

PERIODIC SOLUTIONS FOR A CLASS OF SINGULARLY PERTURBED SYSTEMS [†]

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Abstract. In this paper we provide conditions to ensure the existence, for $\epsilon > 0$ sufficiently small, of periodic solutions of given period $T > 0$ for a class of singularly perturbed first order differential systems. Here $\epsilon > 0$ is the perturbation parameter. Our approach, based on the topological degree theory and the averaging theory, permits to weaken the conditions in ([5], Theorem 2) under which the existence of periodic solutions is proved.

Keywords: periodic solutions, averaging method, singular perturbations, topological degree.

AMS (MOS) subject classification: 34C25, 34C29, 34D15, 47H11.

Introduction

The starting point for the present work is the paper by K. Schneider [5], devoted to the extension of the theory of vibrational stabilizability to singularly perturbed first order control problems. A basic tool for such extension is represented by the averaging theory. In order to apply this theory the author assumes that an appropriate coordinate transformation of the slow variable is a periodic diffeomorphism of fixed period. Indeed, this transformation reduces the considered system to the standard form for the application of the classical averaging principle, (see, for instance, [2]); this, in turn, permits to prove the existence of periodic solutions for sufficiently small values of the perturbation parameter $\epsilon > 0$. The change of variable is introduced by means of a differential equation which is supposed to have a T -periodic solution for any initial condition in a suitable ball (assumption (A₃) of [5]).

Following [1], [4] and [6] we propose here an approach which combines the topological degree theory and the averaging theory. This approach allows us to assume the periodicity condition for the coordinate transformation only on the boundary of a given domain of the state space of the slow variable and then to derive, for any $\epsilon > 0$ sufficiently small, the existence of a periodic solution for the system whose slow component is contained in the interior of the domain. We consider here the class of singularly perturbed systems introduced by Schneider in his paper. However, in this paper we do not look at the problem

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from the control point of view, but rather we treat the general problem of the existence of periodic solutions of the considered class of singularly perturbed first order differential systems. The main result together with the preliminaries and assumptions are presented in Section 1. In Section 2 we illustrate the main result by an example of singularly perturbed system in \mathbf{R}^3 in which the boundary of the above mentioned domain is a circumference, which represents the slow periodic solution of the system at $\epsilon = 0$.

Finally, we would like to remark that as a direct consequence of our result we obtain a result of the classical theory of ordinary differential equation ([3], Theorem 3.1, p. 362).

1. Assumptions and Results

In this paper we consider the following system of differential equations

$$\begin{cases} \dot{x}(t) = \epsilon\phi(t, x(t), y(t)) + \psi_1(t, x(t)), \\ \dot{y}(t) = \psi_2(t, x(t), y(t)) - Ay(t), \end{cases} \quad (1)$$

where $\phi : \mathbf{R} \times \mathbf{R}^k \times \mathbf{R}^m \rightarrow \mathbf{R}^k$, $\psi_1 : \mathbf{R} \times \mathbf{R}^k \rightarrow \mathbf{R}^k$, and $\psi_2 : \mathbf{R} \times \mathbf{R}^k \times \mathbf{R}^m \rightarrow \mathbf{R}^m$ are continuous functions, which are T -periodic with respect to time t . Moreover, ψ_1 is continuously differentiable with respect to the second variable x . A is a $m \times m$ matrix. We assume that the matrix A has not eigenvalues on the imaginary axis. Consequently the space \mathbf{R}^m can be represented as the direct sum $E_+ \oplus E_-$ of the eigenspaces E_{\pm} corresponding to the eigenvalues with positive and negative parts respectively. If we denote by A_+ and A_- the restrictions of A to E_+ and E_- respectively and by P_+ and P_- the projectors on E_+ and E_- then we have the following dichotomy

$$\|e^{-A_+t}P_+\| \leq ce^{-\delta t}, \quad \|e^{A_-t}P_-\| \leq ce^{-\delta t}, \quad t \geq 0, \quad (2)$$

for some $\delta > 0$. We introduce now in \mathbf{R}^m the norm given by

$$\|x\| = \max\{\|P_+x\|, \|P_-x\|\}.$$

We also assume that for every bounded set $B \subset \mathbf{R}^k$ there exists a constant $M(B)$ such that

$$\|\psi_2(t, x, y)\| \leq M(B) + \gamma\|y\|, \quad (3)$$

where $x \in B$ and

$$\gamma < \frac{\delta}{c}. \quad (4)$$

We denote by $\Omega(t, t_0, \xi)$ the solution of the Cauchy problem

$$\begin{cases} \dot{x}(t) = \psi_1(t, x(t)), & t \in [0, T], \\ x(t_0) = \xi. \end{cases} \quad (5)$$

We state the following property for the solution map $\xi \rightarrow \Omega(t, t_0, \xi)$, which will be useful in the sequel.

Lemma 1 *Let $z \in C^1([0, T], \mathbf{R}^m)$, $f \in C([0, T], \mathbf{R}^m)$ and $b \in \mathbf{R}^m$. If*

$$\int_0^t \frac{\partial \Omega}{\partial z}(s, 0, z(s)) \dot{z}(s) ds + z(0) = b + \int_0^t f(s) ds, \quad (6)$$

then

$$z(t) = b + \int_0^t \frac{\partial \Omega}{\partial z}(0, s, \Omega(s, 0, z(s))) f(s) ds. \quad (7)$$

Proof. Take the derivative of (6) with respect to t obtaining

$$\frac{\partial \Omega}{\partial z}(t, 0, z(t)) \dot{z}(t) = f(t). \quad (8)$$

Derive with respect to ξ the identity

$$\Omega(0, t, \Omega(t, 0, \xi)) = \xi$$

to obtain

$$\frac{\partial \Omega}{\partial z}(0, t, \Omega(t, 0, \xi)) \frac{\partial \Omega}{\partial z}(t, 0, \xi) = I, \quad (9)$$

whenever $\xi \in \mathbf{R}^k$. Applying now $\frac{\partial \Omega}{\partial z}(0, t, \Omega(t, 0, z(t)))$ to (8) we get

$$\dot{z}(t) = \frac{\partial \Omega}{\partial z}(0, t, \Omega(t, 0, z(t))) f(t).$$

Finally, integrating this equation from 0 to t and observing that from (6) we have $z(0) = b$, one has (7). \square

Let us now denote by $\eta_y(t, s, \xi)$ the solution of the Cauchy problem

$$\begin{cases} \dot{z}(t) = \frac{\partial \psi_1}{\partial x}(t, \Omega(t, 0, \xi)) z(t) + \phi(t, \Omega(t, 0, \xi), y(t)), \\ z(s) = 0. \end{cases} \quad (10)$$

where y is a T -periodic continuous function and $t, s \in [0, T]$. For $y = 0$ we write $\eta_0(t, s, \xi)$. In the sequel we will denote by $\|y\|_{C_T}$ the norm of y in the Banach space $C_T(\mathbf{R}^m)$ of T -periodic continuous functions.

We can state the following main result.

Theorem 1 *Assume that there exists an open bounded set $U \subset \mathbf{R}^k$ such that*

(A₁) $\Omega(T, 0, \xi) = \xi$ for all $\xi \in \partial U$;

(A₂) $\eta_y(0, s, \xi) \neq \eta_y(T, s, \xi)$ for all $s \in [0, T]$, $\xi \in \partial U$ and $\|y\|_{C_T} \leq \frac{cM}{\delta - \gamma c}$,
where $M = \sup_{t \in [0, T]} \sup_{\xi \in \bar{U}} \|\Omega(t, 0, \xi)\|$;

(A₃) $\deg(\eta_0(T, T, \cdot) - \eta_0(0, T, \cdot), U, 0) \neq 0$.

Then there exists $\epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0)$ system (1) has at least one T -periodic solution (x, y) such that

$$\Omega(0, t, x(t)) \in U \quad \text{for any } t \in [0, T] \quad \text{and} \quad \|y\|_{C_T} \leq \frac{cM}{\delta - \gamma c}.$$

For the proof of this theorem we need the following lemma.

Lemma 2 Let $\eta(t, s)$ be the solution of the problem

$$\begin{cases} \dot{z}(t) = A(t)z(t) + f(t), \\ z(s) = 0. \end{cases}$$

where $A(t)$ is a T -periodic continuous $k \times k$ matrix and f is a T -periodic continuous function taking values in \mathbf{R}^k . We have that

$$\eta(T, s) - \eta(0, s) = \int_{s-T}^s X^{-1}(\tau) f(\tau) d\tau$$

where $X(t)$ is a fundamental matrix of the linear system

$$\dot{z}(t) = A(t)z(t),$$

such that $X(0) = I$.

Proof of Lemma 2. Since $\eta(t, s) = \int_s^t X(t)X^{-1}(\tau)f(\tau)d\tau$, we have that

$$\eta(T, s) - \eta(0, s) = \int_s^T X(T)X^{-1}(\tau)f(\tau)d\tau - \int_s^0 X^{-1}(\tau)f(\tau)d\tau. \quad (11)$$

Make the change of variable $\tau = u + T$ in the integral

$$J = \int_s^T X(T)X^{-1}(\tau)f(\tau)d\tau,$$

obtaining

$$J = \int_{s-T}^0 X(T)X^{-1}(T+u)f(T+u)du = \int_{s-T}^0 \Phi(T)e^{\Lambda T}X^{-1}(T+u)f(u)du =$$

$$= \int_{s-T}^0 \Phi(T) e^{-\Lambda u} e^{\Lambda(T+u)} X^{-1}(T+u) f(u) du,$$

where $\Phi(t)e^{\Lambda t}$ is a Floquet representation of $X(t)$. Therefore $e^{\Lambda t} X^{-1}(t) = \Phi^{-1}(t)$, then

$$e^{\Lambda(T+u)} X^{-1}(T+u) = e^{\Lambda u} X^{-1}(u)$$

and

$$J = \int_{s-T}^0 \Phi(0) e^{-\Lambda u} e^{\Lambda u} X^{-1}(u) f(u) du = \int_{s-T}^0 X^{-1}(u) f(u) du.$$

Therefore from (11) we have

$$\begin{aligned} \eta(T, s) - \eta(0, s) &= \int_{s-T}^0 X^{-1}(\tau) f(\tau) d\tau + \int_0^s X^{-1}(\tau) f(\tau) d\tau = \\ &= \int_{s-T}^s X^{-1}(\tau) f(\tau) d\tau. \end{aligned} \quad (12)$$

Which is the claim. \square

Remark 1 From (9), (12) and the fact that $\frac{\partial \Omega}{\partial z}(t, 0, \xi)$ is the fundamental matrix of (10) with $\phi \equiv 0$, we can deduce the following formula

$$\eta_0(T, T, \xi) - \eta_0(0, T, \xi) = \int_0^T \frac{\partial \Omega}{\partial z}(0, \tau, \Omega(\tau, 0, \xi)) \phi(\tau, \Omega(\tau, 0, \xi), 0) d\tau$$

for $\xi \in \partial U$.

Proof of Theorem 1. The existence of T -periodic solutions for system (1) is equivalent to the existence of solution pairs $(x, y) \in C([0, T], \mathbf{R}^k) \times C_T(\mathbf{R}^m)$ of the following system

$$\left\{ \begin{array}{l} x(t) = x(T) + \epsilon \int_0^t \phi(s, x(s), y(s)) ds + \int_0^t \psi_1(s, x(s)) ds, \\ y_+(t) = e^{-A_+ t} (I - e^{-A_+ T})^{-1} \int_0^T e^{-A_+(T-s)} P_+ \psi_2(s, x(s), y(s)) ds + \\ \quad + \int_0^t e^{-A_+(t-s)} P_+ \psi_2(s, x(s), y(s)) ds, \\ y_-(t) = e^{A_-(T-t)} (e^{A_- T} - I)^{-1} \int_0^T e^{A_-(T-s)} P_- \psi_2(s, x(s), y(s)) ds - \\ \quad - \int_t^T e^{A_-(s-t)} P_- \psi_2(s, x(s), y(s)) ds, \end{array} \right. \quad (13)$$

where $y_+ = P_+ y$ and $y_- = P_- y$. Consider the change of variable

$$x(t) = \Omega(t, 0, z(t)), \quad t \in [0, T], \quad (14)$$

with inverse given by

$$z(t) = \Omega(0, t, x(t)), \quad t \in [0, T]. \quad (15)$$

Observe that if x is the first coordinate of the solution of system (13), then x is differentiable and therefore from (14) z is also differentiable. Consider

$$\frac{d}{dt}\Omega(t, 0, z(t)) = \frac{\partial\Omega}{\partial t}(t, 0, z(t)) + \frac{\partial\Omega}{\partial z}(t, 0, z(t))\dot{z}(t), \quad (16)$$

since

$$\frac{\partial\Omega}{\partial t}(t, 0, z(t)) = \psi_1(t, \Omega(t, 0, z(t))),$$

from (16) we have that

$$\Omega(t, 0, z(t)) - z(0) = \int_0^t \psi_1(s, \Omega(s, 0, z(s)))ds + \int_0^t \frac{\partial\Omega}{\partial z}(s, 0, z(s))\dot{z}(s)ds,$$

or equivalently,

$$\Omega(t, 0, z(t)) - \int_0^t \psi_1(s, \Omega(s, 0, z(s)))ds = z(0) + \int_0^t \frac{\partial\Omega}{\partial z}(s, 0, z(s))\dot{z}(s)ds. \quad (17)$$

Let (x, y) be a solution of system (13); by using (14), (17) and (7) with $b = \Omega(T, 0, z(T))$ we can rewrite (13) in the following form

$$\left\{ \begin{array}{l} z(t) = \Omega(T, 0, z(T)) + \epsilon \int_0^t \Phi(s, z(s), y(s))ds, \\ y_+(t) = e^{-A_+ t} (I - e^{-A_+ T})^{-1} \cdot \int_0^T e^{-A_+(T-s)} P_+ \psi_2(s, \Omega(s, 0, z(s)), y(s))ds + \\ \quad + \int_0^t e^{-A_+(t-s)} P_+ \psi_2(s, \Omega(s, 0, z(s)), y(s))ds, \\ y_-(t) = e^{A_- (T-t)} (e^{A_- T} - I)^{-1} \cdot \int_0^T e^{A_- s} P_- \psi_2(s, \Omega(s, 0, z(s)), y(s))ds - \\ \quad - \int_0^t e^{A_-(s-t)} P_- \psi_2(s, \Omega(s, 0, z(s)), y(s))ds, \end{array} \right. \quad (18)$$

where $\Phi(\tau, \xi, y) = \frac{\partial\Omega}{\partial z}(0, \tau, \Omega(\tau, 0, \xi))\phi(\tau, \Omega(\tau, 0, \xi), y)$. Therefore the problem of finding T -periodic solutions for (1) is equivalent to the problem of the existence of zeros for the compact vector field

$$G_\epsilon \begin{pmatrix} z \\ y_+ \\ y_- \end{pmatrix} (t) = \begin{pmatrix} z(t) - F_1(\epsilon, z, y_+ + y_-)(t) \\ y_+(t) - F_2(z, y_+ + y_-)(t) \\ y_-(t) - F_3(z, y_+ + y_-)(t) \end{pmatrix}$$

with F_1, F_2 and F_3 defined as the righthand sides of the three equations in (18).

Consider now in $C([0, T], \mathbf{R}^k) \times C_T(E_+) \times C_T(E_-)$ the open set

$$V_U = Z_U \times B(0, r) \times B(0, r), \quad \text{where}$$

$$Z_U = \{z \in C([0, T], \mathbf{R}^k) : z(t) \in U, t \in [0, T]\} \quad \text{and} \quad r > \frac{cM}{\delta - \gamma c}.$$

Consider the homotopy

$$\begin{aligned} H_\epsilon \left(\lambda, \begin{pmatrix} z \\ y_+ \\ y_- \end{pmatrix} \right) (t) = \\ = \begin{pmatrix} z(t) - \Omega(T, 0, z(T)) - \epsilon \int_0^{\mu(\lambda, t)} \Phi(\tau, z(\tau), y_+(\tau) + y_-(\tau)) d\tau \\ y_+(t) - \lambda F_2(z, y_+ + y_-)(t) \\ y_-(t) - \lambda F_3(z, y_+ + y_-)(t) \end{pmatrix} \end{aligned}$$

with $\mu(\lambda, t) = \lambda t + (1 - \lambda)T$ and $\lambda \in [0, 1]$. We prove in the sequel that, for sufficiently small $\epsilon > 0$, H_ϵ is an admissible homotopy on ∂V_U . We argue by contradiction, hence we assume the existence of sequences

$$\lambda_n \rightarrow \lambda_0, \quad \lambda_n \in [0, 1], \quad \epsilon_n \rightarrow 0, \quad \epsilon_n > 0 \quad \text{and} \quad \begin{pmatrix} z_n \\ y_+^n \\ y_-^n \end{pmatrix} \in \partial V_U$$

such that

$$\begin{cases} z_n(t) = \Omega(T, 0, z_n(T)) + \epsilon_n \int_0^{\mu(\lambda_n, t)} \Phi(\tau, z_n(\tau), y_+^n(\tau) + y_-^n(\tau)) d\tau \\ y_+^n(t) = \lambda_n F_2(z_n, y_+^n + y_-^n)(t) \\ y_-^n(t) = \lambda_n F_3(z_n, y_+^n + y_-^n)(t). \end{cases} \quad (19)$$

We first show that

$$\|y_+^n\|_{C_T}, \|y_-^n\|_{C_T} \leq \frac{cM}{\delta - \gamma c}. \quad (20)$$

For this consider

$$\begin{aligned} y_+^n(t) &= \lambda_n \int_0^T \sum_{m=0}^{\infty} e^{-A_+((m+1)T+t-s)} P_+ \psi_2(s, \Omega(s, 0, z_n(s)), y_n(s)) ds + \\ &+ \lambda_n \int_0^t e^{-A_+(t-s)} P_+ \psi_2(s, \Omega(s, 0, z_n(s)), y_n(s)) ds = \lambda_n \int_{-\infty}^t e^{-A_+(t-s)} P_+ \tilde{\psi}_2(s) ds; \end{aligned}$$

and

$$y_-^n(t) = -\lambda_n \int_0^T \sum_{m=0}^{\infty} e^{A_-(m+1)T+s-t} P_- \psi_2(s, \Omega(s, 0, z_n(s)), y_n(s)) ds -$$

$$-\lambda_n \int_t^T e^{A_-(s-t)} P_- \psi_2(s, \Omega(s, 0, z_n(s)), y_n(s)) ds = -\lambda_n \int_t^{+\infty} e^{A_-(s-t)} P_- \tilde{\psi}_2(s) ds,$$

where $y_n = y_+^n + y_-^n$ and $\tilde{\psi}_2$ denotes the T -periodic extension from $[0, T]$ to \mathbf{R} of the function $\psi_2(s, \Omega(s, 0, z_n(s)), y_n(s))$. By using (2) and (3) we have that

$$\|y_+^n(t)\| \leq \frac{cM}{\delta} + \frac{c\gamma}{\delta} \|y\|_{C_T},$$

$$\|y_-^n(t)\| \leq \frac{cM}{\delta} + \frac{c\gamma}{\delta} \|y\|_{C_T},$$

and so (20). By our choice of r we have that

$$\begin{pmatrix} y_+^n \\ y_-^n \end{pmatrix} \notin \partial(B(0, r) \times B(0, r)).$$

Therefore, it must be $z_n \in \partial Z_U$ and so, for any $n \in \mathbf{N}$, there exists $t_n \in [0, T]$ such that $z_n(t_n) \in \partial U$. Assumption (A₁) implies that

$$z_n(t_n) = \Omega(T, 0, z_n(t_n)). \quad (21)$$

Putting $t = T$ and then $t = t_n$ in the first equation of (19) and subtracting the obtained equations one has

$$z_n(T) - z_n(t_n) = \epsilon_n \int_{\mu(\lambda_n, t_n)}^T \Phi(\tau, z_n(\tau), y_n(\tau)) d\tau. \quad (22)$$

Furthermore subtracting (21) from the first equation of (19), where we have replaced t by T we obtain

$$z_n(T) - z_n(t_n) = \Omega(T, 0, z_n(T)) - \Omega(T, 0, z_n(t_n)) + \epsilon_n \int_0^T \Phi(\tau, z_n(\tau), y_n(\tau)) d\tau.$$

Since Ω is differentiable we can rewrite the last equality as follows

$$\begin{aligned} \left(I - \frac{\partial \Omega}{\partial z}(T, 0, z_n(t_n)) \right) (z_n(T) - z_n(t_n)) = \\ = \epsilon_n \int_0^T \Phi(\tau, z_n(\tau), y_n(\tau)) d\tau + o(z_n(t_n), z_n(T) - z_n(t_n)), \end{aligned} \quad (23)$$

where $o(\xi, h)$ is such that

$$\frac{o(\xi, h)}{\|h\|} \rightarrow 0 \quad \text{as} \quad \|h\| \rightarrow 0$$

uniformly with respect to ξ belonging to compact sets. Replacing (22) into (23) and dividing by $\epsilon_n > 0$ we obtain

$$\begin{aligned} & \left(I - \frac{\partial \Omega}{\partial z}(T, 0, z_n(t_n)) \right) \int_{\mu(\lambda_n, t_n)}^T \Phi(\tau, z_n(\tau), y_n(\tau)) d\tau = \\ & = \int_0^T \Phi(\tau, z_n(\tau), y_n(\tau)) d\tau + \frac{o(z_n(t_n), z_n(T) - z_n(t_n))}{\epsilon_n}. \end{aligned} \quad (24)$$

From (22) there exists a constant $C > 0$ such that

$$\|z_n(T) - z_n(t_n)\| \leq C\epsilon_n.$$

Therefore

$$\frac{o(z_n(t_n), z_n(T) - z_n(t_n))}{\epsilon_n} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \quad (25)$$

On the other hand the operators F_2 and F_3 are compact, and so we can, without loss of generality, assume that the sequences $\{y_+^n\}$, $\{y_-^n\}$ and consequently $y_n = y_+^n + y_-^n$ converge. Let $y_n \rightarrow y_0$, thus

$$\|y_0\|_{C_T} \leq \frac{cM}{\delta - c\gamma}.$$

Furthermore, we can also assume that $z_n(t_n) \rightarrow \xi_0$, hence $\xi_0 \in \partial U$. Since

$$\|z_n(t) - z_n(t_n)\| = \epsilon_n \left\| \int_{l_n} \Phi(\tau, z_n(\tau), y_n(\tau)) d\tau \right\|,$$

where l_n is the segment joining t with $\mu(\lambda_n, t_n)$, we obtain

$$z_n(\tau) \rightarrow \xi_0 \quad \text{as} \quad n \rightarrow \infty$$

uniformly with respect to τ .

We are now in a position to pass to the limit in (24) obtaining

$$\left(I - \frac{\partial \Omega}{\partial z}(T, 0, \xi_0) \right) \int_{t_0}^T \Phi(\tau, \xi_0, y_0(\tau)) d\tau = \int_0^T \Phi(\tau, \xi_0, y_0(\tau)) d\tau, \quad (26)$$

where $t_0 = \lim_{n \rightarrow \infty} \mu(\lambda_n, t_n)$, $t_0 \in [0, T]$. Using the following property for the

traslation operator $\frac{\partial \Omega}{\partial z}$:

$$\frac{\partial \Omega}{\partial z}(T, 0, \xi) \frac{\partial \Omega}{\partial z}(0, \tau, \Omega(\tau, 0, \xi)) = \frac{\partial \Omega}{\partial z}(T, \tau, \Omega(\tau, 0, \xi)), \quad (27)$$

we can rewrite (26) as follows

$$\int_{t_0}^T \frac{\partial \Omega}{\partial z}(T, \tau, \Omega(\tau, 0, \xi_0)) \phi(\tau, \Omega(\tau, 0, \xi_0), y_0(\tau)) d\tau + \int_0^{t_0} \Phi(\tau, \xi_0, y_0(\tau)) d\tau = 0,$$

or equivalently, by the definition of η_{y_0} , we can write

$$\eta_{y_0}(T, t_0, \xi_0) + \int_0^{t_0} \Phi(\tau, \xi_0, y_0(\tau)) d\tau = 0. \quad (28)$$

But,

$$\eta_{y_0}(0, t_0, \xi_0) = \int_{t_0}^0 \Phi(\tau, \xi_0, y_0(\tau)) d\tau,$$

hence (28) takes the form

$$\eta_{y_0}(T, t_0, \xi_0) - \eta_{y_0}(0, t_0, \xi_0) = 0,$$

which is a contradiction with assumption (A₂).

In conclusion, we have proved that for all $\epsilon \in (0, \epsilon_0)$ the function H_ϵ is an admissible homotopy on ∂V_U . For $\lambda = 0$ we have

$$H_\epsilon \left(0, \begin{pmatrix} z \\ y_+ \\ y_- \end{pmatrix} \right) (t) = \begin{pmatrix} z(t) - \Omega(T, 0, z(T)) - \epsilon \int_0^T \Phi(\tau, z(\tau), y(\tau)) d\tau \\ y_+(t) \\ y_-(t) \end{pmatrix}.$$

By the reduction property of the topological degree, for any $\epsilon \in (0, \epsilon_0)$, one has

$$\begin{aligned} \deg(G_\epsilon, V_U, 0) &= \deg(H_\epsilon(0, \cdot), V_U, 0) = \\ &= \deg(H_\epsilon(0, \cdot), V_U \cap C_{\text{const}}(\mathbf{R}^k) \times \{0\} \times \{0\}, 0) = \deg(\Phi_{0,\epsilon}, U, 0), \end{aligned} \quad (29)$$

where $C_{\text{const}}(\mathbf{R}^k)$ denotes the space of constant functions with values in \mathbf{R}^k and

$$\Phi_{0,\epsilon}(\xi) = \xi - \Omega(T, 0, \xi) - \epsilon \int_0^T \Phi(\tau, \xi, 0) d\tau.$$

By (A₁) we have

$$\Omega(T, 0, \xi) = \xi \quad \text{for} \quad \xi \in \partial U.$$

Therefore, for $\xi \in \partial U$, we get

$$\Phi_{0,\epsilon}(\xi) = -\epsilon \int_0^T \Phi(\tau, \xi, 0) d\tau.$$

But

$$\deg(\Phi_{0,\epsilon}, U, 0) = \deg(\Phi_{0,1}, U, 0), \quad (30)$$

and by Remark 1 we have

$$\Phi_{0,1}(\xi) = \eta_0(T, T, \xi) - \eta_0(0, T, \xi)$$

Now using (29), (30), assumption (A₃) and the homotopy invariance of the topological degree we obtain

$$\deg(G_\epsilon, V_U, 0) \neq 0.$$

This concludes the proof of the theorem. \square

2. An Example

In this Section we provide an example to illustrate our main result: Theorem 1.

$$\begin{cases} \dot{x}(t) = \epsilon \phi(t, x(t), y(t)) + \psi_1(t, x(t)), \\ \dot{y}(t) = \|x(t)\| - ay(t), \quad a > 0, \quad t \in [0, 2\pi] \end{cases} \quad (31)$$

where $\psi_1(t, x) = g(t, x)\sigma(x) + \psi(x)$, with

$$\psi(x) = \begin{pmatrix} x_2 + x_1(x_1^2 + x_2^2 - 1) \\ -x_1 + x_2(x_1^2 + x_2^2 - 1) \end{pmatrix}$$

and $(x_1, x_2) = x$. Let $x_0 = \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix}$. We assume that

$$\phi : \mathbf{R} \times \mathbf{R}^2 \times \mathbf{R} \rightarrow \mathbf{R}^2, \quad g : \mathbf{R} \times \mathbf{R}^2 \rightarrow \mathbf{R}^2, \quad \sigma : \mathbf{R}^2 \rightarrow \mathbf{R}$$

are continuous differentiable functions. Moreover, the functions ϕ and g are 2π -periodic with respect to time t and σ and g satisfy the following conditions

$$\sigma(x_0(\theta)) = 0 \quad \text{for any} \quad \theta \in [0, 2\pi], \quad (32)$$

$$\dot{\sigma}(x_0(\theta)) = \frac{\partial g}{\partial x}(t, x_0(\theta)) = 0 \quad \text{for any} \quad t, \theta \in [0, 2\pi]. \quad (33)$$

Denote by $U \subset \mathbf{R}^2$ the interior of the unitary circle centered at the origin. At $\epsilon = 0$ the first equation of system (31) has the form

$$\dot{x} = \psi_1(t, x) = g(t, x)\sigma(x) + \psi(x), \quad (34)$$

We denote by $\Omega(t, t_0, \xi)$ the solution of (34) satisfying the initial condition $x(t_0) = \xi$. Observe that due to condition (32) x_0 is the solution of equation (34) satisfying the initial condition $x(0) = \xi$, whenever $\|\xi\| = 1$. Therefore condition (A₁) of Theorem 1 holds true with $T = 2\pi$. To verify conditions (A₂), (A₃) we calculate in the sequel the translation operator $\frac{\partial \Omega}{\partial z}(t, \tau, \Omega(\tau, 0, \xi))$, where $\xi = \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix}$ and $t, \tau, \theta \in [0, 2\pi]$. For this observe that if $\xi = \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix}$ then by [3] and (33) we get

$$\frac{\partial \Omega}{\partial z}(t, \tau, \Omega(\tau, 0, \xi)) = Y(t, \tau, \theta), \quad (35)$$

where $Y : \mathbf{R} \times \mathbf{R} \times \mathbf{R} \rightarrow L(\mathbf{R}^2, \mathbf{R}^2)$, the vector space of the linear operators from \mathbf{R}^2 to \mathbf{R}^2 , is the solution of the following problem

$$\begin{cases} \frac{dY}{dt}(t, \tau, \theta) = \frac{d\psi}{dx}(\Omega(t, 0, \xi))Y(t, \tau, \theta), \\ Y(\tau, \tau, \theta) = I. \end{cases} \quad (36)$$

Therefore the problem is reduced to find $Y(t, \tau, \theta)$. To this aim we show that the function

$$K(t, \theta) = \begin{pmatrix} \cos(t + \theta) & e^{2t} \sin(t + \theta) \\ -\sin(t + \theta) & e^{2t} \cos(t + \theta) \end{pmatrix} \quad (37)$$

satisfies the equation

$$\frac{dK}{dt}(t, \theta) = \frac{d\psi}{dx}(\Omega(t, 0, \xi))K(t, \theta). \quad (38)$$

In fact,

$$\begin{aligned} \frac{d\psi}{dx}(\Omega(t, 0, \xi)) &= \frac{d\psi}{dx}(x_0(t + \theta)) = \\ &= \begin{pmatrix} 2 \sin^2(t + \theta) & 2 \sin(t + \theta) \cos(t + \theta) + 1 \\ 2 \sin(t + \theta) \cos(t + \theta) - 1 & 2 \cos^2(t + \theta) \end{pmatrix} \end{aligned}$$

and

$$\frac{dK}{dt}(t, \theta) = \begin{pmatrix} -\sin(t + \theta) & 2e^{2t} \sin(t + \theta) + e^{2t} \cos(t + \theta) \\ -\cos(t + \theta) & 2e^{2t} \cos(t + \theta) - e^{2t} \sin(t + \theta) \end{pmatrix}.$$

Thus the function

$$Y(t, \tau, \theta) = K(t, \theta)(K(\tau, \theta))^{-1} \quad (39)$$

is the solution of system (36). Let $\xi = \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix}$ and denote by $\eta_y(t, s, \xi)$, $t, s \in [0, 2\pi]$, the solution of

$$\begin{cases} \dot{z}(t) = \frac{\partial \psi_1}{\partial x}(t, \Omega(t, 0, \xi))z(t) + \phi(t, \Omega(t, 0, \xi), y(t)), \\ z(s) = 0. \end{cases}$$

corresponding to the 2π -periodic continuous function y . By Lemma 1, (35), (38) and (39) we have that

$$\begin{aligned}
& \langle \eta_y(2\pi, s, \xi) - \eta_y(0, s, \xi), \psi(\xi) \rangle = \\
& = \left\langle \int_{s-2\pi}^s \frac{\partial \Omega}{\partial z}(0, \tau, \Omega(\tau, 0, \xi)) \phi(\tau, \Omega(\tau, 0, \xi), y(\tau)) d\tau, \psi(\xi) \right\rangle = \\
& = \left\langle \int_{s-2\pi}^s K(0, \theta) (K(\tau, \theta))^{-1} \phi(\tau, x_0(t + \theta), y(\tau)) d\tau, \dot{x}_0(\theta) \right\rangle = \\
& = \int_0^{2\pi} \langle \phi(\tau, x_0(\tau + \theta), y(\tau)), \dot{x}_0(\tau + \theta) \rangle d\tau = \\
& = \int_0^{2\pi} \langle \phi(r - \theta, x_0(r), y(r - \theta)), \dot{x}_0(r) \rangle dr, \quad \theta \in [0, 2\pi].
\end{aligned}$$

Hence if

$$\int_0^{2\pi} \langle \phi(r - \theta, x_0(r), y(r - \theta)), \dot{x}_0(r) \rangle dr > 0, \quad \theta \in [0, 2\pi], \quad \|y\|_{C_T} \leq \frac{1}{a}, \quad (40)$$

then we have

$$\eta_y(2\pi, s, \xi) - \eta_y(0, s, \xi) \neq 0, \quad s \in [0, 2\pi], \quad \xi \in \partial U, \quad \|y\|_{C_T} \leq \frac{1}{a},$$

and so assumption (A₂) is verified. Moreover, using the property $\text{ind}(\psi, U, 0) = 1$ we obtain

$$\text{ind}(\eta_0(2\pi, 2\pi, \cdot) - \eta_y(0, 2\pi, \cdot), U, 0) = 1.$$

Note that condition (40) holds true, for instance, if

$$\langle \phi(t, x_0(r), y(t)), \dot{x}_0(r) \rangle > 0, \quad t, r \in [0, 2\pi], \quad \|y\|_{C_T} \leq \frac{1}{a},$$

hence assumption (A₃) is also satisfied.

In conclusion, Theorem 1 applies to state the existence, for $\epsilon > 0$ sufficiently small, of a 2π -periodic solution (x_ϵ, y_ϵ) to system (31) such that

$$\Omega(0, t, x_\epsilon(t)) \in U, \quad t \in [0, 2\pi], \quad \|y_\epsilon\|_{C_T} \leq \frac{1}{a}.$$

□

Remark 2 We would like to point out that our result, as the proposed example shows, is not a consequence of known results concerning the existence of periodic solutions around an equilibrium point with non zero topological degree. In fact, if, for fixed $\xi \in U$, the function $g(t, \xi)$ is not constant with respect to the time $t \in [0, 2\pi]$, then the differential equation (34) does not have equilibrium points in U .

We end the paper by showing how from our result we can derive a classical result of the theory of ordinary differential equations, cf. ([3], Theorem 3.1, p. 362). Indeed, if we put in system (1), $k = 2$, $\psi_2(t, x) = 0$, $A = 0$, $y = 0$, where $x = (x_1, x_2)$, and $\psi_1(t, x) = Bx = (-x_2, x_1)$, $\phi(t, x, 0) = (0, g(t, -x_1, x_2))$, where g is 2π -periodic with respect to t . Then we have

$$\eta(2\pi, s, \xi(a, \theta)) - \eta(0, s, \xi(a, \theta)) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} H(a, \theta),$$

where $\xi(a, \theta) = (-a \cos \theta, a \sin \theta)$ and

$$H(a, \theta) = \int_0^{2\pi} \begin{pmatrix} (\sin \tau) f(\tau + \theta, a \cos \tau, -a \sin \tau) \\ (\cos \tau) f(\tau + \theta, a \cos \tau, -a \sin \tau) \end{pmatrix} d\tau.$$

In [3] it is assumed the following condition

$$\det |H'(a_0, \theta_0)| \neq 0,$$

for some $a_0 \neq 0$. It is now easy to see that this condition ensures that

$$|\deg(\eta(2\pi, s, \xi(\cdot)) - \eta(0, s, \xi(\cdot)), V, 0)| = 1.$$

where V is a sufficiently small open set containing (a_0, θ_0) .

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