



An averaging method for singularly perturbed systems of semilinear differential inclusions with analytic semigroups

Mikhail Kamenskii^{a,1,*}, Paolo Nistri^{b,*}

^a*Department of Mathematics, Voronezh State University, Voronezh, Russia*
^b*Dipartimento di Ingegneria dell'Informazione, Università degli Studi di Siena, Via Roma 56, 53100 Siena, Italy*

Received 16 July 2001; accepted 1 May 2002

Abstract

We consider a system of two semilinear parabolic inclusions depending on a small parameter $\varepsilon > 0$ which is present both in front of the derivative in one of the two inclusions and in the nonlinear terms to model high-frequency inputs.

The aim is to provide conditions in order to guarantee, for $\varepsilon > 0$ sufficiently small, the existence of periodic solutions and in order to study their behaviour as ε tends to zero. Our assumptions permit the definition of upper semicontinuous, convex valued, compact vector operators whose fixed points represent the sought-after periodic solutions. The existence of fixed points is shown by using topological degree theory arguments.

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Keywords: Periodic solutions; Averaging method; Differential inclusions; Singularly perturbed systems; Analytic semigroups

1. Introduction

This paper deals with the question of the existence and dependence on a small parameter $\varepsilon > 0$ of periodic solutions of a singularly perturbed system of semilinear

* Corresponding author.

E-mail addresses: mikhail@kam.vsu.ru (M. Kamenskii), pnistri@dii.unisi.it (P. Nistri).

¹ Supported by the research project “Qualitative analysis and control of dynamical systems” at the University of Siena and by RFRB grants 02-01-00189 and 02-01-00370.

parabolic inclusions with high-frequency nonlinear inputs. This system has the form

$$\begin{cases} \dot{x}(\tau) \in A_1 x(\tau) + f_1(\tau/\varepsilon, x(\tau), y(\tau)), \\ \varepsilon \dot{y}(\tau) \in A_2 y(\tau) + f_2(\tau/\varepsilon, x(\tau), y(\tau)), \end{cases} \tag{1}$$

where $A_i, i = 1, 2$, are the infinitesimal generators of analytic semigroups acting in the separable Banach spaces $E_i, i = 1, 2$, and $\varepsilon > 0$ is the singular perturbation parameter which also models the high frequency of the inputs $f_i, i = 1, 2$. The operators $A_i^{-1}, i = 1, 2$, are assumed to be compact. The multivalued operators

$$f_i : \mathbf{R} \times E_1 \times E_2 \rightarrow E_i, \quad i = 1, 2$$

are T -periodic with respect to the first variable and they are assumed to be subordinate to the fractional powers $A_i^\alpha, i = 1, 2$, in a sense that will be specified later in (F₁)–(F₃).

Our assumptions permit us to define multivalued operators $\Phi_\varepsilon, \varepsilon > 0$, which are compact, upper semicontinuous, with nonempty compact convex values and thus suitable for the application of classical fixed point theory. Indeed, the fixed points of $\Phi_\varepsilon, \varepsilon > 0$, will represent the T/ε -periodic solutions to (1). The existence of fixed points of $\Phi_\varepsilon, \varepsilon > 0$, is proved by using the topological degree theory. Specifically, when $\varepsilon = 0$ starting from (1) we define in E_1 an averaging multivalued operator which turns out to be compact, upper semicontinuous with compact convex values, and we assume that there exists an open bounded set $U \subset E_1$ such that this operator has in \bar{U} a unique fixed point $x^* \in U$ with nonzero topological degree. Then, by linear homotopies and the reduction property of the topological degree, we prove that for $\varepsilon > 0$ sufficiently small, the compact, upper semicontinuous, vector field $(I - \Phi_\varepsilon)$ has nonzero topological degree and so Φ_ε has a fixed point. Furthermore, the behaviour of such fixed point as $\varepsilon \rightarrow 0$ is established.

Similar topological methods to those presented in this paper are used in [12] to study the dependence of periodic solutions of a system of parabolic inclusion on a large parameter. In particular, the case when the system is influenced by exterior forces of high frequency is considered. The method used to investigate such dependence is based on the averaging principle formulated in terms of the topological index for multivalued maps. In [12] the involved semigroups are not compact and so the topological degree theory for condensing operators, with respect to a suitable measure of noncompactness, is employed.

In [1,11] condensing operators (with respect to different measure of noncompactness) and the relative topological degree theory in locally convex spaces represent the main tools for developing a singular perturbation theory for a system of semilinear differential inclusions in infinite dimensional spaces by means of topological methods. In fact, if we consider the uniform topology with respect to the slow variable and the weak topology for the fast variable, respectively, in $C_T(E_1)$ and $L_T^1(E_2)$ where T stands for T -periodic functions, then we are able to prove that the solution map $\varepsilon \rightarrow \Sigma_\varepsilon$ has nonempty values and is upper semicontinuous at $\varepsilon = 0$ in this topology. In [1] the solution set Σ_ε represents the set of all the solutions of a Cauchy problem associated to the considered system of two differential inclusion. While, in [11] the solution set Σ_ε represents the set of periodic solutions of the system. Singular perturbation methods for partial differential equations are also intensively studied (see e.g. [17,18]).

In finite dimensional spaces several attempts to develop a singular perturbation theory for systems of differential inclusions have been made by using different approaches. We mention here the following papers: [3,4,7,16,19,20].

In all these papers the common problem is to choose suitable topologies and conditions on the system under which the solution map $\varepsilon \rightarrow \Sigma_\varepsilon$ turns out to be upper semicontinuous at $\varepsilon = 0$ in the considered topology. In particular, in [3,19,20] the upper semicontinuity in the uniform topology at $\varepsilon = 0$ of a suitably defined subset of the set of solutions pairs $(x^\varepsilon, y^\varepsilon)$ was established, obtaining a Tikhonov type theorem for singularly perturbed differential inclusions. A similar result has been obtained in [9] in infinite dimensional spaces.

In [4] the convergence in the Kuratowski sense of the solution set Σ_ε as $\varepsilon \rightarrow 0$ to the solution set of the reduced system is obtained by considering as limit of the fast solutions the Radon probability measures.

Applications to optimal control problems of the averaging method for singularly perturbed differential inclusions have been presented in [5,6,8]. Specifically, in [5,6] the problem of the approximation of the optimal solution of the singularly perturbed system by the solution of the reduced system is investigated by means of suitably introduced families of periodic optimization problems. It is proved that a necessary and sufficient condition for the validity of such approximation is that these families admit steady-state solutions. Furthermore, it is shown that if these families have only time-dependent periodic solutions it is possible to use these periodic solutions in order to approximate the solution of the singularly perturbed system.

In [8] an average cost functional in an infinite time interval is considered and the problem of its minimization on the trajectories of a nonlinear control system is treated. An explicit relation between controllability time and required period of suboptimal periodic orbits is provided. Indeed, the existence of suboptimal periodic trajectories can be stated in terms of a region of complete controllability for the control system.

The present paper deals with a system of singularly perturbed semilinear differential inclusions in infinite dimensional spaces, where the linear parts are represented by infinitesimal generators of analytic semigroup with compact inverse. Moreover, the nonlinear inputs are of high frequency with respect to time.

The main contribution of this paper consists in extending to the infinite dimensional case, for the periodic solutions to (1), both the employ of an averaging method and the upper semicontinuity (in the uniform topology) of the solution map. Furthermore, the combined effect of external inputs of high frequency with the presence of singular perturbations is considered. Finally, the methods employed to achieve these results are different from those used in the cited literature; in fact, they are based on the theory of analytic semigroups, the related operators of fractional powers and the topological degree theory for compact vector fields. In this paper, we do not consider any cost functional associated to system (1), which could be considered as the model of a nonlinear control problem in infinite dimensional spaces. Such problems will be considered elsewhere.

The paper is organized as follows. In Section 2 we state the problem, we formulate the assumptions which permit to solve it and we give some preliminary results. In Section 3 we formulate and we prove our main result after introducing four preliminary

lemmas. Specifically, we assume that the limit ($\varepsilon=0$) averaging operator has an unique fixed point x^* with topological degree different from zero in a suitable (boundary fixed points free) open set U containing x^* . Then, by means of an admissible linear homotopy and the relevant properties of the topological degree theory for compact, upper semicontinuous, compact convex valued vector field, we show that Φ_ε has a fixed point $(x^\varepsilon, y^\varepsilon)$ with $x^\varepsilon(t) \in U$. Furthermore $(x^\varepsilon, y^\varepsilon) \rightarrow (x^*, y^0)$ as $\varepsilon \rightarrow 0$, where y^0 is a T -periodic solution of the second equation in (1) with $\varepsilon = 1$ and $x(\tau) \equiv x^*$.

2. Formulation of the problem, assumptions and definitions

In this paper we consider the following system of differential inclusions:

$$\begin{cases} \dot{x}(\tau) \in A_1x(\tau) + f_1(\tau/\varepsilon, x(\tau), y(\tau)), \\ \varepsilon \dot{y}(\tau) \in A_2y(\tau) + f_2(\tau/\varepsilon, x(\tau), y(\tau)), \end{cases} \tag{2}$$

where $A_i, i = 1, 2$, are the infinitesimal generators of analytic semigroups $e^{A_it}, t \geq 0$, with A_i^{-1} compact, acting in the Banach spaces $E_i, i = 1, 2$, and $\varepsilon > 0$ is a (small) singular perturbation parameter.

Observe that, without loss of generality, we can assume that the semigroups $e^{A_it}, t \geq 0, i = 1, 2$, satisfy the inequalities

$$\|e^{A_it}\| \leq ce^{-d_it}$$

for some $d_i > 0$ and $c > 0$. Indeed, it is sufficient to add and subtract convenient linear operators to the right-hand side of the differential inclusions in (1). The multivalued operators $f_i : \mathbf{R} \times E_1 \times E_2 \rightarrow E_i, i = 1, 2$, are assumed T -periodic, $T > 0$, in the first variable.

We pose the following:

Problem. *To study, for $\varepsilon > 0$ fixed, the question of the existence of a T/ε -periodic solution to system (2), and to describe its behaviour as $\varepsilon \rightarrow 0$.*

Theorem 2 will provide sufficient conditions for the solution of the proposed problem.

We recall some basic results from the theory of analytic semigroups which we will use in the sequel (see, for instance, [10,14,15]).

Theorem 1. *The closed operator A with dense domain is the infinitesimal generator of the analytic semigroup e^{At} if and only if the resolvent set of A contains a half-plane $\text{Re } \lambda \leq \sigma_0$ and the resolvent satisfies there the inequality*

$$\|(\lambda I - A)^{-1}\| \leq C(1 + |\lambda|)^{-1} \quad \text{for some } C > 0.$$

If A is the infinitesimal generator of the analytic semigroup e^{At} then

$$e^{At} = -\frac{1}{2\pi i} \int_{R(\beta, \sigma)} e^{\lambda t} (\lambda I - A)^{-1} d\lambda, \quad t > 0,$$

where $R(\beta, \sigma)$ consists of the two rays

$$\lambda = \sigma + \rho e^{-i\beta} \quad \text{and} \quad \lambda = \sigma + \rho e^{i\beta}$$

with $\sigma \leq \sigma_0$, $\arcsin 1/c < \beta < \pi/2$. Observe that the compactness of A^{-1} implies the compactness of the semigroup e^{At} , $t > 0$.

If $\sigma_0 < 0$ then the negative fractional powers of A are defined by the formula

$$A^{-\alpha} = -\frac{1}{2\pi i} \int_{R(\beta, \sigma)} \lambda^{-\alpha} (\lambda I - A)^{-1} d\lambda, \quad 0 < \alpha < 1.$$

The operator

$$A^\alpha e^{At} = -\frac{1}{2\pi i} \int_{R(\beta, \sigma)} \lambda^\alpha e^{\lambda t} (\lambda I - A)^{-1} d\lambda, \quad t > 0$$

satisfies the estimate

$$\|A^\alpha e^{At}\| \leq Ct^{-\alpha}. \tag{3}$$

From this inequality one obtains

$$\|A^{-\alpha}(e^{At} - I)\| \leq Ct^\alpha.$$

In the sequel $K_c E$ will denote the collections of all the nonempty compact convex subsets of E .

We assume that the T -periodic multivalued operators f_i , $i = 1, 2$, are subordinate to the fractional powers A_i^α in the following sense:

- (F₁) the operators $(t, x, y) \rightarrow f_i(t, A_1^{-\alpha}x, A_2^{-\alpha}y)$ act from $\mathbf{R} \times E_1 \times E_2$ to $K_c E_i$ and are bounded on bounded sets;
- (F₂) for every pair $(x, y) \in E_1 \times E_2$ the multivalued maps $f_i(\cdot, A_1^{-\alpha}x, A_2^{-\alpha}y) : [0, T] \rightarrow K_c(E_i)$, $i = 1, 2$, admit at least a measurable selection;
- (F₃) for almost all (a.a.) $t \in [0, T]$ the multivalued maps $f_i(t, A_1^{-\alpha}\cdot, A_2^{-\alpha}\cdot) : E_1 \times E_2 \rightarrow K_c(E_i)$, $i = 1, 2$, are upper semicontinuous.

We would like to point out that the formulation of assumptions (F₁)–(F₃) permits to consider the case when the nonlinearities are defined, with respect to the space variables (x, y) , only on $\text{Dom}(A_1^\alpha) \times \text{Dom}(A_2^\alpha)$. This is important for the applications of our abstract results; in fact this situation occurs, for instance, when A_1 and A_2 are generated by a Laplace operator $\sum_{j=1}^m \partial^2 u / \partial^2 \xi_j$ with Dirichlet conditions, the nonlinearities contain terms of the form u^r and $(\partial u / \partial \xi_j)^{q_j}$ and we take as E_1, E_2 some suitable space L^p . For the singlevalued case see the details in [14].

We also remark that, since we have assumed that A_1^{-1} and A_2^{-1} are compact (and consequently the same is true for $A_1^{-\alpha}$ and $A_2^{-\alpha}$), the condition that $f_i(t, A_1^{-\alpha}x, A_2^{-\alpha}y) \subset E_i$ is compact is inessential for the validity of our results. Nevertheless we have assumed here this condition, since, under the assumption that $f_i(t, A_1^{-\alpha}x, A_2^{-\alpha}y) \in K_c(E_i)$, for the basic facts concerning the integration of the multivalued maps we can directly refer to the general theory as presented, for instance, in [2,13].

Finally, it is not hard to see that the problem of finding T/ε -periodic solutions to system (2) is equivalent to that of finding T -periodic solutions of the following system obtained from (2) after the change of variable $t = \tau/\varepsilon$.

$$\begin{cases} x'(t) \in \varepsilon A_1 x(t) + \varepsilon f_1(t, x(t), y(t)), \\ y'(t) \in A_2 y(t) + f_2(t, x(t), y(t)). \end{cases} \tag{4}$$

The equivalence between (2) and (4) is understood in the sense that the set of T/ε -periodic solutions to system (2) coincides with the set of T -periodic solutions to system (4) after the change of variable $t = \tau/\varepsilon$.

In what follows, $C_T(E_1 \times E_2)$ will denote the Banach space of T -periodic continuous functions defined on $[0, T]$ with values in $E_1 \times E_2$.

We define multivalued operators φ_ε^i , $i = 1, 2$, on $C_T(E_1 \times E_2)$ as follows:

$$\begin{aligned} \varphi_\varepsilon^i(x, y) &= \{v_i : [0, T] \rightarrow E_i, \text{ measurable and } T\text{-periodic} : \\ & v_i(t) \in f_i(t, A_1^{-\alpha} x(t), A_2^{-\alpha} y(t)) \text{ for a.a. } t \in [0, T]\}. \end{aligned} \tag{5}$$

Now we can introduce the multivalued operators

$$\Phi_\varepsilon : C_T(E_1 \times E_2) \rightrightarrows C_T(E_1 \times E_2)$$

by

$$\Phi_\varepsilon(x, y) = \{(\Pi_1^\alpha(\varepsilon)v_1, \Pi_2^\alpha v_2) : v_i \in \varphi_\varepsilon^i(x, y), i = 1, 2\},$$

where

$$\begin{aligned} \Pi_1^\alpha(\varepsilon)v_1(t) &:= e^{\varepsilon A_1 t} (I - e^{\varepsilon A_1 T})^{-1} \int_0^T A_1^\alpha e^{\varepsilon A_1 (T-s)} \varepsilon v_1(s) ds \\ &+ \int_0^t A_1^\alpha e^{\varepsilon A_1 (t-s)} \varepsilon v_1(s) ds, \\ \Pi_2^\alpha v_2(t) &:= e^{A_2 t} (I - e^{A_2 T})^{-1} \int_0^T A_2^\alpha e^{A_2 (T-s)} v_2(s) ds \\ &+ \int_0^t A_2^\alpha e^{A_2 (t-s)} v_2(s) ds. \end{aligned}$$

We can now prove the following result.

Proposition 1. *Assume that the operators A_i^{-1} , $i = 1, 2$, are compact. Then $\Phi_\varepsilon, \varepsilon > 0$, are well-defined compact, upper semicontinuous operators on $C_T(E_1 \times E_2)$ with compact convex values in $C_T(E_1 \times E_2)$.*

Proof. First observe that (F_1) implies that the functions v_i , $i = 1, 2$, defined by (5) are both measurable and uniformly bounded. Therefore, by using (3), we can show that $\Pi_1^\alpha(\varepsilon)v_1 \in C_T(E_1)$ and $\Pi_2^\alpha v_2 \in C_T(E_2)$. We prove in the sequel only that $\Pi_2^\alpha v_2 \in C_T(E_2)$ and we denote all the constants by the same letter c ; the other assertion can be proved

in a similar way. Indeed,

$$\|A_2^\beta \Pi_2^\alpha v_2(t)\| \leq c \left[(1 - e^{-d_2 t})^{-1} \int_0^T \frac{ds}{(T-s)^{\alpha+\beta}} + \int_0^t \frac{ds}{(t-s)^{\alpha+\beta}} \right] \|v_2\|_{L_T^\infty}.$$

Moreover, if $t_1 < t_2$ one has

$$\begin{aligned} & \|\Pi_2^\alpha v_2(t_2) - \Pi_2^\alpha v_2(t_1)\| \\ & \leq c \left\{ \left[(1 - e^{-d_2 t})^{-1} \int_0^T \frac{ds}{(T-s)^{\alpha+\beta}} + \int_0^{t_1} \frac{ds}{(t-s)^{\alpha+\beta}} \right] \right. \\ & \quad \cdot \sup_s \|(e^{(t_2-t_1)A_2} - I)A_2^{-\beta} v_2(s)\| + \left. \int_{t_1}^{t_2} \frac{ds}{(t_2-s)^\alpha} \|v_2\|_{L_T^\infty} \right\}, \end{aligned}$$

where $0 < \beta < 1 - \alpha$, and $\|\cdot\|_{L_T^\infty}$ denotes the norm in the Banach space of T -periodic functions $L_T^\infty(E_2)$. By using the compactness of $A_2^{-\beta}$, the boundedness of v_2 and the fact that an analytic semigroup is a C_0 -semigroup we obtain that $\Pi_2^\alpha v_2 \in C_T(E_2)$. Furthermore, if $\Omega \subset L_T^\infty(E_2)$ is such that from $v \in \Omega$ it follows $\|v\|_{L_T^\infty} \leq M$, for some constant $M > 0$, then $\Pi_2^\alpha \Omega$ is a relatively compact set in $C_T(E_2)$.

Let us now prove that Φ_ε is an upper semicontinuous map from $C_T(E_1 \times E_2)$ in $K_c(C_T(E_1 \times E_2))$. It is easy to see, compare e.g. ([12, Proposition 2.1]), that if $x_n \rightarrow x_0$ and $y_n \rightarrow y_0$ in $C_T(E_i)$, $i = 1, 2$, respectively, then the sequences $\{v_i^n\}$ are weakly compact in $L_T^1(E_i)$ and if v_i^0 is a limit point of $\{v_i^n\}$ we have that

$$v_i^0(t) \in f_i(t, A_1^{-\alpha} x_0(t), A_2^{-\alpha} y_0(t)) \quad \text{for a.a. } t \in [0, T].$$

By (F₁) we have that the sequences $\{v_i^n\}$, $i = 1, 2$, are uniformly bounded and so by (3), for any $\varepsilon > 0$, we have that

$$\Pi_1^\alpha(\varepsilon)v_1^n \rightarrow \Pi_1^\alpha(\varepsilon)v_1^0$$

and

$$\Pi_2^\alpha v_2^n \rightarrow \Pi_2^\alpha v_2^0.$$

Finally, it is immediate to verify that, under our assumptions, Φ_ε has convex, compact values. \square

Following [1,14] we give the definition of T -periodic solution to (4).

Definition 1. By a T -periodic solution to (4) we mean a fixed point of the multivalued operator $\Phi_\varepsilon : C_T(E_1 \times E_2) \rightarrow K_c(C_T(E_1 \times E_2))$.

The conditions under which we will show the existence of T -periodic solutions of (4) will be expressed in terms of an averaging operator which we introduce in the sequel. For this, we assume in the rest of the paper the following condition:

(A) for any nonempty, bounded set $\Omega \subset E_1$ the set Y_Ω of all the T -periodic solutions of the differential inclusion

$$y'(t) \in A_2 y(t) + \lambda f_2(t, x, y(t)), \quad \text{for a.a. } t \in [0, T], \lambda \in [0, 1], x \in \Omega, \tag{6}$$

is nonempty and bounded.

For fixed $x \in E_1$, let

$V = \{v : [0, T] \rightarrow E_1, \text{ measurable and } T\text{-periodic:}$

$$v(t) \in f_1(t, A_1^{-\alpha}x, A_2^{-\alpha}y(t)) \text{ for a.a. } t \in [0, T]\},$$

where $y \in Y_x^1$ and Y_x^1 is the solution set of (6) corresponding to $\lambda = 1$ and $\Omega = \{x\}$.

Finally, we define the multivalued averaging operator $A_1^{-1+\alpha}F_1(A_1^{-\alpha}\cdot) : E_1 \rightarrow K_c(E_1)$ as follows:

$$A_1^{-1+\alpha}F_1(A_1^{-\alpha}x) := \overline{\text{co}} \left\{ A_1^{-1+\alpha} \frac{1}{T} \int_0^T v(s) ds : v \in V \right\}.$$

We can prove the following.

Proposition 2. $A_1^{-1+\alpha}F_1(A_1^{-\alpha}\cdot) : E_1 \rightarrow K_c(E_1)$ is a well defined, compact upper semicontinuous operator.

Proof. Let $\Omega \subset E_1$ be a bounded set, then the set Y_Ω^1 is bounded and $f_1(t, A_1^{-\alpha}x, A_2^{-\alpha}y(t))$, $y \in Y_\Omega^1$, is uniformly bounded. Using the compactness of $A_1^{-1+\alpha}$ we obtain the compactness of $A_1^{-1+\alpha}F_1(A_1^{-\alpha}\cdot)$.

Since the application $x \rightarrow Y_x^1$ is u.s.c, then from $x_n \rightarrow x_0$ and $y_n \in Y_{x_n}^1$, $y_n \rightarrow y_0$ we obtain that the functions $v_n(t) \in f_1(t, A_1^{-\alpha}x_n, A_2^{-\alpha}y_n(t))$ for a.a. $t \in [0, T]$ are weakly convergent in $L_T^1(E_1)$ to a function $v_0(t) \in f_1(t, A_1^{-\alpha}x_0, A_2^{-\alpha}y_0(t))$ for a.a. $t \in [0, T]$.

Therefore, $A_1^{-1+\alpha}(1/T) \int_0^T v_n(s) ds \rightarrow A_1^{-1+\alpha}(1/T) \int_0^T v_0(s) ds$ and so the operator $x \rightarrow \psi_x = \{A_1^{-1+\alpha}(1/T) \int_0^T v(s) ds : v \in V\}$ is upper semicontinuous.

On the other hand, the operator $\overline{\text{co}}(\cdot)$ maintains the upper semicontinuity. \square

3. Main result

We are now in the position to formulate our main result.

Theorem 2. Assume that for some open bounded set $U \subset E_1$ the inclusion

$$x \in A_1^{-1+\alpha}F_1(A_1^{-\alpha}x)$$

has a unique solution $x^* \in \bar{U}$ with $x^* \notin \partial U$ and

$$\text{deg}(I - A_1^{-1+\alpha}F_1(A_1^{-\alpha}\cdot), U) \neq 0. \tag{7}$$

Then, for sufficiently small $\varepsilon > 0$, system (4) has a T -periodic solution $(x^\varepsilon, y^\varepsilon)$ such that $x^\varepsilon \in U$, $x^\varepsilon \rightarrow x^*$, $y^\varepsilon \rightarrow y^0$ as $\varepsilon \rightarrow 0$, where y^0 is a T -periodic solution to (6) corresponding to $x = x^*$ and $\lambda = 1$.

In order to prove Theorem 2 we introduce the following operators $P(\varepsilon)$, $P_\alpha(\varepsilon)$ acting from $C_T(E_1)$ or from $L_T^p(E_1)$, $1 \leq p \leq +\infty$, with values in E_1 :

$$P(\varepsilon)v = \varepsilon(I - e^{\varepsilon A_1 T})^{-1} \int_0^T e^{\varepsilon A_1(T-s)} v(s) \, ds,$$

$$P_\alpha(\varepsilon)v = \varepsilon(I - e^{\varepsilon A_1 T})^{-1} \int_0^T A_1^\alpha e^{\varepsilon A_1(T-s)} v(s) \, ds.$$

The proof of Theorem 2 relies on the following four lemmas which describe the properties of the operators $P(\varepsilon)$ and $P_\alpha(\varepsilon)$. Recall that all the constants will be denoted by the same letter c .

Lemma 1. *The operators $P(\varepsilon)$, $\varepsilon > 0$, are uniformly bounded with respect to ε as operators acting from $C_T(E_1)$ or from $L_T^p(E_1)$, $1 \leq p \leq \infty$, to E_1 .*

Proof. Since the semigroup $e^{A_1 t}$, $t \geq 0$, satisfies the inequality

$$\|e^{A_1 t}\| \leq c e^{-d_1 t}$$

we have

$$(I - e^{\varepsilon A_1 T})^{-1} = \sum_{k=0}^{\infty} e^{\varepsilon A_1 k T} \tag{8}$$

and

$$\|(I - e^{\varepsilon A_1 T})^{-1}\| \leq c(1 - e^{-\varepsilon d_1 T})^{-1}. \tag{9}$$

Therefore,

$$\|P(\varepsilon)v\| \leq c\|v\|_{C_T}$$

and

$$\|P(\varepsilon)v\| \leq c\|v\|_{L_T^p}, \quad 1 \leq p \leq \infty. \quad \square$$

Lemma 2. *Let $q = p/(p - 1)$, where $1 < p < 1/\alpha$. The operators $P_\alpha(\varepsilon)$ are uniformly bounded with respect to $\varepsilon > 0$ as operators acting from $C_T(E_1)$ or from $L_T^q(E_1)$ to E_1 .*

Proof. Write $(I - e^{\varepsilon A_1 T})^{-1}$ as follows:

$$(I - e^{\varepsilon A_1 T})^{-1} = \sum_{k=0}^m e^{\varepsilon A_1 k T} + (I - e^{\varepsilon A_1 T})^{-1} e^{\varepsilon A_1(m+1)T},$$

where $m = [1/\varepsilon]$ is the integer part of $1/\varepsilon$. Then

$$\begin{aligned} P_\alpha(\varepsilon)v &= \varepsilon \sum_{k=0}^m \int_0^T A_1^\alpha e^{\varepsilon A_1((k+1)T-s)} v(s) \, ds \\ &\quad + \varepsilon(I - e^{\varepsilon A_1 T})^{-1} \int_0^T A_1^\alpha e^{\varepsilon A_1((m+1)T-s)} v(s) \, ds. \end{aligned}$$

If $v \in C_T(E_1)$ or $v \in L_T^\infty(E_1)$ then, by using (3), we obtain

$$\begin{aligned} \|P_\alpha(\varepsilon)v\| &\leq c \left[\varepsilon^{1-\alpha} \sum_{k=0}^m [((k+1)T)^{1-\alpha} - (kT)^{1-\alpha}] + (\varepsilon mT)^{-\alpha} \right] \|v\|_{L_T^\infty} \\ &= c[(\varepsilon(m+1)T)^{1-\alpha} + (\varepsilon mT)^{-\alpha}] \|v\|_{L_T^\infty} \leq c\|v\|_{L_T^\infty}. \end{aligned}$$

If $v \in L_T^q(E_1)$ where $q = p/(p-1)$, $1 < p < 1/\alpha$, then, using again (3), we obtain

$$\begin{aligned} \left\| \varepsilon \int_0^T A_1^\alpha e^{\varepsilon A_1(T-s)} v(s) \, ds \right\| &\leq c\varepsilon^{1-\alpha} \int_0^T \frac{\|v(s)\|}{(T-s)^\alpha} \, ds \\ &\leq c\varepsilon^{1-\alpha} T^{(1-\alpha p)/p} \|v\|_{L_T^q}, \end{aligned} \tag{10}$$

$$\begin{aligned} &\left\| \varepsilon \sum_{k=1}^m \int_0^T A_1^\alpha e^{\varepsilon A_1((k+1)T-s)} v(s) \, ds \right\| \\ &\leq c\varepsilon^{1-\alpha} \sum_{k=1}^m \frac{1}{k^\alpha} \|v\|_{L_T^1} \leq c\varepsilon^{1-\alpha} \left(1 + \int_1^m \frac{1}{\tau^\alpha} \, d\tau \right) \|v\|_{L_T^q} \leq c(\varepsilon m)^{1-\alpha} \|v\|_{L_T^q} \end{aligned} \tag{11}$$

and

$$\begin{aligned} &\left\| \varepsilon(I - e^{AT})^{-1} \int_0^T A_1^\alpha e^{\varepsilon A_1((m+1)T-s)} v(s) \, ds \right\| \\ &\leq c(\varepsilon m)^{-\alpha} \|v\|_{L_T^1} \leq c(\varepsilon m)^{-\alpha} \|v\|_{L_T^q}. \end{aligned} \tag{12}$$

Collecting (10)–(12) we get

$$\begin{aligned} \|P(\varepsilon)v\| &\leq c[\varepsilon^{1-\alpha} + (\varepsilon[1/\varepsilon])^{1-\alpha} + (\varepsilon[1/\varepsilon])^{-\alpha}] \|v\|_{L_T^q} \\ &\leq c\|v\|_{L_T^q}. \quad \square \end{aligned}$$

Lemma 3. *If $v \in C_T(E_1)$ or $v \in L_T^p(E_1)$, $1 \leq p \leq \infty$, then*

$$P(\varepsilon)v \rightarrow -A_1^{-1} \frac{1}{T} \int_0^T v(s) \, ds \quad \text{as } \varepsilon \rightarrow 0.$$

Proof. By Lemma 1 it is enough to prove Lemma 3 only when $v \in C_T(E_1)$. In order to do this observe that for $w \in D(A_1)$ one has

$$\frac{e^{\varepsilon A_1 T} - I}{\varepsilon T} w - A_1 w \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

By applying to this relation the uniformly bounded operator $\varepsilon(I - e^{\varepsilon A_1 T})^{-1} A_1^{-1}$ we obtain

$$-\frac{1}{T} A_1^{-1} w - \varepsilon(I - e^{\varepsilon A_1 T})^{-1} w \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \tag{13}$$

From (9) we have that (13) holds true for all $w \in E_1$. Since the set $\{v(s) : s \in [0, T]\}$ is compact one has

$$e^{\varepsilon A_1(T-s)} v(s) \rightarrow v(s)$$

uniformly with respect to s . Therefore

$$\varepsilon(I - e^{\varepsilon A_1 T})^{-1} \int_0^T e^{\varepsilon A_1(T-s)} v(s) \, ds \rightarrow -\frac{1}{T} A_1^{-1} \int_0^T v(s) \, ds. \quad \square$$

Lemma 4. *If $v \in L_T^q(E_1)$, where $q = p/(p - 1)$, $1 < p < 1/\alpha$, then*

$$P_\alpha(\varepsilon)v \rightarrow -A_1^{-1+\alpha}(1/T) \int_0^T v(s) \, ds \quad \text{as } \varepsilon \rightarrow 0.$$

Proof. By Lemma 2 it is enough to prove Lemma 4 in the case when v belongs to a dense subset D of $C_T(E_1)$. We choose as D the subset of $C_T(E_1)$ of all functions v for which $A_1^\alpha v$ are continuous. Therefore, by Lemma 3 we have

$$P_\alpha(\varepsilon)v = P(\varepsilon)A_1^\alpha v \rightarrow -\frac{1}{T} A_1^{-1} \int_0^T A_1^\alpha v(s) \, ds = -\frac{1}{T} A_1^{-1+\alpha} \int_0^T v(s) \, ds. \quad \square$$

Finally, we prove Theorem 2.

Proof of Theorem 2. Define the operator $\Phi_0 : C_T(E_1 \times E_2) \rightarrow C_T(E_1 \times E_2)$ as follows:

$$\begin{aligned} \Phi_0(x, y) = \{ & (A_1^{-1+\alpha} F_1(A_1^{-\alpha} x(0)), \Pi_2^\alpha \bar{v}_2) : \bar{v}_2(t) \in f_2(t, A_1^{-\alpha} x(0), A_2^{-\alpha} y_2(t)) \\ & \text{for a.a. } t \in [0, T] \}. \end{aligned}$$

Let $r > 0$ be such that for $x \in \tilde{U}$ inclusions (6) do not have solutions y such that

$$\|y\| \geq r. \tag{14}$$

We prove now that, for $\varepsilon > 0$ sufficiently small, Φ_ε and Φ_0 are linearly homotopic in $\tilde{U} \times B(0, r)$, where $\tilde{U} = \{x : x \in C_T(U)\}$. Assume the contrary, then there exist sequences $\varepsilon_n \rightarrow 0$, $\mu_n \in [0, 1]$, $\mu_n \rightarrow \mu_0$ and $(x_n, y_n) \in \partial(\tilde{U} \times B(0, r))$ such that

$$\begin{aligned} x_n(t) &= \mu_n \Pi_1^\alpha(\varepsilon_n)v_1^n(t) + (1 - \mu_n)w_n, \\ y_n(t) &= \mu_n \Pi_2^\alpha v_2^n(t) + (1 - \mu_n)\Pi_2^\alpha \bar{v}_2^n(t), \end{aligned} \tag{15}$$

where

$$v_1^n(t) \in f_1(t, A_1^{-\alpha} x_n(t), A_2^{-\alpha} y_n(t)) \quad \text{for a.a. } t \in [0, T],$$

$$w_n \in A_1^{-1+\alpha} F_1(A_1^{-\alpha} x_n(0)),$$

$$v_2^n(t) \in f_2(t, A_1^{-\alpha} x_n(t), A_2^{-\alpha} y_n(t)) \quad \text{for a.a. } t \in [0, T],$$

$$\bar{v}_2^n(t) \in f_2(t, A_1^{-\alpha} x_n(0), A_2^{-\alpha} y_n(t)) \quad \text{for a.a. } t \in [0, T].$$

Observe that $v_1^n \in L_T^\infty(E_1)$; $v_2^n, \bar{v}_2^n \in L_T^\infty(E_2)$ and that the sequences $\{w_n\}$ and $\{y_n\}$ are compact. We are going now to study the properties of the sequence $\{x_n\}$. We begin

by showing that the sequence $\{x_n(0)\}$ is compact in E_1 . By using the T -periodicity of the function x_n , $n \in \mathbf{N}$, we obtain from the first equality of (15)

$$x_n(0) = \mu_n e^{\varepsilon_n A_1 [1/\varepsilon_n] T} \varepsilon_n (I - e^{\varepsilon_n A_1 T})^{-1} \int_0^T A_1^\alpha e^{\varepsilon_n A_1 (T-s)} v_1^n(s) ds + \mu_n \int_0^{\varepsilon_n [1/\varepsilon_n] T} A_1^\alpha e^{A_1 (\varepsilon_n [1/\varepsilon_n] T - \tau)} v_1^n(\tau/\varepsilon_n)(\tau) d\tau + (1 - \mu_n) w_n, \tag{16}$$

where $[1/\varepsilon_n]$ denotes the integer part of $1/\varepsilon_n$.

Applying Lemma 2 and the compactness of the semigroup $e^{A_1 t}$ for $t > 0$ we get the compactness of the first sequence of the right-hand side of (16). Furthermore, by (3) and the compactness of $e^{A_1 t}$, $t > 0$, we obtain the compactness of the second sequence of the right-hand side of (16). Therefore, it follows that $\{x_n(0)\}$ is compact.

By letting $t = 0$ in the first equality of (15) we get

$$x_n(0) = \mu_n \varepsilon_n (I - e^{\varepsilon_n A_1 T})^{-1} \int_0^T A_1^\alpha e^{\varepsilon_n A_1 (T-s)} v_1^n(s) ds + (1 - \mu_n) w_n.$$

Since $\{x_n(0)\}$ and $\{(1 - \mu_n) w_n\}$ are compact sequences the sequence

$$\left\{ \mu_n \varepsilon_n (I - e^{\varepsilon_n A_1 T})^{-1} \int_0^T A_1^\alpha e^{\varepsilon_n A_1 (T-s)} v_1^n(s) ds \right\}$$

is also compact.

Now we evaluate $\|x_n(t) - x_n(0)\|$ for $t \in [0, T]$. From (15) again we have

$$x_n(t) - x_n(0) = (e^{\varepsilon_n A_1 t} - I) z_n + \mu_n \varepsilon_n \int_0^t A_1^\alpha e^{\varepsilon_n A_1 (t-s)} v_1^n(s) ds,$$

where $z_n = \mu_n \varepsilon_n (I - e^{\varepsilon_n A_1 T})^{-1} \int_0^T A_1^\alpha e^{\varepsilon_n A_1 (T-s)} v_1^n(s) ds$.

By the compactness of $\{z_n\}$ and the fact that an analytic semigroup is a C_0 -semigroup we obtain

$$(e^{\varepsilon_n A_1 t} - I) z_n \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By (3) we have

$$\left\| \varepsilon_n \int_0^T A_1^\alpha e^{\varepsilon_n A_1 (t-s)} v_1^n(s) ds \right\| \leq c \varepsilon^{1-\alpha} \|v_1^n\|_{L_T^\infty} \rightarrow 0$$

as $n \rightarrow \infty$.

By passing to a subsequence if necessary we have that $x_n(0) \rightarrow x^0$, $w_n \rightarrow w^0 \in A_1^{-1+\alpha} F_1(A_1^{-\alpha} x^0)$, $y_n \rightarrow y^0$. Therefore, for $n \rightarrow \infty$ from (15) we obtain

$$\begin{aligned} x^0 &\in A_1^{-1+\alpha} F_1(A_1^{-\alpha} x^0), \\ y^0 &\in \Pi_2^\alpha v_2^0, \end{aligned} \tag{17}$$

where $v_2^0 \in f_2(t, A_1^{-\alpha} x^0, A_2^{-\alpha} y^0(t))$ for a.a. $t \in [0, T]$.

Clearly $(x^0, y^0) \in \partial(\tilde{U} \times B(0, r))$, but our choice of r implies that $y^0 \notin \partial B(0, r)$. Therefore $x^0 \in \partial \tilde{U}$ which is a contradiction with the first inclusion of (17) and the choice of U .

Consider now the homotopy given by

$$\Phi_0^\lambda(x, y) = \{(A_1^{-1+\alpha}F_1(A_1^{-\alpha}x(0)), \lambda\Pi_2^\alpha \bar{v}_2 : \\ v_2(t) \in f_2(t, A_1^{-\alpha}x(0), y(t)) \text{ for a.a. } t \in [0, T]\},$$

where $\lambda \in [0, 1]$. By assumption (A) this is an admissible homotopy joining $I - \Phi_0$ with $I - \bar{\Phi}_0$, where $\bar{\Phi}_0(x, y) = (A_1^{-1+\alpha}F_1(A_1^{-\alpha}x(0)), 0)$.

Then, by using the reduction property of the topological degree, we get

$$\deg(I - \bar{\Phi}_0, \tilde{U} \times B(0, r)) = \deg(I - A_1^{-1+\alpha}F_1A_1^{-\alpha}, U) \neq 0.$$

Therefore, for $\varepsilon > 0$ sufficiently small, we have the existence of a solution $(x^\varepsilon, y^\varepsilon) \in C_T(E_1 \times E_2)$ to system (4). Finally, the convergence, as $\varepsilon \rightarrow 0$, of $(x^\varepsilon, y^\varepsilon)$ to (x^*, y^0) , where y^0 is a T -periodic solution to (6) corresponding to $x = x^*$ and $\lambda = 1$, can be proved as we have done before for (x_n, y_n) by taking $\mu_n = 1$ for any $n \in \mathbf{N}$. \square

Remark 1. It is immediate to see that the assumption of the uniqueness of $x^* \in U$ is not necessary for the validity of our result. Indeed, if we assume that $A_1^{-1+\alpha}F_1(A_1^{-\alpha}\cdot)$ has a set $\Sigma_0 \neq \emptyset$ of fixed points in U and we assume condition (7) then the sequence $(x^\varepsilon, y^\varepsilon)$ converges to (x^*, y^0) as $\varepsilon \rightarrow 0$, where $x^* \in \Sigma_0$ and y_0 is a T -periodic solution to (6) corresponding to $x = x^*$ and $\lambda = 1$.

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