

PERIODIC SOLUTIONS FOR SINGULARLY
PERTURBED SYSTEMS IN INFINITE
DIMENSIONAL SPACES WITH HYSTERESIS:
AN AVERAGING APPROACH

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Abstract: We consider a singularly perturbed system of two semilinear parabolic equations with nonlinear high frequency inputs which contain the output of a Krasnosel'skii-Pokrovskii hysteresis operator. For sufficiently small values of the perturbation parameter $\varepsilon > 0$, we show the existence of periodic solutions, and we describe their behaviour as $\varepsilon \rightarrow 0$. The employed approach is based on an averaging method together with topological degree arguments.

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1. Introduction

The paper deals with the problem of the existence and dependence on a small parameter $\varepsilon > 0$ of periodic solutions of a singularly perturbed system of semi-linear parabolic equations with high frequency nonlinear inputs containing the output $w(\cdot)$ of an hysteresis operator Γ of Krasnosel'skii-Pokrovskii type acting on the slow variable x , cf. Krasnosel'skii et al [11] and Macki et al [19]. The system has the form

$$\begin{cases} x'(\tau) + A_1 x(\tau) = f_1(\tau/\varepsilon, x(\tau), y(\tau), w(\tau)) , \\ \varepsilon y'(\tau) + A_2 y(\tau) = f_2(\tau/\varepsilon, x(\tau), y(\tau), w(\tau)) , \\ w(\tau) = \Gamma[0, w_0] P x(\tau) , \end{cases} \quad (S)$$

where A_i , $i = 1, 2$, are infinitesimal generators of analytic semigroups acting in the separable Banach spaces E_i , $i = 1, 2$, and $\varepsilon > 0$ is the singular perturbation parameter which also models the high frequency of the inputs f_i , $i = 1, 2$. The operators A_i^{-1} , $i = 1, 2$, are assumed to be completely continuous. The nonlinear operators

$$f_i : \mathbf{R} \times E_1 \times E_2 \times E_3 \rightarrow E_i , \quad i = 1, 2$$

are T -periodic with respect to the first variable and they are assumed to be subordinate to the fractional powers A_i^α , $i = 1, 2$, in a sense that will be specified later in F_1 - F_2). Here $P : E_1 \rightarrow E_0$ is a linear operator, E_0, E_1 are Banach spaces and $\Gamma : \mathbf{R}_+ \times E_3 \times C([0, t], E_0) \rightarrow C([0, t], E_3)$, $t > 0$, is a Krasnosel'skii-Pokrovskii hysteresis operator, which models the interaction of the dynamics represented by the singularly perturbed system and an external device which drives this dynamics through that of the slow variable x . Therefore, the output $w(\cdot)$ of the hysteresis operator Γ can be considered as a feedback control parameter acting on the dynamical system. Our assumptions on Γ ensure, in particular, that $w(\cdot)$ is a continuous function which takes values in a compact set $K \subset E_3$.

The aim of the paper is to provide conditions to guarantee, for $\varepsilon > 0$ sufficiently small, the existence of T -periodic solution to (S) and to study their behaviour as $\varepsilon \rightarrow 0$. Specifically, we first define, for $\varepsilon > 0$, a quasitranslation operator $\tilde{U}_T^\varepsilon : E_1 \times K \times E_2 \rightarrow E_1 \times K \times E_2$ along the mild solutions to (S), then we show that it is well defined, completely continuous and that the fixed points of \tilde{U}_T^ε are the initial conditions of εT -periodic solutions to (S).

Furthermore, when $\varepsilon = 0$, from system (S) we introduce an averaging operator $\Psi : E_1 \times K \rightarrow E_1 \times K$ which is proved to be completely continuous and we assume that it has topological degree different from zero in some relatively open

set N_q of $Q = E_1 \times K$. Finally, by the homotopy invariance and the reduction properties of the topological degree, we prove that, for $\varepsilon > 0$ sufficiently small, the compact vector field $(I - \tilde{U}_T^\varepsilon)$ has topological degree different from zero in $N_q \times B(0, r)$, where $B(0, r) \subset E_2$ is the ball centered at 0 with radius r . Therefore, the quasitranslation operator has fixed points in $N_q \times B(0, r)$. Moreover, the behaviour of such fixed points as $\varepsilon \rightarrow 0$ is established. A possible application in order to illustrate the meaning of the obtained results in the abstract setting is also provided.

A similar approach was employed in Kamenskii et al [8] and [9] to establish the existence and the behaviour of periodic solutions for singularly perturbed systems of semilinear differential inclusions with analytic and C_0 -semigroups respectively. Topological approaches based on the averaging method are widely employed to deal with singularly perturbed inclusions in finite dimensional spaces, Dontchev et al [2], Grammel [6], and some related optimal periodic control problems, Gaitsgory [4], [5] and Grammel [7]. Several papers are also devoted to the study of the existence of periodic and almost periodic solutions for ordinary and partial differential equations with hysteresis of various type, see for instance, Fečkan [3], Krejčí [14], Macki et al [18], Brokate et al [1], Krejčí [16], [15], [17] and Visintin [21].

The most relevant contribution of this paper is that of extending to systems of singularly perturbed equations in infinite dimensional spaces the use of the averaging method in presence both of inputs of high frequency and of an hysteresis operator which introduces in the dynamics a continuous feedback effect through the dynamics of the slow variable x .

The paper is organized as follows. In Section 2 we state the problem, we formulate the assumptions which permit to solve it and we give some definitions and preliminary results. In Section 3 we formulate and we prove our main result: Theorem 1. Finally, in Section 4 we present an application illustrating our results.

2. Formulation of the Problem, Assumptions and Definitions

In this paper we consider the following system

$$\begin{cases} x'(\tau) + A_1 x(\tau) = f_1(\tau/\varepsilon, x(\tau), y(\tau), w(\tau)) , \\ w(\tau) = \Gamma[0, w_0] P x(\tau) , \\ \varepsilon y'(\tau) + A_2 y(\tau) = f_2(\tau/\varepsilon, x(\tau), y(\tau), w(\tau)) , \end{cases} \quad (S)$$

where $\varepsilon > 0$ is a small parameter, $-A_1$ and $-A_2$ are generators of analytic semigroups of linear operators $e^{-A_1 t}$ and $e^{-A_2 t}$, acting in the Banach spaces

E_1 and E_2 . $P : E_1 \rightarrow E_0$ is a linear operator, E_0, E_1 are Banach spaces, $\Gamma : \mathbf{R}_+ \times E_3 \times C([0, t], E_0) \rightarrow C([0, t], E_3)$, $t > 0$, is a Krasnosel'skii-Pokrovskii hysteresis operator, and $(Px(0), w_0) \in E_0 \times E_3$. The operators A_1^{-1} and A_2^{-1} are assumed to be completely continuous, consequently $A_1^{-\beta}$ and $A_2^{-\beta}$, $\beta > 0$, are also completely continuous operators, see Krasnosel'skii et al [13]. The nonlinear operators $f_i, i = 1, 2$ are assumed T-periodic in the first variable.

We assume the following conditions:

A) There exist positive constants C, d such that

$$\|e^{-A_i t}\| \leq C e^{-dt}, \quad t > 0, \quad i = 1, 2.$$

The nonlinear operators $f_i, i = 1, 2$, are subordinate to the fractional powers $A_i^{-\alpha}, i = 1, 2$, in the following sense.

F₁) The operators $(t, x, y, w) \rightarrow f_i(t, A_1^{-\alpha}x, A_2^{-\alpha}y, w)$, acting from $R_+ \times E_1 \times E_2 \times E_3$ to $E_i, i = 1, 2$, are continuous and satisfy Lipschitz condition with respect to the variables (x, y, w) , i.e.

$$\begin{aligned} & \|f_i(t, A_1^{-\alpha}x_1, A_2^{-\alpha}y_1, w_1) - f_i(t, A_1^{-\alpha}x_2, A_2^{-\alpha}y_2, w_2)\| \\ & \leq L(R)(\|x_1 - x_2\| + \|y_1 - y_2\| + \|w_1 - w_2\|), \end{aligned}$$

whenever $i = 1, 2; \|x_j\| \leq R, \|y_j\| \leq R, \|w_j\| \leq R, j = 1, 2, t \in [0, T]$.

F₂) For every $R > 0$ there exists a positive constant ρ_R such that

$$\|f_i(t, A_1^{-\alpha}x, A_2^{-\alpha}y, w)\| \leq \rho_R(1 + \|x\| + \|y\|), \quad (\|w\| \leq R), \quad i = 1, 2.$$

C₁) For every $t_0 \in [0, +\infty)$ the nonlinear hysteresis operator $\Gamma[t_0, \cdot](\cdot)$ acts from $E_3 \times C([t_0, t], E_0)$ to $C([t_0, t], E_3)$, $t \geq t_0$, and if $u(t) \equiv u_*$, then $\Gamma[t_0, w_0]u(t) \equiv w_0$.

C₂) Γ satisfies the semigroup property

$$\Gamma[t_0, w_0]u(t) = \Gamma[t_1, \Gamma[t_0, w_0]u(t_1)]u(t) \quad (t_0 \leq t_1 \leq t).$$

C₃) For every $t_0 \in [0, +\infty)$, the operator Γ satisfies the Lipschitz condition, i.e.

$$\begin{aligned} \max_{s \in [t_0, t]} \|\Gamma[t_0, w_1]u^1(s) - \Gamma[t_0, w_2]u^2(s)\| & \leq k(\|w_1 - w_2\|_{E_3} \\ & + \|u^1 - u^2\|_{C([t_0, t], E_0)}), \quad t \geq t_0. \end{aligned}$$

C_4) There exists a nonempty, convex, compact set $K \subset E_3$ such that for every $w_0 \in K$, $t_0 \in R_+$ we have

$$\Gamma[t_0, w_0]u(t) \in K, \text{ for every } t \geq t_0 \text{ and for every } u \in C([t_0, t], E_0).$$

C_5) For every $t_0 \in [0, +\infty)$ we have $\Gamma[t_0, w_0]u(t_0) = w_0$.

C_6) Let $t_0 \in [0, +\infty)$, $v(t) = x^*(t + t_0 - t_1)$ for $t \geq t_1$. Then

$$\Gamma[t_0, w_0]x^*(t) = \Gamma[t_1, w_0]v(t - t_0 + t_1) \quad (t \geq t_0).$$

C_7) Let $t_0 \in [0, +\infty)$, $\beta > 0$, $v(t) = x^*(\beta t + (1 - \beta)t_0)$ for $t \geq t_0$. Then

$$\Gamma[t_0, w_0]x^*(t) = \Gamma[t_0, w_0]v(\beta t + (1 - \beta)t_0) \quad (t \geq t_0).$$

D_1) For any $x \in E_1$, $w \in K$ and $\lambda \in [0, 1]$ there exists a unique T -periodic solution $\hat{y}_{x,w}^\lambda$ of the integral equation

$$\begin{aligned} y(t) = e^{-A_2 t} (I - e^{-A_2 T})^{-1} \lambda \int_0^T A_2^\alpha e^{-A_2(T-s)} f_2(s, A_1^{-\alpha} x, A_2^{-\alpha} y(s), w) ds \\ + \lambda \int_0^t A_2^\alpha e^{-A_2(t-s)} f_2(s, A_1^{-\alpha} x, A_2^{-\alpha} y(s), w) ds. \end{aligned} \quad (D_1)$$

D_2) For any nonempty, bounded set $\Omega \subset E_1$ the set $\{\hat{y}_{x,w}^\lambda : x \in \Omega, w \in K, \lambda \in [0, 1]\}$ is bounded.

P_1) The linear operator P , acting from E_1 to E_0 , is such that $PA_1^{-\alpha}$ is continuous.

By the change of variable $\tau/\varepsilon = t$ system (S) takes the form

$$\begin{cases} \bar{x}'(t) + \varepsilon A_1 \bar{x}(t) = \varepsilon f_1(t, \bar{x}(t), \bar{y}(t), \bar{w}(t)), \\ \bar{w}(t) = \Gamma[0, w_0] P \bar{x}(t), \\ \bar{y}'(t) + A_2 \bar{y}(t) = f_2(t, \bar{x}(t), \bar{y}(t), \bar{w}(t)). \end{cases} \quad (S')$$

Denoting $(\bar{x}, \bar{w}, \bar{y})$ again by (x, w, y) we obtain

$$\begin{cases} x'(t) + \varepsilon A_1 x(t) = \varepsilon f_1(t, x(t), y(t), w(t)) , \\ w(t) = \Gamma[0, w_0] P x(t) , \\ y'(t) + A_2 y(t) = f_2(t, x(t), y(t), w(t)). \end{cases} \quad (1)$$

Following Krasnosel'skii et al [13], we say that $(x_\varepsilon, w_\varepsilon, y_\varepsilon)$ is a mild solution of (1) on $[0, T]$ with the initial conditions

$$x(0) = x_0, \quad w(0) = w_0, \quad y(0) = y_0, \quad (2)$$

if $x_\varepsilon, w_\varepsilon, y_\varepsilon$ are continuous functions defined on the interval $[0, T]$ with values in E_1, E_3, E_2 respectively, such that $\bar{f}_i(s) = f_i(s, x_\varepsilon(s), y_\varepsilon(s), w_\varepsilon(s)), i = 1, 2$, is bounded on $[0, T]$, and satisfying the equations

$$x_\varepsilon(t) = e^{-\varepsilon A_1 t} x_0 + \int_0^t e^{-\varepsilon A_1(t-s)} \varepsilon f_1(s, x_\varepsilon(s), y_\varepsilon(s), w_\varepsilon(s)) ds, \quad (3)$$

$$w_\varepsilon(t) = \Gamma[0, w_0] P x_\varepsilon(t), \quad (4)$$

$$y_\varepsilon(t) = e^{-A_2 t} y_0 + \int_0^t e^{-A_2(t-s)} f_2(s, x_\varepsilon(s), y_\varepsilon(s), w_\varepsilon(s)) ds, \quad t \in [0, T]. \quad (5)$$

By a T -periodic solution of (1) we mean a continuous T -periodic function $(x_\varepsilon, w_\varepsilon, y_\varepsilon)$ satisfying (3), (4) and (5) on $[0, +\infty)$.

Definition 1. (The Quasitranlation Operator) For every $x_0 \in E_1, w_0 \in K, y_0 \in E_2$ and $\varepsilon > 0$ we assume that there exists an unique solution $(\tilde{x}_\varepsilon, \tilde{w}_\varepsilon, \tilde{y}_\varepsilon)$ on $[0, T]$ of the following system

$$x(t) = e^{-\varepsilon A_1 t} x_0 + \int_0^t A_1^\alpha e^{-\varepsilon A_1(t-s)} \varepsilon f_1(s, A_1^{-\alpha} x(s), A_2^{-\alpha} y(s), w(s)) ds, \quad (6)$$

$$w(t) = \Gamma[0, w_0] P A_1^{-\alpha} x(t), \quad (7)$$

$$y(t) = e^{-A_2 t} y_0 + \int_0^t A_2^\alpha e^{-A_2(t-s)} f_2(s, A_1^{-\alpha} x(s), A_2^{-\alpha} y(s), w(s)) ds. \quad (8)$$

We define the quasitranlation operator $\tilde{U}_T^\varepsilon : E_1 \times K \times E_2 \rightarrow E_1 \times K \times E_2$ as follows

$$\tilde{U}_T^\varepsilon(x_0, w_0, y_0) = (\tilde{x}_\varepsilon(T), \tilde{w}_\varepsilon(T), \tilde{y}_\varepsilon(T)).$$

Remark 1. a. We will show in Proposition 2 that \tilde{U}_T^ε is well defined by proving that, under our assumptions, system (6), (7) and (8) has indeed an unique solution on $[0, T]$ for every $x_0 \in E_1, w_0 \in K, y_0 \in E_2$ and $\varepsilon > 0$.

b. If $(x_\varepsilon, w_\varepsilon, y_\varepsilon)$ is a T -periodic solution of system (1), then from the following inequalities for analytic semigroups (see e.g. Pazy [20])

$$\|A_i^\alpha e^{-A_i t}\| \leq \frac{C_i(\alpha)}{t^\alpha}, \quad \alpha \in (0, 1), \quad t > 0, \quad i = 1, 2, \quad (AS)$$

it follows that

$$x(t) \in D(A_1^\alpha), \quad y(t) \in D(A_2^\alpha),$$

for every $\alpha \in (0, 1)$, and every $t \in [0, T]$. Hence, in particular the initial values $x(0)$ and $y(0)$ of the T -periodic solution of system (1) satisfies the condition $x(0) \in D(A_1^\alpha)$, $y(0) \in D(A_2^\alpha)$.

For notational simplicity in what follows we omit the dependence on $\varepsilon > 0$ of the solutions to various systems. We have the following result.

Proposition 1. *Every fixed point (x_0, w_0, y_0) of the quasitranslation operator defines the initial conditions $x(0) = A_1^{-\alpha}x_0$, $w(0) = w_0$ and $y(0) = A_2^{-\alpha}y_0$ of a T -periodic solution of system (1).*

Proof. Let $(x_0, w_0, y_0) = \tilde{U}_T^\varepsilon(x_0, w_0, y_0)$, then $(x_0, w_0, y_0) = (z_1(T), z_2(T), z_3(T))$, where (z_1, z_2, z_3) is the solution of (6), (7) and (8). We have

$$\begin{aligned}
 x_0 &= e^{-\varepsilon A_1 T} x_0 + \int_0^T A_1^\alpha e^{-\varepsilon A_1(T-s)} \varepsilon f_1(s, A_1^{-\alpha} z_1(s), A_2^{-\alpha} z_3(s), z_2(s)) ds, \\
 w_0 &= \Gamma[0, w_0] P A_1^{-\alpha} z_1(T), \\
 y_0 &= e^{-A_2 T} y_0 + \int_0^T A_2^\alpha e^{-A_2(T-s)} f_2(s, A_1^{-\alpha} z_1(s), A_2^{-\alpha} z_3(s), z_2(s)) ds.
 \end{aligned}$$

Denote by $\tilde{z}_1, \tilde{z}_2, \tilde{z}_3$ the T -periodic extensions of the functions z_1, z_2, z_3 respectively. Let $t \in [T, 2T]$, then

$$\begin{aligned}
 \Gamma[0, w_0] P A_1^{-\alpha} \tilde{z}_1(t) &= \Gamma[T, \Gamma[0, w_0] P A_1^{-\alpha} \tilde{z}_1(T)] P A_1^{-\alpha} \tilde{z}_1(t) \\
 &= \Gamma[T, \Gamma[0, w_0] P A_1^{-\alpha} z_1(T)] P A_1^{-\alpha} \tilde{z}_1(t) = \Gamma[T, w_0] P A_1^{-\alpha} \tilde{z}_1(t) \\
 &= \Gamma[0, w_0] P A_1^{-\alpha} z_1(t - T) = z_2(t - T) = \tilde{z}_2(t).
 \end{aligned}$$

By repeating the same arguments for $t \in [(n + 1)T, (n + 2)T], n \in N$, we obtain

$$\tilde{z}_2(t) = \Gamma[0, w_0] P A_1^{-\alpha} \tilde{z}_1(t),$$

for all $t \geq 0$. Now, by using standard arguments, see Krasnosel'skii et al [13], it is easy to show that $x = A_1^{-\alpha} \tilde{z}_1, y = A_2^{-\alpha} \tilde{z}_3$ satisfy (3) and (5) for all $t \geq 0$. \square

Remark 2. The vice versa is also true. In fact, let (x, w, y) be a T -periodic solution of (1). Let $x(0) = x_0, w(0) = w_0, y(0) = y_0$, then $(A_1^\alpha x_0, w_0, A_2^\alpha y_0)$ is a fixed point of the quasitranslation operator. Hence, the problem of finding T -periodic solutions of (1) is equivalent to that of finding fixed points of the quasitranslation operator.

Using the compactness of the set K and F_2) it is easy to show that the solutions of (6), (7) and (8) on $[0, T]$ for $(x_0, w_0, y_0) \in E_1 \times K \times E_2$ are bounded in $C([0, T], E_1) \times C([0, T], E_3) \times C([0, T], E_2)$. Moreover, by the compactness

of A_i^{-1} , $i = 1, 2$, and that of the set K we have the compactness of the operator defined by the right hand side of the equations (6), (7) and (8). Finally, by using the Leray-Schauder topological degree and its properties we derive the existence of solutions on $[0, T]$; the Lipschitz conditions F_1) and C_3) provide the uniqueness of the solution. Thus we have the following result.

Proposition 2. *Assume that conditions $A), F_1), F_2), C_1) - C_7), P_1)$ are satisfied for some $\alpha \in (0, 1)$. Then, for any $\varepsilon > 0$, \tilde{U}_T^ε is a well-defined operator on $E_1 \times K \times E_2$ with values in $E_1 \times K \times E_2$.*

We can prove the following result.

Proposition 3. *Assume that conditions $A), F_1), F_2), C_1) - C_7), P_1)$ are satisfied for some $\alpha \in (0, 1)$. Then the operators \tilde{U}_T^ε , $\varepsilon > 0$, are completely continuous.*

Proof. Let $B_1 \times K \times B_2 \subset E_1 \times K \times E_2$ be a bounded set, we first show that $\tilde{U}_T^\varepsilon(B_1, K, B_2)$ is a relatively compact set in $E_1 \times K \times E_2$. For this, consider the set

$$S = \{(x, w, y) \in C([0, T], E_1) \times C([0, T], E_3) \times C([0, T], E_2) : \\ x(0) = x_0, w(0) = w_0, y(0) = y_0, (x_0, w_0, y_0) \in B_1 \times K \times B_2\},$$

where

$$x(t) = e^{-\varepsilon A_1 t} x_0 + \int_0^t A_1^\alpha e^{-\varepsilon A_1(t-s)} \varepsilon f_1(s, A_1^{-\alpha} x(s), A_2^{-\alpha} y(s), w(s)) ds, \\ w(t) = \Gamma[0, w_0] P A_1^{-\alpha} x(t), \\ y(t) = e^{-A_2 t} y_0 + \int_0^t A_2^\alpha e^{-A_2(t-s)} f_2(s, A_1^{-\alpha} x(s), A_2^{-\alpha} y(s), w(s)) ds.$$

Again by the compactness of K and F_2) we have that S is bounded in $C([0, T], E_1) \times C([0, T], E_3) \times C([0, T], E_2)$. Moreover,

$$x(T) = e^{-\varepsilon A_1 T} x_0 \\ + A_1^{-\beta} \int_0^T A_1^{\alpha+\beta} e^{-\varepsilon A_1(T-s)} \varepsilon f_1(s, A_1^{-\alpha} x(s), A_2^{-\alpha} y(s), w(s)) ds, \\ w(T) = \Gamma[0, w_0] P A_1^{-\alpha} x(T),$$

$$y(T) = e^{-A_2 T} y_0 \\ + A_2^{-\beta} \int_0^T A_2^{\alpha+\beta} e^{-A_2(T-s)} f_2(s, A_1^{-\alpha} x(s), A_2^{-\alpha} y(s), w(s)) ds,$$

where $\beta \in (0, 1-\alpha)$. Since the operators $e^{-\varepsilon A_1 T}$, $\varepsilon > 0$ and $e^{-A_2 T}$ are completely continuous, the sets

$$\{e^{-\varepsilon A_1 T} x_0 : x_0 \in B_1\} \quad \text{and} \quad \{e^{-A_2 T} y_0 : y_0 \in B_2\}$$

are relatively compact in E_1 and E_2 respectively. The relative compactness of the sets

$$\left\{ A_1^{-\beta} \int_0^T A_1^{\alpha+\beta} e^{-\varepsilon A_1(T-s)} \varepsilon f_1(s, A_1^{-\alpha} x(s), A_2^{-\alpha} y(s), w(s)) ds : (x, w, y) \in S \right\}$$

and

$$\left\{ A_2^{-\beta} \int_0^T A_2^{\alpha+\beta} e^{-A_2(T-s)} f_2(s, A_1^{-\alpha} x(s), A_2^{-\alpha} y(s), w(s)) ds : (x, w, y) \in S \right\}$$

follows from the boundedness of the set S and (AS) . While the relative compactness of the set $\{w(T) : (x, w, y) \in S\} \subset K$ follows from the compactness of $K \subset E_3$. The continuity of the operator \tilde{U}_T^ε follows from the Lipschitz conditions (F_1) , (C_3) and (AS) . \square

Definition 2. (The Averaging Operator) Now, we define the averaging operator $\Psi : E_1 \times K \rightarrow E_1 \times K$, as follows

$$\Psi(x, w) = (F_1(x, w), \bar{\Gamma}(x, w)),$$

where

$$F_1(x, w) = A_1^{-1+\alpha} \frac{1}{T} \int_0^T f_1(s, A_1^{-\alpha} x, A_2^{-\alpha} y(s), w) ds, \quad y = \hat{y}_{x,w}^1,$$

$$\bar{\Gamma}(x, w) = \Gamma[0, w] P A_1^{-\alpha} \bar{u}, \quad \bar{u} \in C([0, T], E_1), \quad \bar{u}(t) = x \quad \text{for any } t \in [0, T].$$

We have also the following.

Proposition 4. Assume that conditions $(A), (F_1), (F_2), (C_1) - (C_7), (P_1), (D_1), (D_2)$ are satisfied for some $\alpha \in (0, 1)$. Then the operator Ψ is completely continuous.

Proof. It is immediate to see that our assumptions ensure that the operator $\bar{\Gamma}$ is completely continuous. Let Ω be a bounded set and consider the set

$\hat{y}_{\Omega,K}^1 = \{\hat{y}_{x,w}^1 : x \in \Omega, w \in K\}$, which is bounded by condition D_2). The set $\hat{y}_{\Omega,K}^1$ is compact in $C([0, T], E_2)$; in fact, if $y \in \hat{y}_{\Omega,K}^1$ then

$$y(t) = e^{-A_2 t} (I - e^{-A_2 T})^{-1} \int_0^T A_2^\alpha e^{-A_2(T-s)} f_2(s, A_1^{-\alpha} x, A_2^{-\alpha} y(s), w) ds \\ + \int_0^t A_2^\alpha e^{-A_2(t-s)} f_2(s, A_1^{-\alpha} x, A_2^{-\alpha} y(s), w) ds,$$

where the right hand side defines a completely continuous operator, cf. Krasnosel'skii et al [13].

Now if $(x_n, w_n) \rightarrow (x_0, w_0)$ then $\hat{y}_{\{(x_n, w_n)\}}^1 \rightarrow \hat{y}_{\{(x_0, w_0)\}}^1$, as fixed points of a completely continuous operator, and by conditions F_1) and C_3) one has $F_1(x_n, w_n) \rightarrow F_1(x_0, w_0)$ and $\bar{\Gamma}(x_n, w_n) \rightarrow \bar{\Gamma}(x_0, w_0)$. \square

3. Main Result

Let N be an open bounded subset of $E_1 \times E_3$, let $Q = E_1 \times K$ and $N_q = N \cap Q$. In what follows $C_T(E_i), i = 1, 2, 3$, will denote the Banach space of all the T -periodic continuous functions taking values in E_i equipped with the sup-norm. Moreover, $\|\cdot\|_C$ and \xrightarrow{C} will denote the norm and the convergence in any of these spaces respectively. We are now in the position to formulate our main result.

Theorem 1. *Assume that the conditions $A), F_1), F_2), C_1) - C_7), D_1), D_2), P_1)$ are satisfied for some $\alpha \in (0, 1)$ and assume the existence of a relatively open bounded set N_q in Q such that $\deg(I - \Psi, N_q) \neq 0$. Then, for sufficiently small $\varepsilon > 0$, system (6), (7) and (8) has at least one T -periodic solution $(x_\varepsilon, w_\varepsilon, y_\varepsilon) \in C_T(E_1) \times C_T(E_3) \times C_T(E_2)$ such that $(x_\varepsilon(0), w_\varepsilon(0)) \in N_q$. Moreover, for every sequence $\varepsilon_n \rightarrow 0$ the limit points (x_*, w_*, y^*) of the sequence $\{(x_{\varepsilon_n}, w_{\varepsilon_n}, y_{\varepsilon_n})\}$ are such that $(x_*, w_*) \in N_q$ is a fixed point of the operator Ψ and $y^* = \hat{y}_{x_*, w_*}^1$.*

Proof. From D_2) it follows that there exists $r > 0$ such that $\|y\|_C < r$ for all $y \in \hat{y}_{N_q}^{[0,1]}$. Now, we define the auxiliary operator $F_2 : E_1 \times K \times E_2 \rightarrow E_2$ as follows

$$F_2(x, w, y_0) = \bar{y}(T),$$

where \bar{y} is the solution of

$$y(t) = e^{-A_2 t} y_0 + \int_0^t A_2^\alpha e^{-A_2(t-s)} f_2(s, A_1^{-\alpha} x, A_2^{-\alpha} y(s), w) ds, \quad (9)$$

Let $\Psi^* : E_1 \times K \times E_2 \rightarrow E_1 \times K \times E_2$ be the operator given by

$$\Psi^*(x, w, y) = ((I - e^{-\varepsilon A_1 T})F_1 + e^{-\varepsilon A_1 T}x, \bar{\Gamma}(x, w), F_2(x, w, y)).$$

We will prove that, for $\varepsilon > 0$ sufficiently small, \tilde{U}_T^ε and Ψ^* are linearly homotopic on $\partial(N_q \times B(0, r))$. Here $B(0, r)$ denotes the ball in E_2 centered at 0 with radius r . Assume the contrary, then there exist sequences $\varepsilon_n \rightarrow 0$, $\lambda_n \rightarrow \lambda_0$, $\lambda_n \in [0, 1]$ and $\{(x_n, y_n, w_n)\} \in \partial(N_q \times B(0, r))$ such that

$$\begin{aligned} x_n &= \lambda_n(e^{-\varepsilon_n A_1 T}x_n + \int_0^T A_1^\alpha e^{-\varepsilon_n A_1(T-s)}\varepsilon_n v_1^n(s) ds) \\ &\quad + (1 - \lambda_n)((I - e^{-\varepsilon_n A_1 T})\nu_n + e^{-\varepsilon_n A_1 T}x_n), \end{aligned} \quad (10)$$

$$w_n = \lambda_n \Gamma[0, w_n] P A_1^{-\alpha} x^n(T) + (1 - \lambda_n) \bar{\Gamma}(x_n, w_n), \quad (11)$$

$$\begin{aligned} y_n &= \lambda_n(e^{-A_2 T}y_n + \int_0^T A_2^\alpha e^{-A_2(T-s)}v_2^n(s) ds) \\ &\quad + (1 - \lambda_n)(e^{-A_2 T}y_n + \int_0^T A_2^\alpha e^{-A_2(T-s)}\bar{v}_2^n(s) ds), \end{aligned} \quad (12)$$

where

$$\begin{aligned} \nu_n &= F_1(x_n, w_n), \\ v_1^n(s) &= f_1(s, A_1^{-\alpha}x^n(s), A_2^{-\alpha}y^n(s), w^n(s)), \\ v_2^n(s) &= f_2(s, A_1^{-\alpha}x^n(s), A_2^{-\alpha}y^n(s), w^n(s)) \\ \bar{v}_2^n(s) &= f_2(s, A_1^{-\alpha}x_n, A_2^{-\alpha}\bar{y}^n(s), w_n), \end{aligned}$$

and $(x^n(t), y^n(t), w^n(t))$ is such that

$$x^n(t) = e^{-\varepsilon_n A_1 t}x_n + \int_0^t A_1^\alpha e^{-\varepsilon_n A_1(t-s)}\varepsilon_n v_1^n(s) ds, \quad (13)$$

$$w^n(t) = \Gamma[0, w_n] P A_1^{-\alpha} x^n(t), \quad (14)$$

$$y^n(t) = e^{-A_2 t}y_n + \int_0^t A_2^\alpha e^{-A_2(t-s)}\varepsilon_n v_2^n(s) ds, \quad (15)$$

and $\bar{y}^n(s)$ is the solution of (9), corresponding to $x = x_n$, $w = w_n$, $y_0 = y_n$. From (10) we get

$$x_n = \lambda_n(I - e^{-\varepsilon_n A_1 T})^{-1} \int_0^T A_1^\alpha e^{-\varepsilon_n A_1(T-s)}\varepsilon_n v_1^n(s) ds + (1 - \lambda_n)\nu_n. \quad (16)$$

From (12) we get

$$\begin{aligned} y_n &= (I - e^{-A_2 T})^{-1} [\lambda_n \int_0^T A_2^\alpha e^{-A_2(T-s)} v_2^n(s) ds \\ &\quad + (1 - \lambda_n) \int_0^T A_2^\alpha e^{-A_2(T-s)} \bar{v}_2^n(s) ds]. \end{aligned} \quad (17)$$

Since the sequence $\{(x_n, w_n, y_n)\}$ is bounded in $E_1 \times E_3 \times E_2$, the sequence of functions $\{(x^n, w^n, y^n)\}$ is bounded in $C_T(E_1) \times C_T(E_3) \times C_T(E_2)$. Assumption F_2 implies that the sequences $\{v_1^n\}$ and $\{v_2^n\}$ are also bounded. By (AS) and F_2 we have also the boundedness of the sequences $\{\bar{y}^n\}$ and $\{\bar{v}_2^n\}$. Without loss of generality we can assume that $\|v_1^n\|_C \leq M$, $\|v_2^n\|_C \leq M$, $\|\bar{v}_2^n\|_C \leq M$, for some $M > 0$. Since F_1 is completely continuous, then the sequence $\{\nu_n\}$ is compact in E_1 . By passing to a subsequence, if necessary, we have that $\nu_n \rightarrow \nu_0$ in E_1 .

Let us now prove the relative compactness of the sequence $\{y_n\}$, where

$$\begin{aligned} y_n &= (I - e^{-A_2 T})^{-1} [\lambda_n A_2^{-\beta} \int_0^T A_2^{\alpha+\beta} e^{-A_2(T-s)} v_2^n(s) ds \\ &\quad + (1 - \lambda_n) A_2^{-\beta} \int_0^T A_2^{\alpha+\beta} e^{-A_2(T-s)} \bar{v}_2^n(s) ds], \quad \beta \in (0, 1 - \alpha). \end{aligned}$$

The sets $\{\int_0^T A_2^{\alpha+\beta} e^{-A_2(T-s)} v_2^n(s) ds\}$ and $\{\int_0^T A_2^{\alpha+\beta} e^{-A_2(T-s)} \bar{v}_2^n(s) ds\}$ are bounded in E_2 , hence $\{y_n\}$ is relatively compact. By passing to a subsequence, if necessary, we have that $y_n \rightarrow y_0$ in E_2 .

Finally, we show the relative compactness of $\{x_n\}$, for this consider the first term of the right-hand side of (16).

$$\begin{aligned} &(I - e^{-\varepsilon_n A_1 T})^{-1} \int_0^T A_1^\alpha e^{-\varepsilon_n A_1(T-s)} \varepsilon_n v_1^n(s) ds \\ &= \int_0^T A_1^\alpha e^{-\varepsilon_n A_1(T-s)} \varepsilon_n v_1^n(s) ds \\ &\quad + \sum_{k=1}^m e^{-\varepsilon_n A_1 k T} \int_0^T A_1^\alpha e^{-\varepsilon_n A_1(T-s)} \varepsilon_n v_1^n(s) ds \\ &\quad + e^{-\varepsilon_n A_1(m+1)T} (I - e^{-\varepsilon_n A_1 T})^{-1} \int_0^T A_1^\alpha e^{-\varepsilon_n A_1(T-s)} \varepsilon_n v_1^n(s) ds. \end{aligned}$$

Let $m = [\frac{1}{\varepsilon_n}]$, where $[\frac{1}{\varepsilon_n}]$ denotes the integer part of $\frac{1}{\varepsilon_n}$. Then $m = \frac{1}{\varepsilon_n} - \theta$, where $\theta \in [0, 1)$. In the following we use (AS) in order to estimate the right

hand side of the previous equation. In fact, the first integral can be estimated as follows

$$\left\| \int_0^T A_1^\alpha e^{-\varepsilon_n A_1(T-s)} \varepsilon_n v_1^n(s) ds \right\| \leq \varepsilon_n^{1-\alpha} \frac{C(\alpha)}{1-\alpha} MT^{1-\alpha}.$$

Analogously, consider the second term

$$\begin{aligned} & \left\| \sum_{k=1}^m e^{-\varepsilon_n A_1 k T} \int_0^T A_1^\alpha e^{-\varepsilon_n A_1(T-s)} \varepsilon_n v_1^n(s) ds \right\| \\ & \leq \sum_{k=1}^m \int_0^T \frac{C(\alpha)}{(\varepsilon_n((k+1)T-s))^\alpha} \varepsilon_n M ds \leq C(\alpha) MT^{1-\alpha} \varepsilon_n^{1-\alpha} \sum_{k=1}^m \frac{1}{k^\alpha} \\ & \leq C(\alpha) MT^{1-\alpha} \varepsilon_n^{1-\alpha} \left(1 + \int_1^m \frac{1}{\tau^\alpha} d\tau \right) \\ & = C(\alpha) MT^{1-\alpha} \varepsilon_n^{1-\alpha} \left(1 + \frac{m^{1-\alpha}}{1-\alpha} - \frac{1}{1-\alpha} \right) \\ & \leq \frac{C(\alpha)}{1-\alpha} MT^{1-\alpha} (\varepsilon_n m)^{1-\alpha} \leq \frac{C(\alpha)}{1-\alpha} MT^{1-\alpha}. \end{aligned}$$

Finally, consider the third term

$$\begin{aligned} & \left\| e^{-\varepsilon_n A_1(m+1)T} (I - e^{-\varepsilon_n A_1 T})^{-1} \int_0^T A_1^\alpha e^{-\varepsilon_n A_1(T-s)} \varepsilon_n v_1^n(s) ds \right\| \\ & \leq \frac{\varepsilon_n}{1 - e^{-\varepsilon_n d_1 T}} \int_0^T \frac{C(\alpha) M}{(\varepsilon_n((m+2)T-s))^\alpha} ds \\ & \leq \frac{\varepsilon_n}{1 - e^{-\varepsilon_n d_1 T}} \frac{C(\alpha) MT^{1-\alpha}}{(\varepsilon_n(m+1))^\alpha} \leq C \left(\frac{1}{\varepsilon_n} - \theta + 1 \right)^{-\alpha} \\ & = C(1 - \varepsilon_n \theta + \varepsilon_n)^{-\alpha} \leq C. \end{aligned}$$

Let $G_n = (I - e^{-\varepsilon_n A_1 T})^{-1} \int_0^T A_1^\alpha e^{-\varepsilon_n A_1(T-s)} \varepsilon_n v_1^n(s) ds$, clearly the sequence $\{G_n\}$ is bounded in E_1 . Let us now show the relative compactness of $\{G_n\}$. For this we rewrite $\{G_n\}$ as follows

$$\begin{aligned} G_n &= A_1^{-\beta} \int_0^T A_1^{\alpha+\beta} e^{-\varepsilon_n A_1(T-s)} \varepsilon_n v_1^n(s) ds \\ &+ A_1^{-\beta} \left(\sum_{k=1}^m e^{-\varepsilon_n A_1 k T} \int_0^T A_1^{\alpha+\beta} e^{-\varepsilon_n A_1(T-s)} \varepsilon_n v_1^n(s) ds \right) \\ &+ e^{-\varepsilon_n A_1(m+1)T} (I - e^{-\varepsilon_n A_1 T})^{-1} \int_0^T A_1^\alpha e^{-\varepsilon_n A_1(T-s)} \varepsilon_n v_1^n(s) ds. \end{aligned}$$

Since the sets

$$\left\{ \int_0^T A_1^{\alpha+\beta} e^{-\varepsilon_n A_1(T-s)} \varepsilon_n v_1^n(s) ds \right\},$$

$$\left\{ \sum_{k=1}^m e^{-\varepsilon_n A_1 k T} \int_0^T A_1^{\alpha+\beta} e^{-\varepsilon_n A_1(T-s)} \varepsilon_n v_1^n(s) ds \right\}$$

are bounded, then the sets

$$\left\{ A_1^{-\beta} \int_0^T A_1^{\alpha+\beta} e^{-\varepsilon_n A_1(T-s)} \varepsilon_n v_1^n(s) ds \right\} \quad \text{and}$$

$$\left\{ A_1^{-\beta} \left(\sum_{k=1}^m e^{-\varepsilon_n A_1 k T} \int_0^T A_1^{\alpha+\beta} e^{-\varepsilon_n A_1(T-s)} \varepsilon_n v_1^n(s) ds \right) \right\}$$

are relatively compact in E_1 . The last term can be rewritten as

$$e^{-\varepsilon_n A_1(m+1)T} (I - e^{-\varepsilon_n A_1 T})^{-1} \int_0^T A_1^\alpha e^{-\varepsilon_n A_1(T-s)} \varepsilon_n v_1^n(s) ds$$

$$= e^{-\varepsilon_n A_1(\frac{1}{\varepsilon_n} - \theta + 1)T} G_n = e^{-A_1 T} e^{-\varepsilon_n A_1(1-\theta)T} G_n.$$

By using the compactness of $e^{-A_1 T}$ we obtain that the set

$$\left\{ e^{-\varepsilon_n A_1(m+1)T} (I - e^{-\varepsilon_n A_1 T})^{-1} \int_0^T A_1^\alpha e^{-\varepsilon_n A_1(T-s)} \varepsilon_n v_1^n(s) ds \right\}$$

is relatively compact in E_1 . Hence, the sequence x_n is relatively compact. Without loss of generality we can assume that $x_n \rightarrow x_0$ in E_1 . Now we evaluate $\|x^n(t) - x_n\|$ in E_1 as follows

$$\|x^n(t) - x_n\| \leq \|(e^{-\varepsilon_n A_1 t} - I)x_n\| + \left\| \int_0^t A_1^\alpha e^{-\varepsilon_n A_1(t-s)} \varepsilon_n v_1^n(s) ds \right\|.$$

The first term of the righthand side tends to zero, since analytic semigroups are also C_0 -semigroups. For the second term we have

$$\left\| \int_0^t A_1^\alpha e^{-\varepsilon_n A_1(t-s)} \varepsilon_n v_1^n(s) ds \right\| \leq \int_0^t \frac{C(\alpha) \varepsilon_n M}{(\varepsilon_n(t-s))^\alpha} ds$$

$$\leq \varepsilon_n^{1-\alpha} M \frac{C(\alpha)}{1-\alpha} T^{1-\alpha} \rightarrow 0$$

as $n \rightarrow \infty$. Therefore, $x^n \xrightarrow{C} x_0$.

Since $w_n \in K$, then $\{w_n\}$ is relatively compact. By passing to a subsequence, if necessary, we have that $w_n \rightarrow w_0$ in E_3 . Consider the sequence $\{w^n\}$. By using the continuity of the operator Γ we get

$$\Gamma[0, w_n]PA_1^{-\alpha}x^n \xrightarrow{C} \Gamma[0, w_0]PA_1^{-\alpha}x_0 = w_0, \text{ that is } w^n \xrightarrow{C} w_0.$$

Finally, consider the sequences $\{y^n\}$ and $\{\bar{y}^n\}$. Since $\{y_n\}$ is relatively compact and $\{v_2^n\}$ is bounded in $C_T(E_2)$, then the sequence $\{y^n\}$ is relatively compact. Let $y^n \xrightarrow{C} y^0$. Assumption F_1) imply that

$$\begin{aligned} & \|e^{-A_2t}y_n + \int_0^t A_2^\alpha e^{-A_2(t-s)} f_2(s, A_1^{-\alpha}x^n(s), A_2^{-\alpha}y^n(s), w^n(s)) ds \\ & \quad - e^{-A_2t}y_0 + \int_0^t A_2^\alpha e^{-A_2(t-s)} f_2(s, A_1^{-\alpha}x_0, A_2^{-\alpha}y^0(s), w_0) ds\| \\ & \leq \|y_n - y_0\| + \frac{C(\alpha)}{1-\alpha} T^{1-\alpha} L(\|x^n - x_0\|_C + \|y^n - y^0\|_C + \|w^n - w_0\|_C) \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$. Hence,

$$y^0(t) = e^{-A_2t}y_0 + \int_0^t A_2^\alpha e^{-A_2(t-s)} f_2(s, A_1^{-\alpha}x_0, A_2^{-\alpha}y^0(s), w_0) ds,$$

and

$$\begin{aligned} y^0(T) &= e^{-A_2T}y_0 \\ & \quad + \int_0^T A_2^\alpha e^{-A_2(T-s)} f_2(s, A_1^{-\alpha}x_0, A_2^{-\alpha}y^0(s), w_0) ds. \end{aligned} \quad (18)$$

Similarly, for $\bar{y}^n(t)$, we get $\bar{y}^n \xrightarrow{C} y^*$, where $y^*(t)$ is the solution of the equation

$$y(t) = e^{-A_2t}y_0 + \int_0^t A_2^\alpha e^{-A_2(t-s)} f_2(s, A_1^{-\alpha}x_0, A_2^{-\alpha}y(s), w_0) ds. \quad (19)$$

Since f_2 is a Lipschitz function, then the equation (19) has a unique solution, thus $y^* = y^0$. Therefore, from (17) as $n \rightarrow \infty$ we obtain

$$\begin{aligned} y_0 &= (I - e^{-A_2T})^{-1}[\lambda_0 \int_0^T A_2^\alpha e^{-A_2(T-s)} f_2(s, A_1^{-\alpha}x_0, A_2^{-\alpha}y^0(s), w_0) ds \\ & \quad + (1 - \lambda_0) \int_0^T A_2^\alpha e^{-A_2(T-s)} f_2(s, A_1^{-\alpha}x_0, A_2^{-\alpha}y^0(s), w_0) ds]. \end{aligned}$$

From (18) we obtain

$$\int_0^T A_2^\alpha e^{-A_2(T-s)} f_2(s, A_1^{-\alpha} x_0, A_2^{-\alpha} y^0(s), w_0) ds = y^0(T) - e^{-A_2 T} y_0.$$

Thus,

$$y_0 = (I - e^{-A_2 T})^{-1} [\lambda_0 (y^0(T) - e^{-A_2 T} y_0) + (1 - \lambda_0) (y^0(T) - e^{-A_2 T} y_0)].$$

Finally, we get

$$y_0 = y^0(T), \text{ i.e. } y^0 = \hat{y}_{x_0, w_0}^1.$$

Consider now (16), in Kamenskii et al [8] it is shown that

$$\varepsilon (I - e^{-\varepsilon A_1 T})^{-1} \int_0^T A_1^\alpha e^{-\varepsilon A_1(T-s)} v(s) ds \rightarrow A_1^{-1+\alpha} \frac{1}{T} \int_0^T v(s) ds, \quad \text{as } \varepsilon \rightarrow 0 \quad (20)$$

for every $v \in C_T(E_1)$. Since f_1 is a Lipschitz function, $v_1^n \xrightarrow{C} v_1^0$, where $v_1^0(s) = f_1(s, A_1^{-\alpha} x_0, A_2^{-\alpha} y^0(s), w_0)$. Hence,

$$\begin{aligned} & \| (I - e^{-\varepsilon_n A_1 T})^{-1} \int_0^T A_1^\alpha e^{-\varepsilon_n A_1(T-s)} \varepsilon_n v_1^n(s) ds - A_1^{-1+\alpha} \frac{1}{T} \int_0^T v_1^0(s) ds \| \\ & \leq \| (I - e^{-\varepsilon_n A_1 T})^{-1} \int_0^T A_1^\alpha e^{-\varepsilon_n A_1(T-s)} \varepsilon_n v_1^0(s) ds - A_1^{-1+\alpha} \frac{1}{T} \int_0^T v_1^0(s) ds \| \\ & \quad + \| (I - e^{-\varepsilon_n A_1 T})^{-1} \int_0^T A_1^\alpha e^{-\varepsilon_n A_1(T-s)} \varepsilon_n (v_1^n(s) - v_1^0(s)) ds \|. \end{aligned}$$

By (20) the first term tends to zero. Since the operators

$$\varepsilon_n (I - e^{-\varepsilon_n A_1 T})^{-1} \int_0^T A_1^\alpha e^{-\varepsilon_n A_1(T-s)} (\cdot)$$

are bounded and $v_1^n \xrightarrow{C} v_1^0$, then also the second term tends to zero. Thus,

$$(I - e^{-\varepsilon_n A_1 T})^{-1} \int_0^T A_1^\alpha e^{-\varepsilon_n A_1(T-s)} \varepsilon_n v_1^n(s) ds \rightarrow A_1^{-1+\alpha} \frac{1}{T} \int_0^T v_1^0(s) ds.$$

Since $y^0 = \hat{y}_{x_0, w_0}^1$, it results that $A_1^{-1+\alpha} \frac{1}{T} \int_0^T v_1^0(s) ds = F_1(x_0, w_0)$. Furthermore, from (16) as $n \rightarrow \infty$ we obtain

$$x_0 = \lambda_0 F_1(x_0, w_0) + (1 - \lambda_0) F_1(x_0, w_0) = F_1(x_0, w_0). \quad (21)$$

Finally, for $n \rightarrow \infty$ from (11) we get

$$w_0 = \lambda_0 \Gamma[0, w_0] P A_1^{-\alpha} x_0(t) + (1 - \lambda_0) \bar{\Gamma}(x_0, w_0) = \bar{\Gamma}(x_0, w_0). \quad (22)$$

Since the set $\partial(N_q \times B(0, r))$ is closed, then $(x_0, w_0, y_0) \in \partial(N_q \times B(0, r))$. But, by our choice of r we have that $y_0 \notin \partial(B(0, r))$. Therefore $(x_0, w_0) \in \partial N_q$ which is a contradiction with (21), (22) and the choice of N_q .

Consider now the homotopy defined as follows

$$H_0^\lambda(x, w, y) = ((I - e^{-\varepsilon A_1 T})F_1(x, w) + e^{-\varepsilon A_1 T}x, \bar{\Gamma}(x, w), e^{-A_2 T}y + \lambda \int_0^T A_2^\alpha e^{-A_2(T-s)} \bar{v}_2(s) ds),$$

where $\bar{v}_2(s) = f_2(s, A_1^{-\alpha}x, A_2^{-\alpha}\bar{y}(s), w)$, $\bar{y}(s)$ is the solution of (9) with $y(0) = y$. For $\lambda = 0$ we have

$$H^0(x, w, y) = ((I - e^{-\varepsilon A_1 T})F_1(x, w) + e^{-\varepsilon A_1 T}x, \bar{\Gamma}(x, w), e^{-A_2 T}y).$$

Now we define the auxiliary operators $\tilde{H}^0 : E_1 \times K \times E_2 \rightarrow E_1 \times K \times E_2$, $\hat{H}^0 : E_1 \times K \rightarrow E_1 \times K$, $D : E_1 \times E_3 \times E_2 \rightarrow E_1 \times E_3 \times E_2$ as follows

$$\begin{aligned} \tilde{H}^0(x, w, y) &= ((I - e^{-\varepsilon A_1 T})F_1(x, w) + e^{-\varepsilon A_1 T}x, \bar{\Gamma}(x, w), 0), \\ \hat{H}^0(x, w) &= ((I - e^{-\varepsilon A_1 T})F_1(x, w) + e^{-\varepsilon A_1 T}x, \bar{\Gamma}(x, w)), \\ D(x, w, y) &= (0, 0, e^{-A_2 T}y). \end{aligned}$$

Since $I - H^0 = (I - D)(I - \tilde{H}^0)$, by the properties of the topological degree, (see Krasnosel'skii et al [12]), we have

$$\begin{aligned} |\deg(I - H^0, N_q \times B(0, r))| &= |\text{ind}(D, 0)| \cdot |\deg(I - \tilde{H}^0, N_q \times B(0, r))| \\ &= |\deg(I - \tilde{H}^0, N_q \times B(0, r))|. \end{aligned} \quad (23)$$

By the reduction property of the topological degree we have

$$|\deg(I - \tilde{H}^0, N_q \times B(0, r))| = |\deg(I - \hat{H}^0, N_q)|.$$

Let us consider now the operator \hat{H}^0 . Since $(I - \hat{H}^0) = (I - B)(I - \Psi)$, where $B : E_1 \times E_3 \rightarrow E_1 \times E_3$, is given by $B(x, w) = (e^{-\varepsilon A_1 T}x, 0)$, we have

$$|\deg(I - \hat{H}^0, N_q \times B(0, r))| = |\text{ind}(B, 0)| \cdot |\deg(I - \Psi, N_q)| \neq 0.$$

Then, from (23)

$$|\deg(I - H^0, N_q \times B(0, r))| \neq 0.$$

Since $\tilde{U}_T^\varepsilon, \Psi^*, H^0$, are homotopic in $N_q \times B(0, r)$ we obtain

$$\begin{aligned} |\deg(I - \tilde{U}_T^\varepsilon, N_q \times B(0, r))| &= |\deg(I - \Psi^*, N_q \times B(0, r))| \\ &= |\deg(I - H^0, N_q \times B(0, r))| \neq 0. \end{aligned}$$

Hence, for sufficiently small $\varepsilon > 0$ there exists at least one T -periodic solution $(x_\varepsilon, w_\varepsilon, y_\varepsilon)$ of (6), (7) and (8) such that $(x_\varepsilon(0), w_\varepsilon(0)) \in N_q$. Now, we consider a sequence $\varepsilon_n \rightarrow 0$. Let $(x_{\varepsilon_n}, w_{\varepsilon_n}, y_{\varepsilon_n})$ be a sequence of T -periodic solutions of system (6), (7) and (8). Letting $x_{\varepsilon_n}(0) = x_n, w_{\varepsilon_n}(0) = w_n, y_{\varepsilon_n}(0) = y_n$, we get

$$\begin{aligned} x_n &= e^{-\varepsilon_n A_1 T} x_n \\ &\quad + \int_0^T A_1^\alpha e^{-\varepsilon_n A_1 (T-s)} \varepsilon_n f_1(s, A_1^{-\alpha} x_{\varepsilon_n}(s), A_2^{-\alpha} y_{\varepsilon_n}(s), w_{\varepsilon_n}(s)) ds, \\ w_n &= \Gamma[0, w_n] P A_1^{-\alpha} x_{\varepsilon_n}(T), \end{aligned}$$

y_n

$$= e^{-A_2 T} y_n + \int_0^T A_2^\alpha e^{-A_2 (T-s)} f_2(s, A_1^{-\alpha} x_{\varepsilon_n}(s), A_2^{-\alpha} y_{\varepsilon_n}(s), w_{\varepsilon_n}(s)) ds.$$

By repeating the arguments of the first part of the proof we can show the existence of a subsequence of $(x_{\varepsilon_n}, w_{\varepsilon_n}, y_{\varepsilon_n})$ converging to (x_*, w_*, y^*) , where $(x_*, w_*) \in N_q$ is a fixed point of the operator Ψ and $y^* = \hat{y}_{x_*, w_*}^1$. This concludes the proof. \square

Remark 3. Condition $F_2)$ can be replaced by the following conditions.

$F_2')$ For any $x_0 \in E_1, w_0 \in K, y_0 \in E_2, \varepsilon > 0$ the set $\{(x_\lambda, w_\lambda, y_\lambda) : \lambda \in [0, 1]\}$, where $(x_\lambda, w_\lambda, y_\lambda)$ is a solution of the following system, with $t \in [0, T]$,

$$\begin{aligned} x(t) &= e^{-\varepsilon A_1 t} x_0 + \lambda \int_0^t A_1^\alpha e^{-\varepsilon A_1 (t-s)} \varepsilon f_1(s, A_1^{-\alpha} x(s), A_2^{-\alpha} y(s), w(s)) ds, \\ w(t) &= \Gamma[0, w_0] P A_1^{-\alpha} x(t), \\ y(t) &= e^{-A_2 t} y_0 + \lambda \int_0^t A_2^\alpha e^{-A_2 (t-s)} f_2(s, A_1^{-\alpha} x(s), A_2^{-\alpha} y(s), w(s)) ds, \end{aligned}$$

is bounded.

F_2'') For any bounded set $\Omega \subset E_1 \times K \times E_2$ the solutions of (6)-(7)-(8), with $(x_0, w_0, y_0) \in \Omega$, and $\varepsilon \in (0, \varepsilon_0]$, for some $\varepsilon_0 > 0$, are uniformly bounded.

F_2''') For any bounded set $\Omega \subset E_1 \times K \times E_2$ the solutions of (9), with $(x, w, y_0) \in \Omega$, and $\varepsilon \in (0, \varepsilon_0]$ are uniformly bounded.

4. Application

Let us consider now the singularly perturbed system of semilinear parabolic equations of the following form

$$\frac{\partial z_1}{\partial t} = \frac{\partial^2 z_1}{\partial \xi^2} + P_1(z_1, z_2) + b \sin \frac{t}{\varepsilon}, \quad t \geq 0, \xi \in [0, l], l > 0, \quad (24)$$

$$\varepsilon \frac{\partial z_2}{\partial t} = \frac{\partial^2 z_2}{\partial \xi^2} + P_2(z_1, z_2) + b \sin \frac{t}{\varepsilon}, \quad t \geq 0, \xi \in [0, l], \quad (25)$$

with the following boundary conditions

$$\frac{\partial z_i}{\partial \xi}(t, 0) = 0, \quad i = 1, 2, \quad (26)$$

$$\nu \frac{\partial z_1}{\partial \xi}(t, l) + z_1(t, l) = q(t), \quad (27)$$

$$\nu \frac{\partial z_2}{\partial \xi}(t, l) + z_2(t, l) = 0, \quad (28)$$

where q is defined as follows

$$\beta q'(t) = v(t) - q(t), \quad (29)$$

where

$$v(t) = \Gamma[0, v_0]z_1(t, 0). \quad (30)$$

We assume that P_1, P_2 are polynomials in the variables z_1, z_2 and the hysteresis nonlinearity Γ is defined, (compare Krasnosel'skii et al [11]), by means of the following input-output relation $v(t) = \Gamma[t_0, v(t_0)]u(t)$ where

$$v(t) = \begin{cases} \min\{h, u(t) - u(t_0) + v(t_0)\}, & \text{for all } t \text{ such that} \\ & u(t) \text{ is nondecreasing,} \\ \max\{-h, u(t) - u(t_0) + v(t_0)\}, & \text{for all } t \text{ such that} \\ & u(t) \text{ is nonincreasing} \end{cases}$$

and $h > 0$. It is easy to see that $v(t) \in [-h, h]$. Thus, in this case, the compact set K of condition C_4) turns out to be $[-h, h]$.

Now, if we consider the change the variables $x_1(t, \xi) = z_1(t, \xi) - g(t, \xi)$, $x_2(t, \xi) = z_2(t, \xi)$, where $g(t, \xi) = \xi^2 q(t) / (2l\nu + l^2)$, then the system takes the form

$$\begin{aligned} \frac{\partial x_1}{\partial t} &= \frac{\partial^2 x_1}{\partial \xi^2} + \xi^2 \frac{v(t) - q(t)}{\beta(2l\nu + l^2)} + \frac{2q(t)}{2l\nu + l^2} \\ &+ P_1\left(\left(x_1(t, \xi) + \frac{\xi^2 q(t)}{2l\nu + l^2}\right), x_2(t, \xi)\right) + b \sin \frac{t}{\varepsilon}, \end{aligned} \quad (24')$$

$$\varepsilon \frac{\partial x_2}{\partial t} = \frac{\partial^2 x_2}{\partial \xi^2} + P_2\left(\left(x_1(t, \xi) + \frac{\xi^2 q(t)}{2l\nu + l^2}\right), x_2(t, \xi)\right) + b \sin \frac{t}{\varepsilon}, \quad (25')$$

$t \geq 0$, $\xi \in [0, l]$ with the boundary conditions

$$\frac{\partial x_i}{\partial \xi}(t, 0) = 0, \quad i = 1, 2, \quad (26')$$

$$\nu \frac{\partial x_i}{\partial \xi}(t, l) + x_i(t, l) = 0, \quad i = 1, 2, \quad (27')$$

and the hysteresis relations (29)-(30), and

$$v(t) = \Gamma[0, v_0]x_1(t, 0). \quad (30')$$

Now, let $E_1 = L_p[0, l] \times R^1$, $E_2 = L_p[0, l]$, $E_3 = R^1$, $E_0 = R^1$, $x(t) = (x_1(t, \cdot), q(t))$, $y(t) = x_2(t, \cdot)$ and

$$A_1 = \left(-\frac{d^2}{d\xi^2}, \frac{1}{\beta}\right),$$

with the boundary conditions (26'), (27') for $i = 1$. We define the operator $P : E_1 \rightarrow E_0$ as follows $P(x_1, q) = x_1(0)$. Finally, let A_2 be the operator defined by the differential operator $-d^2/d\xi^2$ together with boundary conditions (26'), (27') for $i = 2$. Since, as it is shown in Krasnosel'skii et al [13], it is possible to choose p and $\alpha \in (0, 1)$ in such a way that the nonlinearities defined by P_1 and P_2 are subordinated to $A_1^{-\alpha}$ and $A_2^{-\alpha}$, then our Theorem 1 applies.

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