\geq \beta & (44) \\ Re\lambda_{\max}(A_{12} H_2^{-1} H_1 - A_{11}) &\geq \beta + \gamma. & (45) \end{aligned}$$

Then there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$ the solution $(x(t,\epsilon), u(t,\epsilon))$ of (30)–(31) satisfying $(x(0,\epsilon), u(0,\epsilon)) = (x_0, u_0)$ is such that

$$\begin{aligned} |x(t,\epsilon)| &\leq \delta + a_0 e^{-\beta t} & (46) \\ u(t,\epsilon) &= -\frac{1}{\epsilon} H \dot{x}(t,\epsilon) + u_0 & (47) \end{aligned}$$

where $t \in [0, \infty)$ and a_0 is a positive constant depending only on the data.

As in [5] the proof consists in showing that, under assumptions (44)–(45), system (30)–(31) fulfills all of the assumptions of the Singular Perturbation Theory on an infinite time interval. In particular, for $\epsilon = 0$ it turns out that, substituting the equivalent control

$$\begin{aligned} u_{eq}(t,x) &= -(H_2 B_2)^{-1} H \left[\begin{pmatrix} J_p e^{J_p t} & 0 \\ 0 & 0 \end{pmatrix} e_0 \right. \\ &\quad \left. + Ax + Bv(x,t) \right] & (48) \end{aligned}$$

into (30) we obtain the exponential stability of its zero solution. Denote by $(\tilde{x}(t), \tilde{u}(t))$, where $\tilde{u}(t) = u_{eq}(t, \tilde{x})$, the solution of (30)–(31) corresponding to $\epsilon = 0$ and satisfying $(\tilde{x}(0), \tilde{u}(0)) = (x_0, u_0)$.

We can summarize all the previous considerations as follows.

Corollary 1. Under the assumptions of Theorem 1 and Theorem 2 we have

$$\lim_{\epsilon \to 0} \lim_{q \to \infty} (x(t,q,\epsilon), u(t,q,\epsilon)) = (\tilde{x}(t), \tilde{u}(t)) \quad (49)$$

uniformly in $[0, \infty)$.

Remark 2. By using the same approach, one can regulate x to a prescribed neighbourhood of any differentiable and bounded trajectory $x_{ref}(t)$, $t \in [0, \infty)$ by letting $z = \dot{x} - x_{ref}$ in (21).

4. Numerical example

Consider the system

$$
\begin{aligned}
\dot{x} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v & (50) \\
y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x & (51)
\end{aligned}
$$

with $x(0) = (1\ 1)^T$. The disturbance is $v(t) = \sin t$. The parameters for the LTR design are

$$q = 100,\ R = 1,\ W = 1 \qquad (52)$$

and for the exponential term $K(q) = G(q)$, $S_0 = -1$ and $q_0 = 10^3$. The controller has been designed according to the procedure illustrated in the previous section. The complete control scheme is shown in Fig. 1, where $\epsilon = 0.01$ and $H = (1\ 1)$.

In Fig. 2 it is shown the input signal u: the difference between classic LTR observer design (dashed line) and modified design by adding the exponential term is apparent in the observer initial transient.

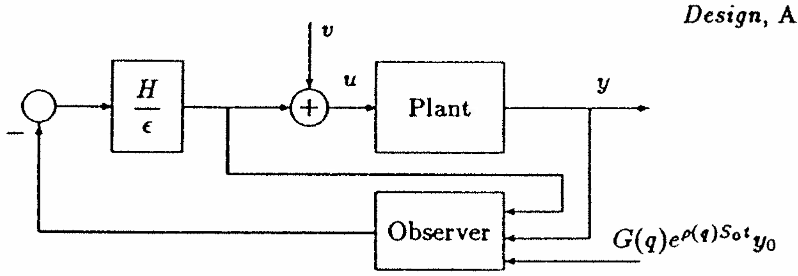

Figure 1: Feedback control configuration

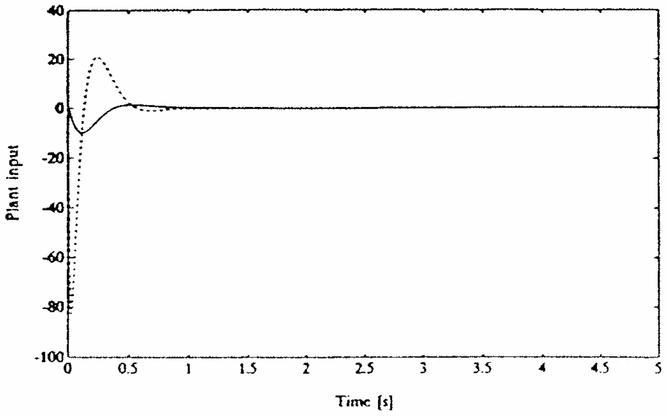

Figure 2: Plant input

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