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An Averaging Method for Singularly Perturbed Systems of Semilinear Differential Inclusions with C_0-Semigroups

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Abstract. We consider a system of two semilinear differential inclusions with infinitesimal generators of C_0-semigroups. The nonlinear terms are of high frequency with respect to time and periodic with a specified period. Moreover, they are condensing in the state variables (x, y) with respect to a suitable measure of noncompactness. The goal of the paper is to give sufficient conditions to guarantee, for $\epsilon > 0$ sufficiently small, the existence of periodic solutions and to study their behaviour as $\epsilon \to 0$. The main tool to achieve this is the topological degree theory for uppersemicontinuous, condensing vector fields.

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1. Introduction

In [5] the authors treated the problem of the existence and dependence on a small parameter $\epsilon > 0$ of periodic solutions of a singularly perturbed system of semilinear parabolic inclusions with high frequency nonlinear inputs having the form

$$\begin{aligned} \dot{x}(\tau) &\in A_1 x(\tau) + f_1(\tau/\epsilon, x(\tau), y(\tau)), \\ \epsilon \dot{y}(\tau) &\in A_2 y(\tau) + f_2(\tau/\epsilon, x(\tau), y(\tau)), \end{aligned} \tag{1}$$

where A_i, $i = 1, 2$, are the infinitesimal generators of analytic semigroups $e^{A_i t}$, $t \geqslant 0$, acting in separable Banach spaces E_i with compact inverse A_i^{-1}. The nonlinear multivalued operators f_i: $\mathbf{R} \times E_1 \times E_2 \multimap E_i$, $i = 1, 2$, are assumed T-periodic with respect to the first variable and subordinate to the fractional powers A_i^{α}, $i = 1, 2$, with respect to x and y respectively, (see [5]). For fixed $\epsilon > 0$ we look for T/ϵ-periodic solutions to (1).

These assumptions allowed us to define uppersemicontinuous, compact multivalued operators Φ_ϵ, $\epsilon > 0$, with nonempty, compact convex values in such a way that the periodic solutions to (1) turn out to be the fixed points of Φ_ϵ. In [5] to show the existence of fixed points and to study their dependence on $\epsilon > 0$ we have applied to the upper semicontinuous, compact vector field $I - \Phi_\epsilon$ the related topological degree theory. Specifically, this was done by defining at $\epsilon = 0$ a suitable uppersemicontinuous, compact averaging operator with nonempty compact convex values, by assuming that its topological degree is different from zero in some open bounded set and then by showing that $I - \Phi_\epsilon$ and the averaging operator are linearly homotopic.

The aim of this paper is to deal with the same problem for (1) in the case when A_i, $i = 1, 2$, are the infinitesimal generators of C_0-semigroups $e^{A_i t}$, $t \geqslant 0$. As we will see, in this case, due to the lack of compactness of the operators Φ_ϵ we are led to consider conditions on $e^{A_i t}$ and f_i, $i = 1, 2$, which ensure the uppersemicontinuity and the condensivity of these operators with respect to a suitably introduced measure of noncompactness. Indeed, the approach to solve the proposed problem in this paper is similar to that employed in [5] and recalled above, but the assumptions and the proofs of the relevant results are substantially different.

In fact, we still convert the problem of finding periodic solutions to (1) to a fixed point problem for appropriate multivalued operators Φ_ϵ, $\epsilon > 0$, and we define at $\epsilon = 0$ a suitable averaging operator. Then in Section 2 we provide sufficient conditions on the semigroups $e^{A_i t}$ and the multivalued nonlinear operators f_i, $i = 1, 2$, to guarantee that both Φ_ϵ and the averaging operator are uppersemicontinuous and condensing operators with nonempty convex values.

Furthermore, we prove that there exist linear admissible homotopies between Φ_ϵ, for $\epsilon > 0$ sufficiently small, and the averaging operator, which is assumed with topological degree different from zero with respect to an open bounded set $U \subset E_1$. Here the topological degree theory is that for uppersemicontinuous condensing vector fields. Therefore, for $\epsilon > 0$ sufficiently small there exists a fixed point (x^ϵ, y^ϵ) of Φ_ϵ which represents a T/ϵ-periodic solution of system (1) with $x_\epsilon(t) \in U$ for any $t \in [0, T]$ and $t = \tau/\epsilon$. Moreover, we prove that $(x^\epsilon, y^\epsilon) \to (x^*, y^0)$ as $\epsilon \to 0$, where $x^* \in U$ is a fixed point of the averaging operator and y^0 is a T-periodic solution of the second equation in (1) with $\epsilon = 1$ corresponding to $x(\tau) \equiv x^*$. Condensing operators and related topological degree theory have been employed by the authors in the studying of the dependence of periodic solutions of a system of parabolic inclusions on a large parameter, see [7], and in the development of a singular perturbation theory for systems of semilinear differential inclusions in infinite dimensional spaces [1, 6].

The paper is organized as follows. In Section 2 we state the problem, we formulate the assumptions and we give some preliminary results. In Section 3 we prove our main result: Theorem 1. Finally, in Section 4 we consider the case of systems of high dimension and the Appendix ends the paper. In the Appendix we provide a proof of the equivalence between the exponential stability of a matrix of the form $-\Gamma + M$ and the condition that $\rho(\Gamma^{-1}M) < 1$, where Γ is a diagonal matrix with positive elements and M is a matrix with all non-negative elements. The proof is based on the elementary theory of linear dynamical systems and the authors decided to provide it since they did not find an explicit proof of this equivalence in the literature.

2. Formulation of the Problem, Assumptions, Definitions and Preliminary Results

In this paper we consider the following system of differential inclusions

$$\begin{aligned} \dot{x}(\tau) &\in A_1 x(\tau) + f_1(\tau/\epsilon, x(\tau), y(\tau)), \\ \epsilon \dot{y}(\tau) &\in A_2 y(\tau) + f_2(\tau/\epsilon, x(\tau), y(\tau)), \end{aligned} \tag{1}$$

where A_i, $i = 1, 2$, are the infinitesimal generators of C_0-semigroups $e^{A_i t}$, $t \geqslant 0$, acting in separable Banach spaces E_i, $i = 1, 2$, and $\epsilon > 0$ is a small singular perturbation parameter. The multivalued operators $f_i\colon \mathbf{R} \times E_1 \times E_2 \multimap E_i$, $i = 1, 2$, are assumed T-periodic, $T > 0$, in the first variable. Our problem is the following

PROBLEM. To give conditions under which, for $\epsilon > 0$ fixed, system (1) admits T/ϵ-periodic solutions and to study their behaviour as $\epsilon \to 0$.

Our main result, Theorem 1, will solve this problem under the following conditions:

(F_1) for any $(x, y) \in E_1 \times E_2$ the multivalued maps $f_i(\cdot, x, y)\colon [0, T] \to K_c(E_i)$, $i = 1, 2$, have a measurable selection. (Here and in the sequel $K_c(E)$ will denote the collection of all the nonempty compact convex subsets of E);

(F_2) for almost all (a.a.) $t \in [0, T]$ the multivalued maps $f_i(t, \cdot, \cdot)\colon E_1 \times E_2 \to K_c(E_i)$, $i = 1, 2$, are uppersemicontinuous;

(F_3) for any bounded set $\Omega_i \subset E_i$, $i = 1, 2$, the following inequalities hold

$$\chi_{E_i}(f_i([0, T] \times \Omega_1 \times \Omega_2)) \leqslant m_{i_1} \chi_{E_1}(\Omega_1) + m_{i_2} \chi_{E_2}(\Omega_2),$$

where χ_{E_i}, $i = 1, 2$, denotes the Hausdorff measure of noncompactness in the space E_i, $i = 1, 2$, respectively.

Furthermore assume that

$$\text{(A)} \quad \|e^{A_i t}\|^{(\chi)} \leqslant e^{-\gamma_i t}, \quad t \geqslant 0, \quad \text{for some } \gamma_i > 0,\ i = 1, 2;$$

and

$$\rho\left(\begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}^{-1} \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \right) < 1. \tag{2}$$

Here $\rho(Q)$ denotes the spectral radius of the matrix Q and $\|B\|^{(\chi)} := \chi(BS)$, where B is a bounded linear operator from a Banach space E into itself and S is the unit sphere in E.

Following [5, 7] and [8], our approach consists in converting our problem into a fixed point problem for a suitable defined operator Φ_ϵ, $\epsilon > 0$, acting from

$C_T(E_1 \times E_2)$ to $K_c(C_T(E_1 \times E_2))$, where $C_T(E_1 \times E_2)$ stands for the Banach space of T-periodic functions (x, y) taking values in $E_1 \times E_2$. In order to do this, after the change of variable $t = \tau/\epsilon$, we first rewrite system (1) in the following form

$$\begin{aligned} \dot{x}(t) &\in \epsilon A_1 x(t) + \epsilon f_1(t, x(t), y(t)), \\ \dot{y}(t) &\in A_2 y(t) + f_2(t, x(t), y(t)) \end{aligned} \tag{3}$$

with $t \in [0, T]$. Therefore the problem of finding T/ϵ-periodic solutions to system (1) is equivalent to that of finding T-periodic solutions to system (3). Then we define, for $\epsilon > 0$ given, a multivalued operator Φ_ϵ: $C_T(E_1 \times E_2) \to K_c(C_T(E_1 \times E_2))$ as is shown in the sequel. For any pair $(x, y) \in C_T(E_1 \times E_2)$ we consider the set $\varphi(x, y)$ of all the pairs (v_1, v_2) of the measurable T-periodic selections $v_i(\cdot)$ of $f_i(\cdot, x(\cdot), y(\cdot))$, $i = 1, 2$, namely

$$\varphi(x, y) := \{(v_1, v_2) : v_i(t) \in f_i(t, x(t), y(t)),\ i = 1, 2,\ \text{for a.a. } t \in [0, T] \\ v_i \text{ measurable and } T\text{-periodic } \}.$$

Given $\varphi(x, y)$ the nonempty compact convex set $\Phi_\epsilon(x, y) \subset C_T(E_1 \times E_2)$ is defined by

$$\Phi_\epsilon(x, y) = \{(\Pi_1(\epsilon)v_1, \Pi_2 v_2) : (v_1, v_2) \in \varphi(x, y)\},$$

where

$$\begin{aligned} \Pi_1(\epsilon)v_1(t) :=\ & e^{\epsilon A_1 t}(I - e^{\epsilon A_1 T})^{-1} \int_0^T e^{\epsilon A_1 (T-s)} \epsilon v_1(s)\, ds + \\ & + \int_0^t e^{\epsilon A_1 (t-s)} \epsilon v_1(s)\, ds, \\ \Pi_2 v_2(t) :=\ & e^{A_2 t}(I - e^{A_2 T})^{-1} \int_0^T e^{A_2 (T-s)} v_2(s)\, ds + \\ & + \int_0^t e^{A_2 (t-s)} v_2(s)\, ds. \end{aligned} \tag{4}$$

Observe that, without loss of generality, we can assume that the semigroups of linear operators $e^{A_i t}$, $t \geqslant 0$, satisfy the estimates

$$\|e^{A_i t}\| \leqslant c e^{-dt}, \quad t \geqslant 0,\ i = 1, 2 \tag{5}$$

for some $d > 0$. In fact, we can always reduce our considerations to this situation by adding and subtracting suitable linear operators to A_1 and A_2, and observing that the resulting nonlinearities still satisfy $(F_1) \div (F_3)$.

This remark shows that the linear operators $(I - e^{\epsilon A_1 T})^{-1}$ and $(I - e^{A_2 T})^{-1}$ exist.

DEFINITION 1. Let $\Omega \subset C_T(E_1 \times E_2)$ be a bounded set. Let φ_i, $i = 1, 2$, be measurable functions (see [4], Theorem 4.2.4) defined by

$$\varphi_i(\Omega)(t) = \chi_{E_i}(P_i \Omega(t)),$$

where $\Omega(t) = \{(x(t), y(t)) : (x, y) \in \Omega\}$ and P_i is the natural projection of $E_1 \times E_2$ on E_i. Finally, let

$$\Psi(\Omega) = \lim_{\delta\to 0} \sup_{w\in\Omega} \max_{|t_1-t_2|\leqslant\delta} \|w(t_1) - w(t_2)\|_{E_1\times E_2}.$$

We define a measure of noncompactness $\nu(\cdot)$ in $C_T(E_1 \times E_2)$ as follows

$$\nu(\Omega) = (\varphi_1(\Omega)(t), \varphi_2(\Omega)(t), \Psi(\Omega)).$$

Observe that the values of this measure of noncompactness belong to the space $M_T(\mathbf{R}^2)\times\mathbf{R}$, where $M_T(\mathbf{R}^2)$ is the space of T-periodic, measurable functions with values in $\mathbf{R}^2$. It is easy to see that ν is monotone, compactly invariant and regular but it is not semiadditive. The ordering related to the monotonicity is defined as follows: $\mathbf{R}^2$ is ordered by the cone $\mathbf{R}^2_+$ of positive coordinates, while the space $M_T(\mathbf{R}^2)$ is ordered by the cone

$$K = \{y \in M_T(\mathbf{R}^2);\ y(t) \in \mathbf{R}^2_+ \text{ for a.a. } t \in \mathbf{R}\}$$

and the space $M_T(\mathbf{R}^2) \times \mathbf{R}$ is ordered by $K \times [0, \infty)$. As it is shown in the Proposition 5 of the Appendix, inequality (2) is equivalent to the exponential stability of the matrix

$$-\begin{pmatrix} \epsilon\gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} + \begin{pmatrix} \epsilon\, m_{11} & \epsilon\, m_{12} \\ m_{21} & m_{22} \end{pmatrix} \tag{6}$$

for any $\epsilon > 0$. Therefore we have the following proposition (see [4], Theorem 6.1.2).

PROPOSITION 2. *Assume* $(F_1)\div(F_3)$ *and* (A). *Then, for* $\epsilon > 0$, *the operator* Φ_ϵ: $C_T(E_1 \times E_2) \to K_c(C_T(E_1 \times E_2))$ *is uppersemicontinuous and condensing with respect to the measure of noncompactness* $\nu(\cdot)$.

Note that Proposition 5 of the Appendix also shows that

$$\gamma_2 > m_{22}. \tag{7}$$

We will formulate conditions in the sequel which ensure the existence of T-periodic solutions for system (3) in terms of the averaging operator which we are now going to define. For this we assume the following condition

(M) for every nonempty bounded set $B \subset E_1$, the set Y_B of all the T-periodic solutions of the following inclusion

$$y'(t) \in A_2 y(t) + \lambda f_2(t, x, y(t)), \tag{8}$$

where $\lambda \in [0, 1]$ and $x \in B$ is nonempty and bounded.

For $x \in E_1$, let V be the set of all the measurable selections of the multivalued map $f_1(t, x, y(t))$ where $y \in Y_x^1$ and Y_x^1 denotes the solution set of (8) for $B = \{x\}$ and $\lambda = 1$.

DEFINITION 3. We define the multivalued averaging operator $A_1^{-1}F\colon E_1 \multimap E_1$ as follows

$$A_1^{-1}F(x) = \overline{\text{co}}\left\{A_1^{-1}\frac{1}{T}\int_0^T v(s)\,\mathrm{d}s\colon\ v \in V\right\}.$$

We can prove the following result.

PROPOSITION 4. *Assume the conditions* $(\mathrm{F}_1)\div(\mathrm{F}_3)$, (A) *and* (M). *Then the operator $A_1^{-1}F$ is uppersemicontinuous with nonempty compact convex values and it is χ_{E_1}-condensing.*

Proof. First of all observe that conditions $(\mathrm{F}_1)\div(\mathrm{F}_3)$ together with inequality (7) ensure that the set Y_x^1 is compact and the application $x \multimap Y_x^1$ is uppersemicontinuous (see [4], Theorem 3.5.2). Therefore $A_1^{-1}F$ is the composition of two uppersemicontinuous multivalued maps with two continuous applications and so $A_1^{-1}F$ is uppersemicontinuous. Moreover the compactness of Y_x^1 implies the compactness of $A_1^{-1}F(x)$, the convexity follows directly from the definition. We prove now that $A_1^{-1}F$ is χ_{E_1}-condensing. For this, let $B_1 \subset E_1$ be a bounded set for which

$$\chi_{E_1}(B_1) \leqslant \chi_{E_1}(A_1^{-1}F(B_1)). \tag{9}$$

We will show that B_1 is a relatively compact set. Let $\xi_1 = \chi_{E_1}(B_1)$ and $\xi_2 = \sup_{t\in[0,T]}\varphi_2(Y_{B_1}^1)(t)$. From (F_3), (A) and the definition of $A_1^{-1}F$ we have the estimate

$$\xi_1 \leqslant \frac{m_{11}}{\gamma_1}\xi_1 + \frac{m_{12}}{\gamma_1}\xi_2. \tag{10}$$

Moreover, by using (F_3) and (A) we also have

$$\begin{aligned}\varphi_2(Y_{B_1}^1)(t) \ &\leqslant\ \mathrm{e}^{-\gamma_2 t}(I - \mathrm{e}^{-\gamma_2 T})^{-1}\int_0^T \mathrm{e}^{-\gamma_2(T-s)}(m_{21}\xi_1 + m_{22}\xi_2)\,\mathrm{d}s\, +\\ &\quad + \int_0^t \mathrm{e}^{-\gamma_2(t-s)}(m_{21}\xi_1 + m_{22}\xi_2)\,\mathrm{d}s \leqslant \frac{m_{21}}{\gamma_2}\xi_1 + \frac{m_{22}}{\gamma_2}\xi_2\,;\end{aligned}$$

and then

$$\xi_2 \leqslant \frac{m_{21}}{\gamma_2}\xi_1 + \frac{m_{22}}{\gamma_2}\xi_2. \tag{11}$$

From (10) and (11) we get that the vector $\xi = (\xi_1, \xi_2)$ with nonnegative coordinates satisfies the inequality

$$\xi \leqslant D\xi, \tag{12}$$

where D is the matrix

$$\begin{pmatrix} \dfrac{m_{11}}{\gamma_1} & \dfrac{m_{12}}{\gamma_1} \\ \dfrac{m_{21}}{\gamma_2} & \dfrac{m_{22}}{\gamma_2} \end{pmatrix}.$$

From (2) we have that the eigenvalues of D belong to the open unit circle. Therefore, (12) implies that $\xi = 0$ and so $\xi_1 = \chi_{E_1}(B_1) = 0$, that is B_1 is relatively compact. □

3. Main Result

In order to prove our main result we need a stronger condition on the semigroup $e^{A_1 t}$, $t \geqslant 0$. Specifically, we assume that

$$(\mathrm{A}_1) \quad \|e^{A_1 t}\| \leqslant e^{-\gamma_1 t}, \quad t \geqslant 0,\ \gamma_1 > 0.$$

That is the semigroup $e^{A_1 t}$ is contractive.

For $\epsilon > 0$ and any bounded set $U \subset E_1$, we define the solution set Σ_ϵ^U of (3) as follows

$$\begin{aligned}\Sigma_\epsilon^U \;=\; &\{(x^\epsilon, y^\epsilon) \in C_T(E_1 \times E_2) : (x^\epsilon, y^\epsilon) \text{ is a solution to} \\ &\quad \text{system (3) with } x^\epsilon(t) \in U \text{ for any } t \in [0, T]\}\end{aligned}$$

while at $\epsilon = 0$, we define a solution set Σ_0 as follows

$$\begin{aligned}\Sigma_0 \;=\; &\{(x^0, y^0) \in E_1 \times C_T(E_2) : x^0 \text{ is a solution to next} \\ &\quad \text{system (13) and } y^0 \in Y^1_{x^0}\}.\end{aligned}$$

Now we are in a position to state the main result of this paper.

THEOREM 1. *Assume conditions* (F_1)÷(F_3), (A), (M) *and* (A_1). *Assume that there exists an open bounded set $U \subset E_1$ such that the inclusion*

$$x \in A_1^{-1} F(x) \tag{13}$$

has no solution on ∂U and

$$\deg(I - A_1^{-1} F, U, 0) \neq 0.$$

Then for $\epsilon > 0$ sufficiently small the solution set Σ_ϵ^U is nonempty and the map $\epsilon \multimap \Sigma_\epsilon^U$ is uppersemicontinuous at $\epsilon = 0$.

Proof. As it is shown in [2], there exists uppersemicontinuous, compact multivalued operator $\widehat{A_1^{-1}F}\colon E_1 \to K_c(E_1)$ which is linearly homotopic to $A_1^{-1}F$ on $\overline{U}$. Consider the operator $\Phi_0\colon C_T(E_1 \times E_2) \to K_c(C_T(E_1 \times E_2))$ defined by

$$\begin{aligned}\Phi_0(x, y) \;=\; &\{(\widehat{A_1^{-1}F}(x(0)), \Pi_2 \overline{v}_2) : \overline{v}_2(t) \in f_2(t, x(0), y(t)) \\ &\quad \text{for a.a. } t \in [0, T]\}.\end{aligned}$$

Let $r > 0$ such that for $B = \overline{U}$ the inclusion (8) does not have T-periodic solution y such that

$$\|y\|_{C_T(E_2)} \geqslant r. \tag{14}$$

Define now a set $\widetilde{U} \subset C_T(E_1)$ in the following way

$$\widetilde{U} = \{x \in C_T(E_1) : x(t) \in U, \quad \text{for } t \in [0, T]\}.$$

We want to show now that for $\epsilon > 0$ sufficiently small Φ_ϵ and Φ_0 are linearly homotopic on $\widetilde{U} \times B(0, r)$. If we assume the contrary, then there exist sequences $\epsilon_n \to 0$, $\mu_n \in [0, 1]$, $\mu_n \to \mu_0$, $(x_n, y_n) \in \partial(\widetilde{U} \times B(0, r))$ such that

$$\begin{aligned} x_n(t) &= \mu_n \Pi_1(\epsilon_n) v_1^n(t) + (1 - \mu_n) w_n, \\ y_n(t) &= \mu_n \Pi_2 v_2^n(t) + (1 - \mu_n) \Pi_2 \overline{v}_2^n(t), \end{aligned} \tag{15}$$

where, for a.a. $t \in [0, T]$, we have

$$\begin{aligned} &v_1^n(t) \in f_1(t, x_n(t), y_n(t)), \\ &w_n \in \widehat{A_1^{-1}F}(x_n(0)), \\ &v_2^n(t) \in f_2(t, x_n(t), y_n(t)), \\ &\overline{v}_2^n(t) \in f_2(t, x_n(0), y_n(t)). \end{aligned}$$

Note that $v_1^n \in L_T^\infty(E_1)$ and $v_2^n, \overline{v}_2^n \in L_T^\infty(E_2)$. Moreover, their norms in the respective spaces are uniformly bounded and the sequence $\{w_n\}$ is compact. We prove now the compactness of $\{(x_n, y_n)\}$ in $C_T(E_1 \times E_2)$. We start by proving that $\sup_{t\in[0,T]} \chi_{E_1}(\{x_n(t)\}) = 0$. For this we evaluate

$$\begin{aligned} \chi_{E_1}(\{v_1^n(t)\}) &\leqslant m_{11}\chi_{E_1}(\{x_n(t)\}) + m_{12}\chi_{E_2}(\{y_n(t)\}) \\ &\leqslant m_{11} \sup_{t\in[0,T]} \chi_{E_1}(\{x_n(t)\}) + m_{12} \sup_{t\in[0,T]} \chi_{E_2}(\{y_n(t)\}). \end{aligned} \tag{16}$$

Analogously, we have

$$\chi_{E_2}(\{v_2^n(t)\}) \leqslant m_{21} \sup_{t\in[0,T]} \chi_{E_1}(\{x_n(t)\}) + m_{22} \sup_{t\in[0,T]} \chi_{E_2}(\{y_n(t)\}). \tag{17}$$

By using the results of [3] we have that for any $\delta > 0$ there exist a set $e_\delta \subset [0, T]$, a compact set $K_\delta \subset E_1$ and a sequence $\{g_n\} \subset L_T^\infty(E_1)$ such that $\operatorname{meas} e_\delta < \delta$, $g_n(s) \in K_\delta$ and

$$\|v_1^n(t) - g_n(t)\| \leqslant m_{11} \sup_{t\in[0,T]} \chi_{E_1}(\{x_n(t)\}) + m_{12} \sup_{t\in[0,T]} \chi_{E_2}(\{y_n(t)\}) \tag{18}$$

for $t \in [0, T] \setminus e_\delta$.

For any $t \in [0, T]$ the following set

$$\begin{aligned} \{\widetilde{x}_n(t) : \widetilde{x}_n(t) = \mu_n \epsilon_n \mathrm{e}^{\epsilon_n A_1 t}(I - \mathrm{e}^{\epsilon_n A_1 T})^{-1} \int_0^T \mathrm{e}^{\epsilon_n A_1(T-s)} g_n(s)\, \mathrm{d}s + \\ + \ (1 - \mu_n) w_n\} \end{aligned}$$

is relatively compact in E_1. In fact, the semigroups $\mathrm{e}^{\epsilon_n A_1 t}$ and $\mathrm{e}^{\epsilon_n A_1(T-s)}$ are of class C_0 and so for any $w \in K_\delta$ we have that

$$\mathrm{e}^{\epsilon_n A_1 t} w \to w \quad \text{and} \quad \mathrm{e}^{\epsilon_n A_1(T-s)} w \to w$$

as $n \to \infty$, uniformly with respect to t, s and w.

Therefore, the set

$$K_{1,n}^{\delta} = \left\{ \mathrm{e}^{\epsilon_n A_1 t} \int_0^T \mathrm{e}^{\epsilon_n A_1 (T-s)} g_n(s)\,\mathrm{d}s : t \in [0,T] \right\}$$

is relatively compact. By ([4], Theorem 4.5.1) we have that

$$\epsilon_n (I - \mathrm{e}^{\epsilon_n A_1 T})^{-1} x \to -\frac{1}{T} A_1^{-1} x$$

for all $x \in E_1$ and thus $\bigcup_n \epsilon_n (I - \mathrm{e}^{\epsilon_n A_1 T})^{-1} K_{1,n}^{\delta}$ is a relatively compact set. In conclusion

$$\{\widetilde{x}_n(t)\} \subseteq \bigcup_n (\epsilon_n (I - \mathrm{e}^{\epsilon_n A_1 T})^{-1} K_{1,n}^{\delta} + (1-\mu_n) w_n)$$

is a relatively compact sequence in E_1 for any $t \in [0,T]$.

Now, we want to estimate $\|x_n(t) - \widetilde{x}_n(t)\|$. Since

$$\left\| \epsilon_n \int_0^t \mathrm{e}^{\epsilon_n A_1 (t-s)} v_1^n(s)\,\mathrm{d}s \right\| \leqslant \epsilon_n \|v_1^n\|_\infty \to 0 \tag{19}$$

as $n \to \infty$, it is sufficient to estimate only the term

$$\alpha_n(t) = \left\| \mu_n \epsilon_n (I - \mathrm{e}^{\epsilon_n A_1 T})^{-1} \int_0^T \mathrm{e}^{\epsilon_n A_1 (T+t-s)} [v_1^n(s) - g_n(s)]\,\mathrm{d}s \right\|.$$

By using (16) and (18) for n sufficiently large we obtain

$$\begin{aligned} \alpha_n(t) &\leqslant \frac{\epsilon_n}{1 - \mathrm{e}^{-\epsilon_n \gamma_1 T}} \Big[T (m_{11} \sup_{t\in[0,T]} \chi_{E_1}(\{x_n(t)\}) + \\ &\quad + m_{12} \sup_{t\in[0,T]} \chi_{E_1}(\{y_n(t)\}) + \delta M \Big] \\ &\leqslant \frac{1}{\gamma_1 T (1-\delta)} \Big[T (m_{11} \sup_{t\in[0,T]} \chi_{E_1}(\{x_n(t)\}) + \\ &\quad + m_{12} \sup_{t\in[0,T]} \chi_{E_1}(\{y_n(t)\}) + \delta M \Big]. \end{aligned}$$

By (17) and the arbitrarity of $\delta > 0$ we set

$$\chi_{E_1}(\{x_n(t)\}) \leqslant \frac{m_{11} \sup_{t\in[0,T]} \chi_{E_1}(\{x_n(t)\}) + m_{12} \sup_{t\in[0,T]} \chi_{E_2}(\{y_n(t)\})}{\gamma_1} \tag{20}$$

and so, if we put $\xi_1 = \sup_{t\in[0,T]} \chi_{E_1}(\{x_n(t)\})$ and $\xi_2 = \sup_{t\in[0,T]} \chi_{E_2}(\{y_n(t)\})$, (20) can be rewritten as

$$\xi_1 \leqslant \frac{m_{11}}{\gamma_1} \xi_1 + \frac{m_{12}}{\gamma_1} \xi_2. \tag{21}$$

By standard methods (see, for instance, [7, 8]) one obtains from the second equality of (15)

$$\xi_2 \leqslant \frac{m_{21}}{\gamma_2}\xi_1 + \frac{m_{22}}{\gamma_2}\xi_2. \tag{22}$$

Condition (2) and inequalities (21) and (22) produce again $\xi_1 = \xi_2 = 0$. Then from (17) it follows that $\chi_{E_2}(\{v_2^n(t)\}) = 0$ for a.a. $t \in [0, T]$. The same holds true for $\{\overline{v}_2^n\}$, hence, by ([4], Corollary 5.11) we get the compactness of the sequence $\{y_n\}$. Now from the compactness of $\{x_n(0)\}$, (19) and the C_0-property of the semigroup $e^{A_1 t}$ we obtain

$$\|x_n(t) - x_n(0)\| \to 0$$

as $n \to \infty$ uniformly with respect to t.

Hence, without loss of generality, we can assume that

$$x_n \to x^0,$$
$$y_n \to y^0,$$

where x^0 is a constant function which we identify with its value. Moreover, it follows that $(x^0, y^0) \in \partial(\widetilde{U} \times B(0, r))$ and by our choice of $r > 0$ it turns out that $\|y^0\| \neq r$.

Therefore, we must have $x^0 \in \partial\widetilde{U}$. By ([5], Lemma 3.3) passing to the limit in (15) (see the details in [2]) we obtain

$$x_0 \in A_1^{-1}F(x^0),$$
$$y_0 \in \Pi_2 v_2^0,$$

which is a contradiction, since the first inclusion, by our choice of U, cannot have solution on ∂U. Finally, consider the following homotopy

$$\Phi_0^\mu(x, y) = \{(A_1^{-1}F(x(0)), \mu\Pi_2\overline{v}_2) : \overline{v}_2(t) \in f_2(t, x(0), y(t)) \text{ for a.a. } t \in [0, T]\},$$

where $\mu \in [0, 1]$. This is an admissible homotopy between $I - \Phi_0$ and $I - \overline{\Phi}_0$, where

$$\overline{\Phi}_0(x, y) = (A_1^{-1}F(x(0)), 0).$$

By using the reduction theorem for the topological degree for condensing operators (see, e.g., [2]) we obtain

$$\deg(I - \overline{\Phi}_0, \widetilde{U} \times B(0, r), 0) = \deg(I - A_1^{-1}F, U, 0) \neq 0.$$

Thus, for $\epsilon > 0$, sufficiently small Σ_ϵ^U is nonempty by the solution property of the topological degree. The uppersemicontinuity of the application $\epsilon \to \Sigma_\epsilon^U$ at $\epsilon = 0$

can be proved in the same way as we have proved the convergence of (x_n, y_n) to (x^0, y^0) when in (15) we put $\mu_n = 1$ for any $n \in \mathbf{N}$. □

4. Systems of Higher Dimension

By means of the methods illustrated in the previous section it is possible to obtain the same result for systems of inclusions of higher dimension of the form

$$\begin{aligned} &\dot{x}_i(\tau) \in A_i x_i(\tau) + f_i(\tau/\epsilon, x_1(\tau), \ldots, x_p(\tau), y_1(\tau), \ldots, y_q(\tau)), \\ &\epsilon \dot{y}_j(\tau) \in A_{p+j} y_j(\tau) + f_{p+j}(\tau/\epsilon, x_1(\tau), \ldots, x_p(\tau), y_1(\tau), \ldots, y_q(\tau)) \end{aligned} \tag{23}$$

with $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$.

In this case in the assumption (A) the index i varies from 1 to $p + q$, in (A_1) it varies from 1 to p and the assumptions (F_1) and (F_2) are formulated in terms of the vectors $x = (x_1, x_2, \ldots, x_p)$ and $y = (y_1, y_2, \ldots, y_q)$. Finally, condition (F_3) can be rewritten as follows

$$\chi_{E_i}(f_i([0, T] \times \Omega_1 \times \Omega_2 \times \cdots \times \Omega_{p+q})) \leqslant \sum_{k=1}^{p+q} m_{ik} \chi_{E_k}(\Omega_k)$$

with $i = 1, 2, \ldots, p + q$ and $\Omega_i \subset E_i$ are bounded sets.

Furthermore, condition (2) takes the form

$$\rho(\Gamma^{-1} M) < 1, \tag{24}$$

where Γ and M are $(p + q) \times (p + q)$ matrices with $\Gamma = \mathrm{diag}(\gamma_i)_{i=1}^{p+q}$, $\gamma_i > 0$ and $M = (m_{ij})_{i,j=1}^{p+q}$, $m_{ij} \geqslant 0$. To adapt the arguments of the proof of Theorem 1 to the present situation of system (23) we need only to show that condition (24) is equivalent to the exponential stability of $-\Gamma + M$. This is done in Proposition 5 of the following Appendix. Moreover, there it is also shown that the exponential stability of $-\Gamma + M$ implies that of $-\widetilde{\Gamma} + \widetilde{M}$ which is in turn equivalent to the condition $\rho(\widetilde{\Gamma}^{-1}\widetilde{M}) < 1$, where $\widetilde{\Gamma} = \mathrm{diag}(\gamma_{p+j})_{j=1}^{q}$ and $\widetilde{M} = (m_{p+i,p+j})_{i,j=1}^{q}$. This last condition replaces (7).

Appendix

In this section we provide a proof of the equivalence between the condition $\rho(\Gamma^{-1}M) < 1$ and the fact that the matrix $-\Gamma + M$ is exponentially stable. We conjecture that this is known, but since we were unable to find an explicit proof in the literature and for the sake of completness, we present a proof based on elementary results from the theory of linear differential systems.

PROPOSITION 5. *Assume that* $\Gamma = \mathrm{diag}(\gamma_i)$, $\gamma_i > 0$, $i = 1, 2, \ldots, k$ *and* $M = (m_{ij})_{i,j=1}^{k}$, $m_{ij} \geqslant 0$. *We have that*

$$\rho(\Gamma^{-1}M) < 1 \Leftrightarrow -\Gamma + M \text{ is exponentially stable.}$$

Proof. "⇒" Consider the matrix $-\lambda\Gamma+M$. Observe that if $\lambda = 0$, since $m_{ij} \geqslant 0$ for any $i, j = 1, 2, \ldots, k$, then there exists at least a nonnegative eigenvalue of M. Therefore, for $\lambda > 0$ sufficiently large all the eigenvalues of $-\lambda\Gamma + M$ have negative real part and so there exists $\lambda_{\max} \geqslant 0$ such that

$$\sigma(-\lambda_{\max}\Gamma + M) \cap \{\mathrm{Re}\, z \geqslant 0\} \neq \emptyset,$$

where $\sigma(Q)$ denotes the spectrum of the matrix Q. We want to prove that $\lambda_{\max} < 1$. We argue by contradiction and, hence, assume that $\lambda_{\max} \geqslant 1$. Consider the system

$$y' = (-\lambda_{\max}\Gamma + M)y$$

which has a nontrivial T-periodic solution $y(t)$, for some $T > 0$, which satisfies

$$y(t) = \int_{-\infty}^{t} \mathrm{e}^{-\lambda_{\max}\Gamma(t-s)} My(s)\,\mathrm{d}s.$$

Denote by $\|y\|^v = (\max_t |y_1(t)|, \ldots, \max_t |y_k(t)|)$; since Γ is a diagonal matrix we have that

$$\|y\|^v \leqslant \frac{1}{\lambda_{\max}}\Gamma^{-1}M\|y\|^v \quad \text{with} \quad \frac{1}{\lambda_{\max}} \leqslant 1,$$

and so $\|y\|^v = 0$, which is a contradiction.

Let us now prove "⇐". For this, again we argue by contradiction assuming that $\rho(\Gamma^{-1}M) \geqslant 1$. Applying the Frobenius theorem we obtain the existence of $\lambda = \rho(\Gamma^{-1}M)$ such that $\lambda x = \Gamma^{-1}Mx$ with $x \geqslant 0$ and $x \neq 0$. From this, we get

$$(\lambda - 1)\Gamma x = -\Gamma x + Mx. \tag{25}$$

Consider now the function $y(t)$ given by

$$y(t) = \mathrm{e}^{(-\Gamma+M)t}x$$

which is easily seen to be the solution of the Cauchy problem

$$\begin{aligned} y' &= (-\Gamma + M)y, \\ y(0) &= x. \end{aligned}$$

It is not difficult to show that the matrix $\mathrm{e}^{(-\Gamma+M)t}$ has all the elements nonnegative for any $t \geqslant 0$. From (25) we have

$$y'(t) = \mathrm{e}^{(-\Gamma+M)t}(\lambda - 1)\Gamma x,$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\|y(t)\|^2 = 2(\lambda - 1)\langle \mathrm{e}^{(-\Gamma+M)t}\Gamma x,\ y(t)\rangle \geqslant 0$$

for any t, which is a contradiction of the fact that $y(t) \to 0$ as $t \to +\infty$. This concludes the proof. □

We have also the following

COROLLARY 6. *Assume that the matrix* $-\Gamma + M$ *is exponentially stable. Then for every* $1 \leqslant h \leqslant k$ *the matrix* $-\widetilde{\Gamma} + \widetilde{M}$*, where* $\widetilde{M} = (m_{h+i,h+j})_{i,j=1}^{k-h}$ *and* $\widetilde{\Gamma} = \operatorname{diag}(\gamma_{h+j})_{j=1}^{k-h}$*, is exponentially stable.*

Proof. We argue once again by contradiction, hence we assume that the matrix $-\widetilde{\Gamma}+\widetilde{M}$ is not exponentially stable. Then by Proposition 5, we have $\rho(\widetilde{\Gamma}^{-1}\widetilde{M}) \geqslant 1$. By the theorem of Frobenius we obtain that there exist $\lambda = \rho(\widetilde{\Gamma}^{-1}\widetilde{M})$ and $\widetilde{x} \geqslant 0$, $\widetilde{x} \neq 0$, such that

$$\lambda\widetilde{x} = \widetilde{\Gamma}^{-1}\widetilde{M}\widetilde{x}.$$

Therefore, $(\lambda - 1)\widetilde{\Gamma}\widetilde{x} = (-\widetilde{\Gamma} + \widetilde{M})\widetilde{x}$. Let us consider the k-dimensional vector $x = (0, \ldots, 0, \widetilde{x})$. Taking $y(t) = \mathrm{e}^{(-\Gamma+M)t}x$ we have

$$y'(t) = \mathrm{e}^{(-\Gamma+M)t}(-\Gamma + M)\begin{pmatrix} 0 \\ \widetilde{x} \end{pmatrix} = \mathrm{e}^{(-\Gamma+M)t}\begin{pmatrix} \widehat{x} \\ (\lambda - 1)\widetilde{\Gamma}\widetilde{x} \end{pmatrix}, \tag{26}$$

where all the h-coordinates of $\widehat{x}$ are nonnegative. Since all the elements of the matrix $\mathrm{e}^{(-\Gamma+M)t}$ are nonnegative, by using the same arguments of Proposition 5 we obtain from (26) that $\frac{\mathrm{d}}{\mathrm{d}t}\|y(t)\|^2 \geqslant 0$, which is a contradiction, since $y(t) \to 0$ as $t \to \infty$. □

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