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Small parameter perturbations of nonlinear periodic systems*

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Abstract
In this paper we consider a class of nonlinear periodic differential systems perturbed by two nonlinear periodic terms with multiplicative different powers of a small parameter $\varepsilon > 0$. For such a class of systems we provide conditions that guarantee the existence of periodic solutions of given period $T > 0$. These conditions are expressed in terms of the behaviour on the boundary of an open bounded set U of $\mathbb{R}^n$ of the solutions of suitably defined linearized systems. The approach is based on the classical theory of the topological degree for compact vector fields. An application to the existence of periodic solutions to the van der Pol equation is also presented.

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1. Introduction

This paper represents the continuation of the work started by the authors in [8]. There we considered the existence problem both of periodic solutions and of solutions of the Cauchy problem for a system of ordinary differential equations described by

$$\dot{x} = \psi(t, x) + \varepsilon\phi(t, x), \tag{1}$$

where $\phi, \psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ are continuously differentiable functions, T-periodic with respect to time t, and ε is a small positive parameter.

To solve this problem for $\varepsilon > 0$ sufficiently small a new approach was presented in [8] (theorem 1). Such an approach is based on the linearized system

$$\dot{y} = \frac{\partial\psi}{\partial x}(t, \Omega(t, 0, \xi))y + \phi(t, \Omega(t, 0, \xi)), \tag{2}$$

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where $\xi \in \mathbb{R}^n$ and $\Omega(\cdot, t_0, \xi)$ denotes the solution of (1) at $\varepsilon = 0$ satisfying $x(t_0) = \xi$. Specifically, consider the change of variable

$$z(t) = \Omega(0, t, x(t)) \tag{3}$$

and the solution $\eta(\cdot, s, \xi)$ of (2) such that $y(s) = 0$. If there exists a bounded open set $U \subset \mathbb{R}^n$ such that $\Omega(T, 0, \xi) = \xi$ for any $\xi \in \partial U$, and $\eta(T, s, \xi) - \eta(0, s, \xi) \neq 0$, for any $s \in [0, T]$, and any $\xi \in \partial U$, then (1) has a T-periodic solution for $\varepsilon > 0$ sufficiently small, provided that $\gamma(\eta(T, 0, \cdot)) \neq 0$. Here $\gamma(F, U)$ denotes the rotation number of a continuous map $F : \bar{U} \to \bar{U}$.

The advantage of the proposed approach as compared with the classical averaging method, which is one of the most useful tools for treating this problem, lies in the fact that in order to use this second method for establishing the existence of periodic solutions in perturbed systems of the form (1) one must assume that the change of variable (3) is T-periodic with respect to t for every T-periodic function x such that $\Omega(0, t, x(t)) \in U$, for any $t \in [0, T]$, instead of only on the boundary of the bounded open set U.

The same assumption is necessary in vibrational control problems, [2, 12], to reduce the considered system to the standard form for applying the averaging method. For an extensive list of references on this topic see [5].

Our approach has also been employed in [9] to prove the existence of periodic solutions for a class of first-order singularly perturbed differential systems.

The aim of this paper is to extend the previously outlined approach to a more general class of perturbed systems than (1). To be more precise, we consider here systems of the following form:

$$\dot{x} = \psi(t, x) + \varepsilon^2\phi_1(t, x) + \varepsilon^3\phi_2(t, x, \varepsilon), \tag{4}$$

where the functions $\psi,\ \phi_1 : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \phi_2 : \mathbb{R} \times \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ are continuously differentiable and T-periodic with respect to the first variable and ε is a small positive parameter. We denote again by $\Omega(\cdot, t_0, \xi)$ the solution of the Cauchy problem

$$\begin{aligned} &\dot{x} = \psi(t, x), \\ &x(t_0) = \xi \end{aligned} \tag{5}$$

and by $\eta_i(\cdot, s, \xi)$, $i = 1, 2$, the solution of the Cauchy problems

$$\begin{aligned} &\dot{y} = \frac{\partial\psi}{\partial x}(t, \Omega(t, 0, \xi))y + \phi_1(t, \Omega(t, 0, \xi)), \qquad \text{if } i = 1, \\ &\dot{y} = \frac{\partial\psi}{\partial x}(t, \Omega(t, 0, \xi))y + \phi_2(t, \Omega(t, 0, \xi), 0), \qquad \text{if } i = 2, \\ &y(s) = 0. \end{aligned} \tag{6}$$

In section 2 we prove the main result of this paper: theorem 1. Indeed, under suitable assumptions on $\Omega(T, 0, \xi)$ and $\eta_i(T, s, \xi)$, $i = 1, 2$, for $s \in [0, T]$, $\xi \in \partial U$ and for $\varepsilon > 0$ sufficiently small, we prove the existence of T-periodic solutions of system (4) provided that $\deg(\eta_2(T, 0, \cdot), U) \neq 0$. Moreover, in the case of system (1), as a direct consequence of theorem 2 we get theorem 1 of [8].

In section 3, we apply theorem 1 to autonomous systems in $\mathbb{R}^2$ perturbed by a non-autonomous term of higher order (with respect to $\varepsilon > 0$) to show the existence of periodic solutions in a suitable open set defined by means of the trajectory of the linear part of the autonomous system.

Finally, in section 4 we illustrate the results obtained by means of an example concerning the van der Pol equation. To the best knowledge of the authors this represents a new approach to investigate the existence of periodic solutions for the periodically forced van der Pol equation.

Indeed, many papers in the literature are devoted to the study of the response of the van der Pol equation to a periodic stimulus (of different period), but the methods are quite different; in fact, they are essentially based on asymptotic expansions, Fourier series, singular perturbation theory and averaging methods. We refer to the papers [3, 4, 6, 7] and the references therein.

2. Main results

Consider the system

$$\dot{x} = \psi(t, x) + \varepsilon^2\phi_1(t, x) + \varepsilon^3\phi_2(t, x, \varepsilon), \tag{7}$$

where the functions ψ, $\phi_1 : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $\phi_2 : \mathbb{R} \times \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ are continuously differentiable, T-periodic with respect to time t, and ε is a small positive parameter.

To investigate the existence of T-periodic solutions of system (7) we introduce the compact integral operator $F_\varepsilon : C([0, T], \mathbb{R}^n) \to C([0, T], \mathbb{R}^n)$ defined by

$$F_\varepsilon(x)(t) = x(t) - x(T) - \int_0^t (\psi(\tau, x(\tau)) + \varepsilon^2\phi_1(\tau, x(\tau)) + \varepsilon^3\phi_2(\tau, x(\tau), \varepsilon))\,\mathrm{d}\tau,$$

for any $t \in [0, T]$. Clearly, if $F_\varepsilon(x) = 0$ then x is a T-periodic solution of system (7).

Denote by $\Omega(\cdot, t_0, \xi)$ the solution of the Cauchy problem

$$\begin{aligned} &\dot{x} = \psi(t, x),\\ &x(t_0) = \xi \end{aligned} \tag{8}$$

and by $\eta_i(\cdot, s, \xi)$, $1 = 1, 2$, the solution of the Cauchy problems

$$\begin{aligned} &\dot{y} = \frac{\partial\psi}{\partial x}(t, \Omega(t, 0, \xi))y + \phi_1(t, \Omega(t, 0, \xi)), \qquad \text{if } i = 1,\\ &\dot{y} = \frac{\partial\psi}{\partial x}(t, \Omega(t, 0, \xi))y + \phi_2(t, \Omega(t, 0, \xi), 0), \qquad \text{if } i = 2,\\ &y(s) = 0. \end{aligned} \tag{9}$$

The following lemma provides an explicit representation of the functions η_1 and η_2.

Lemma 1. *Let $\xi \in \mathbb{R}^n$ and $s, t \in [0, T]$. We have that*

$$\eta_1(t, s, \xi) = \frac{\partial\Omega}{\partial z}(t, 0, \xi)\int_s^t \Phi_1(\tau, \xi)\,\mathrm{d}\tau,$$

where

$$\Phi_1(t, \xi) = \frac{\partial\Omega}{\partial z}(0, t, \Omega(t, 0, \xi))\phi_1(t, \Omega(t, 0, \xi))$$

and

$$\eta_2(t, s, \xi) = \frac{\partial\Omega}{\partial z}(t, 0, \xi)\int_s^t \Phi_2(\tau, \xi, 0)\,\mathrm{d}\tau,$$

where

$$\Phi_2(t, \xi, \varepsilon) = \frac{\partial\Omega}{\partial z}(0, t, \Omega(t, 0, \xi))\phi_2(t, \Omega(t, 0, \xi), \varepsilon).$$

Proof. It is sufficient to observe, that the matrix $(\partial\Omega/\partial z)(t, 0, \xi)$ is the fundamental matrix of the linear system

$$\dot{y} = \frac{\partial\psi}{\partial x}(t, \Omega(t, 0, \xi))y.$$

Moreover, $((\partial\Omega/\partial z)(t,0,\xi))^{-1} = (\partial\Omega/\partial z)(0,t,\Omega(t,0,\xi))$. In fact, if we derive the identity

$$\Omega(0,t,\Omega(t,0,\xi)) = \xi$$

with respect to ξ, we obtain

$$\frac{\partial\Omega}{\partial z}(0,t,\Omega(t,0,\xi))\frac{\partial\Omega}{\partial z}(t,0,\xi) = I,$$

whenever $\xi \in \mathbb{R}^n$. Therefore, by the variation of constants formula for a linear non-homogeneous system

$$\dot{y} = \frac{\partial\psi}{\partial x}(t,\Omega(t,0,\xi))y + \phi_1(t,\Omega(t,0,\xi)),$$

we have

$$\begin{aligned}\eta_1(t,s,\xi) &= \int_s^t \frac{\partial\Omega}{\partial z}(t,0,\xi)\left(\frac{\partial\Omega}{\partial z}(\tau,0,\xi)\right)^{-1}\phi_1(\tau,\Omega(\tau,0,\xi))\,\mathrm{d}\tau\\ &= \frac{\partial\Omega}{\partial z}(t,0,\xi)\int_s^t \Phi_1(\tau,\xi)\,\mathrm{d}\tau.\end{aligned}$$

The formula for η_2 is obtained in the same way. □

We have the following main result.

Theorem 1. *Let $U \subset \mathbb{R}^n$ be an open and bounded set. Assume, that*

(A1) $\Omega(T,0,\xi) = \xi$, *for any* $\xi \in \partial U$,
(A2) $\eta_1(T,s,\xi) - \eta_1(0,s,\xi) = 0$, *for any* $s \in [0,T]$ *and any* $\xi \in \partial U$,
(A3) $\eta_2(T,s,\xi) - \eta_2(0,s,\xi) \neq 0$, *for any* $s \in [0,T]$ *and any* $\xi \in \partial U$.

Then, for $\varepsilon > 0$ *sufficiently small,*

$$\deg(F_\varepsilon, W(T,U)) = \deg(\eta_2(T,0,\cdot),U), \tag{10}$$

where $W(T,U) = \{x \in C([0,T],\mathbb{R}^n) : \Omega(0,t,x(t)) \in U,\ \textit{whenever}\ t \in [0,T]\}$.

In order to prove the theorem we need the following lemma.

Lemma 2. *Let* $z \in C^1([0,T],\mathbb{R}^n)$, $f \in C([0,T],\mathbb{R}^n)$ *and* $b \in \mathbb{R}^n$. *If*

$$\int_0^t \frac{\partial\Omega}{\partial z}(s,0,z(s))\dot{z}(s)\,\mathrm{d}s + z(0) = b + \int_0^t f(s)\,\mathrm{d}s, \tag{11}$$

then

$$z(t) = b + \int_0^t \frac{\partial\Omega}{\partial z}(0,s,\Omega(s,0,z(s)))f(s)\,\mathrm{d}s. \tag{12}$$

Proof. Take the derivative of (11) with respect to t and then apply $(\partial\Omega/\partial z)(0,t,\Omega(t,0,z(t)))$ to both sides. Finally, integrating the resulting differential system from 0 to t and observing that from (11) we have $z(0) = b$ one has (12). □

Proof of theorem 1. Let x be a solution of the equation

$$F_\varepsilon(x) = 0. \tag{13}$$

Thus, x is a T-periodic solution to (7). Consider the change of variable

$$x(t) = \Omega(t,0,z(t)), \qquad t \in [0,T] \tag{14}$$

with inverse given by

$$z(t) = \Omega(0, t, x(t)), \qquad t \in [0, T]. \tag{15}$$

Observe that if x is a solution of (13) then it is differentiable; in fact, for any $t \in [0, T]$, we have

$$x(t) = x(T) + \int_0^t (\psi(\tau, x(\tau)) + \varepsilon^2\phi_1(\tau, x(\tau)) + \varepsilon^3\phi_2(\tau, x(\tau), \varepsilon))\, d\tau. \tag{16}$$

Therefore, from (15) z is also differentiable. Consider

$$\frac{d}{dt}\Omega(t, 0, z(t)) = \frac{\partial\Omega}{\partial t}(t, 0, z(t)) + \frac{\partial\Omega}{\partial z}(t, 0, z(t))\dot{z}(t). \tag{17}$$

Since

$$\frac{\partial\Omega}{\partial t}(t, 0, z(t)) = \psi(t, \Omega(t, 0, z(t)))$$

from (17) we have that

$$\Omega(t, 0, z(t)) - z(0) = \int_0^t \psi(s, \Omega(s, 0, z(s)))\, ds + \int_0^t \frac{\partial\Omega}{\partial z}(s, 0, z(s))\dot{z}(s)\, ds$$

or equivalently,

$$\Omega(t, 0, z(t)) - \int_0^t \psi(s, \Omega(s, 0, z(s)))\, ds = z(0) + \int_0^t \frac{\partial\Omega}{\partial z}(s, 0, z(s))\dot{z}(s)\, ds. \tag{18}$$

Using (14), (18) and lemma 2 with $b = \Omega(T, 0, z(T))$ we can rewrite (16) in the following form:

$$G_\varepsilon(z)(t) = 0, \tag{19}$$

where $G_\varepsilon : C([0, T], \mathbb{R}^n) \to C([0, T], \mathbb{R}^n)$ is given by

$$G_\varepsilon(z)(t) = z(t) - \Omega(T, 0, z(T)) - \int_0^t (\varepsilon^2\Phi_1(\tau, z(\tau)) + \varepsilon^3\Phi_2(\tau, z(\tau), \varepsilon))\, d\tau.$$

Therefore, the solutions of equation (13) belonging to the set $W(T, U)$ correspond to the solutions of equation (19) belonging to the set

$$Z = \{z \in C([0, T], \mathbb{R}^n) : z(t) \in U, \text{ whenever } t \in [0, T]\}$$

and by the homeomorphism theorem for compact vector fields (see [10], theorem 26.4) to prove theorem 1 it is enough to show that

$$\deg(G_\varepsilon, Z) = \deg(\eta_2(T, 0, \cdot), U) \tag{20}$$

for $\varepsilon > 0$ sufficiently small.

For this, consider the compact vector field $\tilde{G}_\varepsilon : C([0, T], \mathbb{R}^n) \to C([0, T], \mathbb{R}^n)$ defined as follows

$$\tilde{G}_\varepsilon = I - A_\varepsilon,$$

where

$$A_\varepsilon(z)(t) = \Omega(T, 0, z(T)) + \int_0^T (\varepsilon^2\Phi_1(\tau, z(\tau)) + \varepsilon^3\Phi_2(\tau, z(\tau), \varepsilon))\, d\tau$$

for any $t \in [0, T]$; hence, $A_\varepsilon(z)$ is a constant function in $C([0, T], \mathbb{R}^n)$. Let us show that for $\varepsilon > 0$ sufficiently small the compact vector fields G_ε and $\tilde{G}_\varepsilon$ are homotopic on the boundary of the set Z. To this aim, for $\lambda \in [0, 1]$, we define the following homotopy:

$$\Delta_\varepsilon(\lambda, z)(t) = z(t) - \Omega(T, 0, z(T)) - \int_0^{\lambda t + (1-\lambda)T} (\varepsilon^2\Phi_1(\tau, z(\tau)) + \varepsilon^3\Phi_2(\tau, z(\tau), \varepsilon))\, d\tau,$$

whenever $t \in [0, T]$, which deforms the vector field G_ε to the vector field $\tilde{G}_\varepsilon$. Let us show that Δ_ε does not vanish on the boundary of the set Z for $\varepsilon > 0$ sufficiently small.

Let us assume the contrary. Therefore, there exists a sequence $\{\varepsilon_k\}_{k=1}^{\infty} \subset (0, 1]$ such that $\varepsilon_k \to 0$ as $k \to \infty$ and sequences $\{\lambda_k\}_{k=1}^{\infty} \subset [0, 1]$ and $\{z_k\}_{k=1}^{\infty} \subset \partial Z$ such that

$$z_k(t) = \Omega(T, 0, z_k(T)) + \varepsilon_k^2 \int_0^{\lambda_k t + (1-\lambda_k)T} \Phi_1(\tau, z_k(\tau))\,\mathrm{d}\tau + \varepsilon_k^3 \int_0^{\lambda_k t + (1-\lambda_k)T} \Phi_2(\tau, z_k(\tau), \varepsilon_k)\,\mathrm{d}\tau, \qquad t \in [0, T]. \tag{21}$$

Without loss of generality we can assume that $\lambda_k \to \lambda_0$ and $z_k \to z_0$ in $C([0, T], \mathbb{R}^n)$ as $k \to \infty$. Therefore, $\lambda_0 \in [0, 1]$ and $z_0 \in \partial Z$. Furthermore, there exists a sequence $\{t_k\}_{k=1}^{\infty}$ such that $z_k(t_k) \in \partial U$ and by condition (A1)

$$\Omega(T, 0, z_k(t_k)) = z_k(t_k), \qquad \text{for any } k \in \mathbb{N}. \tag{22}$$

By subtracting from (21) (where t is replaced by T) the same equation with $t = t_k$ we obtain

$$z_k(T) - z_k(t_k) = \varepsilon_k^2 \int_{\lambda_k t_k + (1-\lambda_k)T}^{T} \Phi_1(\tau, z_k(\tau))\,\mathrm{d}\tau + \varepsilon_k^3 \int_{\lambda_k t_k + (1-\lambda_k)T}^{T} \Phi_2(\tau, z_k(\tau), \varepsilon_k)\,\mathrm{d}\tau. \tag{23}$$

By (22) the equation (21) where $t = T$ can be rewritten as

$$z_k(T) - z_k(t_k) = \Omega(T, 0, z_k(T)) - \Omega(T, 0, z_k(t_k)) + \varepsilon_k^2 \int_0^T \Phi_1(\tau, z_k(\tau))\,\mathrm{d}\tau + \varepsilon_k^3 \int_0^T \Phi_2(\tau, z_k(\tau), \varepsilon_k)\,\mathrm{d}\tau$$

or equivalently

$$\left(I - \frac{\partial\Omega}{\partial z}(T, 0, z_k(t_k))\right)(z_k(T) - z_k(t_k)) = \frac{\partial^2\Omega}{\partial z^2}(T, 0, z_k(t_k))(z_k(T) - z_k(t_k))(z_k(T) - z_k(t_k)) + \varepsilon_k^2 \int_0^T \Phi_1(\tau, z_k(\tau))\,\mathrm{d}\tau + \varepsilon_k^3 \int_0^T \Phi_2(\tau, z_k(\tau), \varepsilon_k)\,\mathrm{d}\tau + \mathrm{o}(z_k(t_k), z_k(T) - z_k(t_k)), \tag{24}$$

where the function $\mathrm{o}(\xi, h)$ satisfies

$$\frac{\|\mathrm{o}(\xi, h)\|}{\|h\|^2} \to 0, \qquad \text{as } \|h\| \to 0, \text{ with } h, \xi \in \mathbb{R}^n. \tag{25}$$

Replacing (23) in (24) and dividing by $\varepsilon_k^3 > 0$ after a suitable transformation we obtain

$$\frac{1}{\varepsilon_k} Q_{1k}(z_k) + Q_{2k}(z_k) = P_k + \frac{1}{\varepsilon_k^3}\,\mathrm{o}(z_k(t_k), z_k(T) - z_k(t_k)), \tag{26}$$

where

$$P_k = \varepsilon_k \frac{\partial^2\Omega}{\partial z^2}(T, 0, z_k(t_k)) \times \left(\int_0^{\lambda_k t + (1-\lambda_k)T} \Phi_1(\tau, z_k(\tau))\,\mathrm{d}\tau + \varepsilon_k \int_0^{\lambda_k t + (1-\lambda_k)T} \Phi_2(\tau, z_k(\tau), \varepsilon_k)\,\mathrm{d}\tau\right) \times \left(\int_0^{\lambda_k t + (1-\lambda_k)T} \Phi_1(\tau, z_k(\tau))\,\mathrm{d}\tau + \varepsilon_k \int_0^{\lambda_k t + (1-\lambda_k)T} \Phi_2(\tau, z_k(\tau), \varepsilon)\,\mathrm{d}\tau\right)$$

and

$$Q_{1k}(z_k) = \left(I - \frac{\partial\Omega}{\partial z}(T, 0, z_k(t_k))\right)\int_{\lambda_k t_k+(1-\lambda_k)T}^{T} \Phi_1(\tau, z_k(\tau))\,\mathrm{d}\tau - \int_0^T \Phi_1(\tau, z_k(\tau))\,\mathrm{d}\tau,$$

$$Q_{2k}(z_k) = \left(I - \frac{\partial\Omega}{\partial z}(T, 0, z_k(t_k))\right)\int_{\lambda_k t_k+(1-\lambda_k)T}^{T} \Phi_2(\tau, z_k(\tau), \varepsilon_k)\,\mathrm{d}\tau - \int_0^T \Phi_2(\tau, z_k(\tau), \varepsilon_k)\,\mathrm{d}\tau.$$

It is easy to see that

$$P_k \to 0 \qquad \text{as } k \to \infty \tag{27}$$

and

$$\frac{1}{\varepsilon_k^3}\mathrm{o}(z_k(t_k), z_k(T) - z_k(t_k)) \to 0 \qquad \text{as } k \to \infty. \tag{28}$$

Let us show that

$$\frac{1}{\varepsilon_k} Q_{1k}(z_k) \to 0 \qquad \text{as } k \to \infty. \tag{29}$$

We have

$$\frac{1}{\varepsilon_k} Q_{1k}(z_k) = \frac{Q_{1k}(z_k) - Q_{1k}(c_k) + Q_{1k}(c_k)}{\varepsilon_k} = \frac{\partial Q_{1k}}{\partial z}(c_k)\left(\frac{z_k - c_k}{\varepsilon_k}\right) + \frac{Q_{1k}(c_k)}{\varepsilon_k} + \frac{\mathrm{o}_k(z_k - c_k)}{\varepsilon_k}, \tag{30}$$

where by c_k we denote the constant function $c_k(t) \equiv z_k(t_k)$ for any $t \in [0, T]$, and the function $\mathrm{o}_k(\cdot)$ satisfies

$$\frac{\|\mathrm{o}_k(h)\|}{\|h\|} \to 0, \qquad \text{as } \|h\| \to 0, \quad h \in \mathrm{C}([0, T], \mathbb{R}^n). \tag{31}$$

Using (23)

$$\|z_k(t) - c_k\| \leqslant \varepsilon_k^2 M \tag{32}$$

for any $t \in [0, T]$, where $M > 0$ is a constant, and so the first and third terms in (30) tend to zero as k tends to ∞. Let us prove that

$$Q_{1k}(c_k) = 0, \qquad k \in \mathbb{N}. \tag{33}$$

By lemma 1

$$\left(I - \frac{\partial\Omega}{\partial z}(T, 0, \xi)\right)\int_s^T \Phi_1(\tau, \xi)\,\mathrm{d}\tau - \int_0^T \Phi_1(\tau, \xi)\,\mathrm{d}\tau = \eta_1(0, s, \xi) - \eta_1(T, s, \xi) \tag{34}$$

and by condition (A2) we obtain (33). Thus, (29) holds true and by (27)–(29) we can pass to the limit in (26), obtaining

$$Q_2(z_0) = 0, \tag{35}$$

where $Q_2 = \lim_{k\to\infty} Q_{2k}$. On the other hand by lemma 1 we obtain a result analogous to (34) for Φ_2; thus,

$$Q_2(z_0) = \eta_2(0, s, z_0(t_0)) - \eta_2(T, s, z_0(t_0)), \tag{36}$$

where $s = \lim_{k\to\infty}(\lambda_k t_k + (1 - \lambda_k)T)$. Hence

$$\eta_2(0, s, z_0(t_0)) - \eta_2(T, s, z_0(t_0)) = 0, \tag{37}$$

which contradicts assumption (A3) since $z_0(t_0) \in \partial U$. Therefore, there exists $\varepsilon_0 > 0$ such that

$$\Delta_\varepsilon(\lambda, z) \neq 0, \qquad \lambda \in [0, 1], \quad z \in \partial Z, \quad \varepsilon \in (0, \varepsilon_0) \tag{38}$$

and so

$$\deg(G_\varepsilon, Z) = \deg(\tilde{G}_\varepsilon, Z), \qquad \varepsilon \in (0, \varepsilon_0). \tag{39}$$

Denote by $C_0([0, T], \mathbb{R}^n)$ the subspace of the space $C([0, T], \mathbb{R}^n)$ consisting of all constant functions defined on $[0, T]$ with values in $\mathbb{R}^n$. We have $A_\varepsilon(\partial Z) \subset C_0([0, T], \mathbb{R}^n)$. From (38), with $\lambda = 0$, we have

$$\tilde{G}_\varepsilon(z) \neq 0, \qquad z \in \partial Z, \quad \varepsilon \in (0, \varepsilon_0)$$

and using the inclusion $\partial(Z \cap C_0([0, T], \mathbb{R}^n)) \subset \partial Z$ we conclude that the vector field $\tilde{G}_\varepsilon$ does not have zeros on the boundary of the set $Z \cap C_0([0, T], \mathbb{R}^n)$ when $\varepsilon \in (0, \varepsilon_0)$. Therefore, by the reduction property of the topological degree (see [10], theorem 27.1), we have

$$\deg_{C([0,T],\mathbb{R}^n)}(\tilde{G}_\varepsilon, Z) = \deg_{C_0([0,T],\mathbb{R}^n)}(\tilde{G}_\varepsilon, Z \cap C_0([0, T], \mathbb{R}^n)). \tag{40}$$

Furthermore, since the constant function $z \in Z \cap C_0([0, T], \mathbb{R}^n)$ is a solution of the equation $\tilde{G}_\varepsilon(z) = 0$ if and only if the element $\xi \in U$, given by $\xi = z(t)$, for any $t \in [0, T]$, is a solution of the equation $K_\varepsilon \xi = 0$, where $K_\varepsilon : U \to \mathbb{R}^n$ is defined as follows

$$K_\varepsilon \xi = \xi - \Omega(T, 0, \xi) - \varepsilon^2 \int_0^T \Phi_1(\tau, \xi)\, d\tau - \varepsilon^3 \int_0^T \Phi_2(\tau, \xi, \varepsilon)\, d\tau,$$

then

$$\deg_{C_0([0,T],\mathbb{R}^n)}(\tilde{G}_\varepsilon, Z \cap C_0([0, T], \mathbb{R}^n)) = \deg_{\mathbb{R}^n}(K_\varepsilon, U). \tag{41}$$

Consider now $K_{\varepsilon,2} : U \to \mathbb{R}^n$ defined by

$$K_{\varepsilon,2}\xi = -\varepsilon^3 \int_0^T \Phi_2(\tau, \xi, \varepsilon)\, d\tau.$$

By condition (A1), i.e. $\xi - \Omega(T, 0, \xi) = 0$, for any $\xi \in \partial U$, lemma 1 and assumption (A2) we obtain

$$\begin{aligned} -\int_0^T \Phi_1(\tau, \xi)\, d\tau &= \int_T^0 \Phi_1(\tau, \xi)\, d\tau = \eta_1(0, T, \xi) \\ &= \eta_1(0, T, \xi) - \eta_1(T, T, \xi) = 0, \qquad \text{whenever } \xi \in \partial U. \end{aligned}$$

Therefore $K_\varepsilon\, \xi = K_{\varepsilon,2}\xi$, whenever $\xi \in \partial U$ and thus

$$\deg_{\mathbb{R}^n}(K_\varepsilon, U) = \deg_{\mathbb{R}^n}(K_{\varepsilon,2}, U), \qquad \text{for any } \varepsilon \in (0, \varepsilon_0). \tag{42}$$

Finally, consider the vector field $\tilde{K}_{\varepsilon,2} : U \to \mathbb{R}^n$ defined as follows:

$$\tilde{K}_{\varepsilon,2}\xi = -\int_0^T \Phi_2(\tau, \xi, \varepsilon)\, d\tau.$$

By condition (A2) we have

$$-\int_0^T \Phi_2(\tau, \xi, 0)\, d\tau = \eta_2(0, T, \xi) - \eta_2(T, T, \xi) \neq 0, \qquad \text{for any } \xi \in \partial U. \tag{43}$$

Therefore, there exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that

$$-\int_0^T \Phi_2(\tau, \xi, \varepsilon)\, d\tau \neq 0, \qquad \text{for any } \xi \in \partial U \text{ and any } \varepsilon \in (0, \varepsilon_1)$$

and so the vector fields $K_{\varepsilon,2}$ and $\tilde{K}_{\varepsilon,2}$ are linearly homotopic on the boundary of the set U for any $\varepsilon \in (0, \varepsilon_1)$. Moreover, by condition (43) the vector fields $\tilde{K}_{\varepsilon,2}$ and $\tilde{K}_{0,2}$ are linearly homotopic on ∂U for any $\varepsilon \in (0, \varepsilon_1)$. Therefore,

$$\begin{aligned}\deg(K_{\varepsilon,2}, U) &= \deg(\eta_2(0, T, \xi) - \eta_2(T, T, \xi), U)\\ &= \deg(\eta_2(T, 0, \xi) - \eta_2(0, 0, \xi), U) = \deg(\eta_2(T, 0, \xi), U). \end{aligned} \tag{44}$$

Summarizing (39)–(42) and (44) we obtain (10), which proves theorem 1. □

Consider now the system

$$\dot{x} = \psi(t, x) + \varepsilon\phi(t, x, \varepsilon) \tag{45}$$

and

$$F_\varepsilon(x)(t) = x(t) - x(T) - \int_0^t (\psi(\tau, x(\tau)) + \varepsilon\phi(\tau, x(\tau), \varepsilon))\, \mathrm{d}\tau.$$

Denote by η the solution of the Cauchy problem (9) associated with (45).

As a straightforward consequence of theorem 1 we have the following theorem.

Theorem 2. *Let $U \subset \mathbb{R}^n$ be an open and bounded set. Assume that*

(A1) $\Omega(T, 0, \xi) = \xi$, for any $\xi \in \partial U$,
(A2) $\eta(T, s, \xi) - \eta(0, s, \xi) \neq 0$, for any $s \in [0, T]$ and any $\xi \in \partial U$.

Then, for $\varepsilon > 0$, sufficiently small

$$\deg(F_\varepsilon, W(T, U)) = \deg(\eta(T, 0, \cdot), U). \tag{46}$$

Observe that if $\deg(\eta(T, 0, \cdot), U) \neq 0$ then theorem 1 of [8], concerning the existence of T-periodic solutions of system (45), follows from theorem 2.

3. Small perturbations of autonomous systems

In this section by using the earlier results we provide sufficient conditions to ensure that the topological degree of the integral operator associated to the following system in $\mathbb{R}^2$,

$$\dot{x} = Ax + \varepsilon^2\phi_1(x) + \varepsilon^3\phi_2(t, x, \varepsilon), \tag{47}$$

is different from zero for $\varepsilon > 0$ sufficiently small.

System (47) is regarded here as the perturbed system of the autonomous system

$$\dot{x} = Ax + \varepsilon^2\phi_1(x) \tag{48}$$

by means of the T-periodic perturbation $\varepsilon^3\phi_2(t, x, \varepsilon)$, where A is a 2×2 matrix. At $\varepsilon = 0$ we have the linear system

$$\dot{x} = Ax. \tag{49}$$

We assume, that

(A1) the matrix A has eigenvalues $\mathrm{i}\lambda$ and $-\mathrm{i}\lambda$, where $\lambda > 0$.

Let x_0 be a non-zero periodic solution of system (49), which is the boundary of an open set U_0 of $\mathbb{R}^2$. Observe that such a periodic solution exists by virtue of (A1). Moreover, since the period T of any periodic solution of system (49) is equal to $2\pi/\lambda$, we assume that the function ϕ_2 is $(2\pi/\lambda)$-periodic with respect to the first variable.

Denote by U_δ the open sets whose boundaries are given by the trajectories $(1 + \delta)x_0$ for $\delta \in \mathbb{R}_+$. Since x_0 is a $(2\pi/\lambda)$-periodic solution of system (49) then $(1 + \delta)x_0$, $\delta \in \mathbb{R}_+$, is also a $(2\pi/\lambda)$-periodic solution of this system. Moreover, we have $U_0 \subset U_\delta$.

Define Ω, η_1, η_2 and F_ε as in the previous section with $\psi(t, \xi) = A\xi$ and $T = (2\pi/\lambda)$.

We have the following result.

Theorem 3. *Assume condition (A1). Moreover, assume that for* $\delta \in (0, 1)$ *we have*

(A2) $\eta_1(T, s, \xi) - \eta_1(0, s, \xi) = 0$, *for any* $s \in [0, T]$ *and any* $\xi \in \partial U_0$,
(A3) $\eta_2(T, s, \xi) - \eta_2(0, s, \xi) \neq 0$, *for any* $s \in [0, T]$ *and any* $\xi \in \partial U_0$,
(A4) $\eta_1(T, s, \xi) - \eta_1(0, s, \xi) \neq 0$, *for any* $s \in [0, T]$ *and any* $\xi \in \partial U_\delta$.

Then, for all $\varepsilon > 0$ *sufficiently small*

$$\deg(F_\varepsilon, W(T, U_\delta)\backslash \bar{W}(T, U_0)) = (\deg(\eta_1(T, 0, \cdot), U_\delta) - \deg(\eta_2(T, 0, \cdot), U_0)). \tag{50}$$

Proof. First of all observe that the set $W(T, U_\delta)\backslash \bar{W}(T, U_0)$ is well defined since $U_0 \subset U_\delta$. Moreover, from (A1)–(A3) it follows that theorem 1 is applicable with $U = U_0$ and so for $\varepsilon > 0$ sufficiently small we obtain

$$\deg(F_\varepsilon, W(T, U_0)) = \deg(\eta_2(T, 0, \cdot), U_0). \tag{51}$$

Finally, (A1) and (A4) imply that theorem 2 is applicable with $U = U_\delta$ and so for $\varepsilon > 0$ sufficiently small we obtain

$$\deg(F_\varepsilon, W(T, U_\delta)) = \deg(\eta_1(T, 0, \cdot), U_\delta). \tag{52}$$

Thus, (51) and (52) ensure (50) with $\varepsilon > 0$ sufficiently small. □

In what follows we prove that if we perturb (48) by a nonlinear T_ε-periodic perturbation, where

$$T_\varepsilon \to T \qquad \text{as } \varepsilon \to 0, \tag{53}$$

then (47) has T_ε-periodic solutions. In the case when all the powers of ε on the right-hand side of (47) are the same and $T_\varepsilon \neq T$ this result is known and is called the frequency pulling phenomenon (see [1]). Thus, theorem 4 shows that this phenomenon occurs also in the case of different powers of ε. As T_ε we consider here $T_{\varepsilon,\mu} = 2\pi/\lambda(1 + \varepsilon^3\mu)$, where μ is a scaling parameter. Precisely, we consider the system

$$\dot{x} = Ax + \varepsilon^2\phi_1(x) + \varepsilon^3\phi_2\left(\frac{t}{1 + \varepsilon^3\mu}, x, \varepsilon, \mu\right). \tag{54}$$

Denote by $F_{\varepsilon,\mu}$ the integral operator associated with (54) and denote by $\eta_i(\cdot, s, \xi)$, $i = 1, 2$, the solution of (9) corresponding to (54) for $\mu = 0$.

We can formulate the following result.

Theorem 4. *Assume that for* $\mu = 0$ *and* $T = 2\pi/\lambda$ *all the conditions of theorem 3 are satisfied. Then, there exists* $\mu_0 > 0$ *such that, for every* $\mu \in (-\mu_0, \mu_0)$, *there is an* $\varepsilon_0 > 0$ *such that*

$$\begin{aligned}&\deg(F_{\varepsilon,\mu}, W(T_{\varepsilon,\mu}, U_\delta)\backslash \bar{W}(T_{\varepsilon,\mu}, U_0)) \\ &\qquad = (\deg(\eta_1(T_{\varepsilon,\mu}, 0, \cdot), U_\delta) - \deg(\eta_2(T_{\varepsilon,\mu}, 0, \cdot), U_0))\end{aligned} \tag{55}$$

for any $\varepsilon \in (0, \varepsilon_0)$.

Proof. In (54) consider the change of variable

$$y(t) = x(t(1 + \varepsilon^3\mu)). \tag{56}$$

Thus, we obtain the system

$$\dot{y} = Ay + \varepsilon^2\phi_1(y) + \varepsilon^3\phi_{2,\mu}(t, y, \varepsilon), \tag{57}$$

where

$$\phi_{2,\mu}(t, \xi, \varepsilon) = \phi_2(t, \xi, \varepsilon, \mu) + \mu A\xi + \varepsilon^2\mu\phi_1(\xi) + \varepsilon^3\mu\, \phi_2(t, \xi, \varepsilon, \mu).$$

Since the conditions of theorem 3 hold true for (57) when $\mu = 0$, there exists $\mu_0 > 0$ such that the conditions of theorem 3 hold true for (57) for every fixed $\mu \in (-\mu_0, \mu_0)$. Observe, that the solution of problem (9) with $i = 1$ associated with (54) and the one associated with (57) coincide. Denote by $\eta_{2,\mu}$ the solution of the problem (9) with $i = 2$ associated with (57) and denote by $G_{\varepsilon,\mu}$ the compact operator corresponding to the T-periodic problem for the system (57). By theorem 3, for every $\mu \in (-\mu_0, \mu_0)$, there exists $\varepsilon_0 > 0$ such that

$$\deg(G_{\varepsilon,\mu}, W(T, U_\delta)\backslash\bar{W}(T, U_0)) = (\deg(\eta_1(T, 0, \cdot), U_\delta) - \deg(\eta_{2,\mu}(T, 0, \cdot), U_0)) \tag{58}$$

for $\varepsilon \in (0, \varepsilon_0)$. Since

$$\eta_{2,\mu}(T, 0, \xi) \to \eta_2(T, 0, \xi) \qquad \text{as } \mu \to 0, \quad \text{for any } \xi \in \mathbb{R}^2,$$

we may assume without loss of generality that $\mu_0 > 0$ is chosen in such a way that

$$\deg(\eta_{2,\mu}(T, 0, \cdot), U_0) = \deg(\eta_2(T, 0, \cdot), U_0), \qquad \text{for any } \mu \in (-\mu_0, \mu_0). \tag{59}$$

By (56) the zeros of the operator $G_{\varepsilon,\mu}$ belonging to $W(T, U)$ correspond to the zeros of the operator $F_{\varepsilon,\mu}$ belonging to $W(T_{\varepsilon,\mu}, U)$. Therefore,

$$\deg(F_{\varepsilon,\mu}, W(T_{\varepsilon,\mu}, U_\delta)\backslash\bar{W}(T_{\varepsilon,\mu}, U_0)) = \deg(G_{\varepsilon,\mu}, W(T, U_\delta)\backslash\bar{W}(T, U_0))$$

and by taking into account (58) and (59) we obtain (55). □

4. An application to the existence of periodic solutions of the van der Pol equation

We conclude this paper by illustrating a topological degree approach, based on the above results, to investigate the existence of periodic solutions of the van der Pol equation

$$\ddot{x} - \varepsilon(1 - x^2)\dot{x} + x = 0. \tag{60}$$

Consider the compact operator $F_\varepsilon : \mathrm{C}([0, 2\pi], \mathbb{R}^2) \to \mathrm{C}([0, 2\pi], \mathbb{R}^2)$ given by

$$F_\varepsilon(x)(t) = x(t) - x(2\pi) - \int_0^t \begin{pmatrix} x_2(\tau) \\ -x_1(\tau) + \varepsilon(1 - x_1^2(\tau))x_2(\tau) \end{pmatrix} \mathrm{d}\tau, \tag{61}$$

whose zeros $x = (x_1, x_2)$ correspond to the 2π-periodic solution $(x, \dot{x})$ of system (60). It is known (see for instance [11], section 4.6), that for $\varepsilon > 0$ sufficiently small the operator F_ε has a zero $x_\varepsilon \in C([0, 2\pi], \mathbb{R}^2)$ such that

$$\|x_\varepsilon\|_{C_{2\pi}} \to 2 \qquad \text{as } \varepsilon \to 0.$$

Here $\|\cdot\|_{C_{2\pi}}$ denotes the norm in the Banach space $C([0, 2\pi], \mathbb{R}^2)$.

We have the following result.

Proposition 1. *For $\varepsilon > 0$ sufficiently small and $\delta \in (0, 2)$*

$$\operatorname{ind}(F_\varepsilon, W(2\pi, U_\delta)) = 0, \tag{62}$$

where

$$U_\delta = \{\xi \in \mathbb{R}^2 : \|\xi\| \in (2 - \delta, 2 + \delta)\}.$$

Proof. The proposition is a straightforward consequence of theorem 2. In fact, if we put

$$\psi(t, \xi) = \begin{pmatrix} \xi_2 \\ -\xi_1 \end{pmatrix} \qquad \text{and} \qquad \phi(t, \xi, \varepsilon) = \begin{pmatrix} 0 \\ (1 - \xi_1^2)\xi_2 \end{pmatrix}$$

then, as can be verified, we have that

$$\Omega(t, t_0, \xi) = \begin{pmatrix} \cos(t - t_0) & \sin(t - t_0) \\ -\sin(t - t_0) & \cos(t - t_0) \end{pmatrix} \xi,$$

$$\eta(2\pi, s, \xi) - \eta(0, s, \xi) = \left(\pi - \left(\frac{\pi}{4}\right)(\xi_1^2 + \xi_2^2)\right) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

and so (A1) and (A2) of theorem 2 are satisfied. □

On the basis of the approach presented in section 2 we now perturb the van der Pol equation by means of a higher order (with respect to ε) non-autonomous term to obtain, for $\varepsilon > 0$ sufficiently small, an integral operator with non-zero topological degree in a suitable open set.

In fact, consider the following perturbed system obtained from (60) by adding the forcing term $\varepsilon\sqrt{\varepsilon}\sin(t/(1+\varepsilon\sqrt{\varepsilon}\mu))$:

$$\ddot{x} - \varepsilon(1-x^2)\dot{x} + x + \varepsilon\sqrt{\varepsilon}\sin\left(\frac{t}{1+\varepsilon\sqrt{\varepsilon}\mu}\right) = 0. \tag{63}$$

Consider the following operator $F_{\varepsilon,\mu} : \mathrm{C}([0,2\pi],\mathbb{R}^2) \to \mathrm{C}([0,2\pi],\mathbb{R}^2)$ associated to (63):

$$\begin{aligned} F_{\varepsilon,\mu}(x)(t) = x(t) - x\left(\frac{2\pi}{1+\varepsilon\sqrt{\varepsilon}\mu}\right) \\ - \int_0^t \begin{pmatrix} x_2(\tau) \\ -x_1(\tau) + \varepsilon(1-x_1^2(\tau))x_2(\tau) - \varepsilon\sqrt{\varepsilon}\sin\left(\frac{\tau}{1+\varepsilon\sqrt{\varepsilon}\mu}\right) \end{pmatrix} \mathrm{d}\tau. \end{aligned} \tag{64}$$

Let

$$\phi_1(\xi) = \begin{pmatrix} 0 \\ (1-\xi_1^2)\xi_2 \end{pmatrix}, \qquad \phi_2(t,\xi,\varepsilon) = \begin{pmatrix} 0 \\ -\sin t \end{pmatrix} \qquad \text{and} \qquad T_{\varepsilon,\mu} = \frac{2\pi}{1+\varepsilon\sqrt{\varepsilon}\mu}$$

define the corresponding functions η_1 and η_2.

We are now in a position to formulate the following result.

Proposition 2. *There exists $\mu_0 > 0$ such that for every $\mu \in (-\mu_0, \mu_0)$ and $\delta \in (0,2)$ there exists $\varepsilon_0 > 0$ such that*

$$\deg(F_{\varepsilon,\mu}, W(T_{\varepsilon,\mu}, U_\delta)\backslash\bar{W}(T_{\varepsilon,\mu}, U_0)) = 1 \tag{65}$$

for any $\varepsilon \in (0, \varepsilon_0)$.

Proof. As already observed

$$\eta_1(2\pi, s, \xi) - \eta_1(0, s, \xi) = \left(\pi - \left(\frac{\pi}{4}\right)(\xi_1^2+\xi_2^2)\right)\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} &\eta_1(2\pi, s, \xi) - \eta_1(0, s, \xi) = 0, && \text{for any } s \in [0,2\pi] \text{ and any } \xi \in \partial U_0, \\ &\eta_1(2\pi, s, \xi) - \eta_1(0, s, \xi) \neq 0, && \text{for any } s \in [0,2\pi] \text{ and any } \xi \in \partial U_\delta. \end{aligned}$$

Moreover

$$\deg(\eta_1(2\pi, 0, \cdot), U_\delta) = 1. \tag{66}$$

It is also easy to see that

$$\eta_2(2\pi, s, \xi) - \eta_2(0, s, \xi) = \begin{pmatrix} 0 \\ -\frac{3\pi}{4} \end{pmatrix}$$

and so

$$\deg(\eta_2(2\pi, 0, \cdot), U_0) = 0. \tag{67}$$

By (66), (67) and theorem 4 we obtain (65). □

From a physical point of view the term $\varepsilon\sqrt{\varepsilon}\sin(w_{\varepsilon,\mu}t)$ is equivalent to an external voltage (see [1]). Hence, we have proved that $w_{\varepsilon,\mu} = 1/(1+\varepsilon\sqrt{\varepsilon}\mu)$ belongs to the frequency pulling range for the forced van der Pol oscillator described by (63). This phenomenon is very useful in radio engineering; the classical form of the external voltage is $\varepsilon\sin(w_\varepsilon t)$ and thus our approach allows us to save electricity.

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