

SINGULARLY PERTURBED SECOND ORDER  
DIFFERENTIAL INCLUSIONS BY  
AN AVERAGING METHOD

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**Abstract:** In this article, we consider a system of two singularly perturbed semilinear inclusions of second order. The nonlinear terms are of high frequency with respect to time. We consider different boundary conditions associated to the differential inclusions. We solve the problem of establishing the existence of solutions of this system for  $\varepsilon > 0$  sufficiently small and we describe their behaviour for  $\varepsilon \rightarrow 0$ . To do this, we use an approach based both on the Green functions and on an averaging method. The employed tools are of topological nature: mainly the

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topological degree theory for multivalued upper semicontinuous compact vector fields with convex compact values.

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## 1. Introduction

We deal with the problem of existence and dependence on a small parameter  $\varepsilon > 0$  of the solutions to a system of singularly perturbed multivalued boundary value problems of second order with high frequency nonlinear inputs, which has the form:

$$\begin{cases} i) & x'' - A_1 x \in F_1(\tau/\varepsilon, x, y), \\ ii) & \varepsilon^2 y'' - A_2 y \in F_2(\tau/\varepsilon, x, y), \end{cases}$$

with periodic boundary conditions

$$x(0) = x(\varepsilon), \quad x'(0) = x'(\varepsilon),$$

or Neumann condition

$$x'(0) = x'(\varepsilon) = 0,$$

associated with the first inclusion, and the linear boundary conditions

$$l(y(0), y(\varepsilon), y'(0), y'(\varepsilon)) = 0,$$

associated with the second differential inclusion. The only requirement for the linear boundary condition  $l(y(0), y(\varepsilon), y'(0), y'(\varepsilon)) = 0$  is that there exists a Green function for the linear boundary value problem

$$\begin{cases} i) & y'' - A_2 y = f, \\ ii) & l(y(0), y(\varepsilon), y'(0), y'(\varepsilon)) = 0. \end{cases}$$

Here  $A_1$ ,  $A_2$  are  $n \times n$  and  $m \times m$  real matrices and  $F_1$ ,  $F_2$ , are convex compact set-valued maps satisfying conditions, that will be precised in Section 1.

We first consider the change of variable  $t = \tau/\varepsilon$ , and then we study for the resulting system, the existence of solutions  $(x_\varepsilon, y_\varepsilon)$  for  $\varepsilon > 0$

sufficiently, small and the behaviour of such pairs as  $\varepsilon \rightarrow 0$ . For this we propose an approach based on the study of the Green function for the linear boundary value problem

$$x'' - \varepsilon^2 A_1 x = \varepsilon^2 f,$$

with associated periodic or Neumann boundary conditions and of its limit, when  $\varepsilon \rightarrow 0$ . This investigation, together with the introduction of an averaging operator  $\mathcal{J} : \mathbf{R}^n \mapsto \mathbf{R}$  suitably defined for the first inclusion will permit us in the main result (Theorem 1) to show both the existence of a solution pair  $(x_\varepsilon, y_\varepsilon)$  for  $\varepsilon > 0$  small enough and the behaviour of  $(x_\varepsilon, y_\varepsilon)$  as  $\varepsilon \rightarrow 0$ .

Specifically, first we assume that the topological degree<sup>1</sup> of the vector field  $I - \mathcal{J}$  is different from zero in an open bounded set  $U \subset \mathbf{R}^n$ . Then we prove, by means of two admissible linear homotopies, that the topological degree of the vector field  $I - \Phi_\varepsilon$ , whose zeroes are the solutions  $(x_\varepsilon, y_\varepsilon)$ , is different from zero in an open bounded set  $V$  of  $C([0, 1], \mathbf{R}^n) \times C([0, 1], \mathbf{R}^m)$ , for  $\varepsilon > 0$  sufficiently small. Indeed, under our assumptions, it is possible to convert the problem of finding solutions of our problem into the problem of finding fixed points of a multivalued upper semicontinuous compact vector field  $I - \Phi_\varepsilon$  with compact convex values.

Analogous problems have been considered in Kamenski et al [6], [7] for periodic solutions of systems of first order semilinear differential inclusions in infinite dimensional spaces in the case, when the linear part is represented by analytic and  $C_0$  semigroups respectively. We want to point out that in the present situation of second order differential inclusions, we propose a different approach to determine the limit as  $\varepsilon \rightarrow 0$  of the solution of the linear boundary value problem associated with the first differential inclusion. Indeed, after the change of variable  $t = \tau/\varepsilon$ , as it is easy to see, we obtain the equation

$$x'' - \varepsilon^2 A_1 x = \varepsilon^2 f,$$

where  $f$  is any measurable selection of  $F_1(t, x(t), y(t))$ ,  $t \in [0, 1]$ . Namely we obtain a linear boundary value problem depending on  $\varepsilon > 0$  with periodic or Neumann boundary condition. The proposed approach is

<sup>1</sup>We refer the reader for instance to Górniewicz [4] for suitable definitions of degree for multivalued operator.

based on explicit computation of the corresponding Green function and of its limit as  $\varepsilon \rightarrow 0$ .

Furthermore, observe that for a singularly perturbed system of second order differential equations or inclusions as considered in the present paper, even in the case of Cauchy problem, it is impossible for studying the behaviour of the solutions as  $\varepsilon \rightarrow 0$  to use the classical approach based on the boundary layer. In fact, in this case a condition of Tikhonov type is not satisfied (cf. Tikhonov [11]). We refer to Gasparo et al [3] for a case, where second order perturbation problems for differential equations can be reduced to a Tikhonov-like problem. In Qinde et al [13] an asymptotic expansion of the solution is given for a singularly perturbed second order differential system, without reduction to a Tikhonov-like problem.

Second order singular perturbation problems have been studied in Lions [8] by using variational methods. Averaging method for singular perturbation problems have been also extensively studied, since the pioneer paper of Vololosoov [12] (cf. also Sanders et al [10] and its bibliography). We only mention here Plotnikov [9] and Grammel [5] for the Cauchy problem for differential inclusions and Gaitsgory [1], [2] for optimal control problems.

## 2. Formulation of the Problem and Assumptions

In this paper we consider the following system of second order differential inclusions

$$\begin{cases} i) & x''(\tau) - A_1 x(\tau) \in F_1(\tau/\varepsilon, x(\tau), y(\tau)), \\ ii) & \varepsilon^2 y''(\tau) - A_2 y(\tau) \in F_2(\tau/\varepsilon, x(\tau), y(\tau)), \end{cases} \quad (1)$$

where  $A_1$  and  $A_2$  are  $n \times n$  and  $m \times m$  matrices respectively,  $F_1 : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^m \mapsto \mathbf{R}^n$  and  $F_2 : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^m \mapsto \mathbf{R}^m$  are convex compact valued maps satisfying Carathéodory conditions. Together with (1) we consider the following boundary conditions:

$$\text{periodic boundary conditions } x(0) = x(\varepsilon), x'(0) = x'(\varepsilon), \quad (2)$$

or

$$\text{Neumann boundary conditions } x'(0) = x'(\varepsilon) = 0, \quad (3)$$

for the first inclusion of (1). For the second differential inclusion we consider any linear boundary condition

$$l(y(0), y(\varepsilon), y'(0), y'(\varepsilon)) = 0, \quad (4)$$

such that the linear boundary value problem obtained after the change of variable  $t = \tau/\varepsilon$

$$\begin{cases} i) & y'' - A_2 y = f, \\ ii) & l(y(0), y(1), y'(0), y'(1)) = 0, \end{cases} \quad (5)$$

has an unique solution for any  $f \in L^1([0, 1], \mathbf{R}^m)$  given by

$$y(t) = \int_0^1 G_2(t, s) f(s) ds, \quad (6)$$

where  $G_2(t, s)$  is the Green function of (5). We assume that  $G_2$  is continuous on  $[0, 1] \times [0, 1]$ . We can rewrite (6) in the operator form  $y = \mathcal{G}_2 f$ , where  $\mathcal{G}_2 : L^1([0, 1], \mathbf{R}^m) \mapsto C([0, 1], \mathbf{R}^m)$ . Considering in (1) the change of variable  $t = \tau/\varepsilon$ , we obtain the following boundary value problem

$$\begin{cases} i) & x''(t) - \varepsilon^2 A_1 x(t) \in \varepsilon^2 F_1(t, x(t), y(t)), \\ ii) & y''(t) - A_2 y(t) \in F_2(t, x(t), y(t)), \quad t \in [0, 1], \end{cases} \quad (7)$$

together the boundary conditions

$$x(0) = x(1), \quad x'(0) = x'(1), \quad l(y(0), y(1), y'(0), y'(1)) = 0, \quad (8)$$

or

$$x'(0) = x'(1) = 0, \quad l(y(0), y(1), y'(0), y'(1)) = 0. \quad (9)$$

We denote the problem (7)-(8) or (7)-(9) by  $(\mathcal{P}_\varepsilon)$ . The aim of this paper is to investigate the existence of solutions of  $(\mathcal{P}_\varepsilon)$  and to study their behaviour as  $\varepsilon \rightarrow 0$ . Specifically, assuming that the vector field of a suitably defined averaging operator  $\mathcal{J} : \mathbf{R}^n \mapsto \mathbf{R}^n$  for the problem  $(\mathcal{P}_\varepsilon)$  has topological degree different from 0 in a bounded set  $U \subset \mathbf{R}^n$ , we will prove that for  $\varepsilon > 0$  small enough, problem  $(\mathcal{P}_\varepsilon)$  has a nonempty solution set  $\Sigma_\varepsilon \subset \tilde{U} \times B_{2M}$ , which is upper semicontinuous at  $\varepsilon = 0$ . Here  $\tilde{U} = C([0, 1], U)$  and  $B_{2M} \subset C([0, 1], \mathbf{R}^m)$  is the ball of radius  $2M$  centered at the origin, where  $M$  is the constant of the following assumption **(H<sub>4</sub>)** corresponding to  $\Omega = U$ .

In order to formulate and to prove this result we need the following assumptions.

- (H<sub>0</sub>) The matrix  $A_1$  is invertible ;
- (H<sub>1</sub>) For any  $x \in \mathbf{R}^n$  and any  $y \in \mathbf{R}^m$  the set of measurable selections of the multivalued maps  $F_i(\cdot, x, y)$ ,  $i = 1, 2$ , is nonempty for a.a.  $t \in [0, 1]$  the maps  $F_i(t, \cdot, \cdot)$  are uppersemicontinuous;
- (H<sub>2</sub>) For any pair of bounded sets  $\Omega_1 \subset \mathbf{R}^n$ ,  $\Omega_2 \subset \mathbf{R}^m$  there exists a measurable function  $\nu \in L^1([0, 1], \mathbf{R})$  such that

$$\|F_i(t, x, y)\| \leq \nu(t),$$

for any  $(x, y) \in \Omega_1 \times \Omega_2$ , for a.a.  $t \in [0, 1]$  and  $i = 1, 2$ .

For fixed  $\bar{x} \in \mathbf{R}^n$ , consider the family of boundary value problems

$$\begin{cases} y''(t) - A_2 y(t) \in \lambda F_2(t, \bar{x}, y(t)), & \lambda \in [0, 1], \\ l(y(0), y(1), y'(0), y'(1)) = 0. \end{cases} \quad (10)$$

We assume

- (H<sub>3</sub>) For any  $\bar{x} \in \mathbf{R}^n$  and any  $\lambda \in [0, 1]$  the solution set

$$Y(x, \lambda) \subset C([0, 1], \mathbf{R}^n)$$

to (10) is nonempty;

- (H<sub>4</sub>) For any bounded set  $\Omega \subset \mathbf{R}^n$  there exists a constant  $M > 0$  such that for any  $x \in \Omega$  and any  $\lambda \in [0, 1]$  we have  $\|Y(x, \lambda)\| \leq M$ .

We now introduce the averaging operator for the problem ( $\mathcal{P}_\varepsilon$ ). For this, fix  $x \in \mathbf{R}^n$  and consider the set  $Y(x, 1)$ , then we define  $\mathcal{J} : \mathbf{R}^n \mapsto \mathbf{R}^n$ , as follows

$$\mathcal{J}(x) = \overline{\text{co}}\{ -A_1^{-1} \int_0^1 f(t) dt : f(t) \in F_1(t, x, g(t)), \quad (11) \\ \text{for a.a. } t \in [0, 1] \text{ and } g \in Y(x, 1) \},$$

where  $\overline{\text{co}}$  denotes the closed convex hull. We will denote by  $\text{Fix}(\mathcal{J})$  the set of fixed points of  $\mathcal{J}$ .

It is not hard to verify that from (H<sub>0</sub>)-(H<sub>4</sub>) and the upper semicontinuity of the map  $x \mapsto Y(x, 1)$ , the averaging operator  $\mathcal{J}$  is uppersemicontinuous with nonempty convex compact values.

$$x_\varepsilon = \int_0^1 G_\varepsilon(t, s) w_\varepsilon(s) ds, \quad (14)$$

where  $G_\varepsilon(t, s)$  is the Green function of the considered boundary value problem. Precisely, for the Neumann problem, we have

$$G_\varepsilon(t, s) = \frac{-\varepsilon e^{-\varepsilon\sqrt{p}|t-s|} (1 + e^{-2\varepsilon\sqrt{p}\min\{t,s\}}) (1 + e^{-2\varepsilon\sqrt{p}\max\{t,s\}})}{2(1 - e^{-2\varepsilon\sqrt{p}})},$$

while for the periodic boundary value problem we have

$$G_\varepsilon(t, s) = -\varepsilon \frac{e^{-\varepsilon\sqrt{p}|t-s|} + e^{-\varepsilon\sqrt{p}(1-|t-s|)}}{2\sqrt{p}(1 - e^{-\varepsilon\sqrt{p}})}.$$

It is easy to see that in both the cases we have

$$G_\varepsilon(t, s) \rightarrow -\frac{1}{p} \quad \text{as } \varepsilon \rightarrow 0 \quad (15)$$

uniformly with respect to  $(t, s) \in [0, 1] \times [0, 1]$ .

Using (14), we can rewrite (13) in the equivalent form

$$w_\varepsilon(t) - (A_1 - p) \int_0^1 G_\varepsilon(t, s) w_\varepsilon(s) ds = f(t). \quad (16)$$

Taking into account (15), from (16) as  $\varepsilon \rightarrow 0$  we obtain

$$w_0(t) + \frac{1}{p}(A_1 - p) \int_0^1 w_0(s) ds = f(t). \quad (17)$$

Observe that equation (17) has an unique solution for any  $f \in L^1([0, 1], \mathbf{R}^n)$ . In fact, by integrating (17) from 0 to 1 it is easy to see that

$$\int_0^1 w_0(t) dt = pA_1^{-1} \int_0^1 f(t) dt, \quad (18)$$

and so ,

$$w_0(t) = f(t) - (I - pA_1^{-1}) \int_0^1 f(t) dt.$$

Since (15) holds uniformly with respect to  $t, s$ , we obtain that for  $\varepsilon > 0$  sufficiently small (16) has an unique solution  $w_\varepsilon$  and  $w_\varepsilon \rightarrow w_0$  in

$L^1([0, 1], \mathbf{R}^n)$  as  $\varepsilon \rightarrow 0$ . Therefore, for the function  $x_\varepsilon$  given by (14), we have

$$x_\varepsilon \rightarrow x_0 = -\frac{1}{p} \int_0^1 w_0(t) dt.$$

Finally, from (18) we get

$$x_0 = -A_1^{-1} \int_0^1 f(t) dt. \quad \square$$

**Remark 1.** Since  $w_\varepsilon$  depends continuously on  $f$  the linear operator  $\mathcal{G}_1^\varepsilon : (\varepsilon, f) \rightarrow x_\varepsilon$  defined by (14) is compact and continuous from  $[0, 1] \times L^1([0, 1], \mathbf{R}^n)$  to  $C([0, 1], \mathbf{R}^n)$  and we write  $x_\varepsilon = \mathcal{G}_1^\varepsilon f$ .

*Proof of Theorem 1.* It is evident that the solutions of problem  $(\mathcal{P}_\varepsilon)$  are represented by the fixed points of the nonlinear operator  $\Phi_\varepsilon : C([0, 1], \mathbf{R}^n) \times C([0, 1], \mathbf{R}^m) \mapsto K_c(C([0, 1], \mathbf{R}^n) \times C([0, 1], \mathbf{R}^m))$ , where  $K_c(E)$  denotes the family of nonempty compact convex subsets of  $E$ , defined by the formula

$$\Phi_\varepsilon(x, y) = (\Phi_1^\varepsilon(x, y), \Phi_2(x, y)),$$

where

$$\Phi_1^\varepsilon(x, y) = \{ \mathcal{G}_1^\varepsilon f_1 : f_1(t) \in F_1(t, x(t), y(t)) \text{ for a.a. } t \in [0, 1] \},$$

$$\Phi_2(x, y) = \{ \mathcal{G}_2 f_2 : f_2(t) \in F_2(t, x(t), y(t)) \text{ for a.a. } t \in [0, 1] \}.$$

We consider  $\Omega = U$  in assumption  $(\mathbf{H}_4)$  and let  $M$  be the corresponding constant. Denote by  $V$  the open bounded set of  $C([0, 1], \mathbf{R}^n) \times C([0, 1], \mathbf{R}^m)$  given by  $\tilde{U} \times B_{2M}$ . Recall that  $\tilde{U} = C([0, 1], U)$  and  $B_{2M} \subset C([0, 1], \mathbf{R}^m)$  is the ball centered at the origin of radius  $2M$ .

We first prove that, for  $\varepsilon > 0$  small enough, the vector field  $I - \Phi_\varepsilon$  is homotopic, by means of an affine homotopy, to  $I - \Phi_0$  in  $V$ , where  $\Phi_0$  is the nonlinear operator given by the formula

$$\Phi_0(x, y) = (\tilde{\mathcal{J}}(x(0)), \Phi_2(x, y)),$$

here  $\tilde{\mathcal{J}}(x(0))$  represents the set of constant functions taking values in the set  $\mathcal{J}(x(0))$  defined by (11). We argue by contradiction, thus we assume that there exist sequences  $\varepsilon_n \rightarrow 0$ ,  $\lambda_n \rightarrow \lambda_0$ ,  $\lambda_n \in [0, 1]$ ,  $(x_n, y_n) \in \partial V$ , such that

$$\begin{cases} x_n \in \lambda_n \Phi_1^{\varepsilon_n}(x_n, y_n) + (1 - \lambda_n) \tilde{\mathcal{J}}(x_n(0)), \\ y_n \in \Phi_2(x_n, y_n), \end{cases} \quad (19)$$

or equivalently,

$$\begin{cases} x_n(t) = \lambda_n(\mathcal{G}_1^{\varepsilon_n} f_1^n)(t) + (1 - \lambda_n)v_n, \\ y_n(t) = (\mathcal{G}_2 f_2^n)(t), \end{cases} \quad (20)$$

where  $f_1^n(t) \in F_1(t, x_n(t), y_n(t))$ ,  $f_2^n(t) \in F_2(t, x_n(t), y_n(t))$  for a.a.  $t \in [0, 1]$  and  $v_n \in \mathcal{J}(x_n(0))$ . From the first equation of (20) we have

$$x_n''(t) = \varepsilon_n^2 A_1(x_n(t) - (1 - \lambda_n)v_n) + \varepsilon_n^2 f_1^n(t),$$

for a.a.  $t \in [0, 1]$ .

Since  $x_n \in C([0, 1], \mathbf{R}^n) \subset L^\infty([0, 1], \mathbf{R}^n)$  and, by (H<sub>4</sub>),  $x_n$  and  $y_n$  are uniformly bounded we can deduce

$$\int_0^1 |x_n'(t)|^2 dt \leq c \varepsilon_n^2. \quad (21)$$

Therefore the sequence  $x_n$  is relatively compact in  $C([0, 1], \mathbf{R}^n)$ . From the second equation of (20) we obtain the relative compactness in  $C([0, 1], \mathbf{R}^m)$  of the sequence  $y_n$ . Finally, from the boundedness of  $\mathcal{J}$ , we can derive the relative compactness of the sequence  $v_n$  in  $C([0, 1], \mathbf{R}^n)$ . Without loss of generality we can assume that  $x_n \rightarrow x^*$ ,  $y_n \rightarrow y^*$  and  $v_n \rightarrow v^*$  uniformly in  $[0, 1]$ . Inequality (21) implies that the function  $x^*$  is a constant function. Since all the operators of the right hand side of the inclusions (19) are uppersemicontinuous we can pass to the limit in (20), obtaining

$$\begin{cases} x^* = \lambda_0(-A_1^{-1} \int_0^1 f_1^*(t) dt) + (1 - \lambda_0)v^*, \\ y^*(t) = (\mathcal{G}_2 f_2^*)(t), \end{cases}$$

where  $f_1^*(t) \in F_1(t, x^*(t), y^*(t))$ ,  $f_2^*(t) \in F_2(t, x^*(t), y^*(t))$  for a.a.  $t \in [0, 1]$  and  $v^* \in \mathcal{J}(x^*)$ . But  $y^* \in Y(x^*, 1)$  and so  $-A_1^{-1} \int_0^1 f_1^*(t) dt \in \mathcal{J}(x^*)$  we have

$$\lambda_0(-A_1^{-1} \int_0^1 f_1^*(t) dt) + (1 - \lambda_0)v^* \in \mathcal{J}(x^*),$$

and thus, since  $\text{Fix}(\mathcal{J}) \cap \partial U = \emptyset$ , we obtain

$$x^* \in \text{Fix}(\mathcal{J}) \cap U. \quad (22)$$

But  $(x_n, y_n) \in \partial V$  then  $(x^*, y^*) \in \partial V$ . Condition  $(H_4)$  and the very definition of  $V$  imply that  $x^* \in \partial U$ , which is a contradiction with (22). Therefore

$$\deg(I - \Phi_\varepsilon, V, 0) = \deg(I - \Phi_0, V, 0).$$

Consider now the homotopy

$$H_\lambda(x, y) = (\tilde{\mathcal{J}}(x(0)), \lambda(\mathcal{G}_2 f_2)), \quad \lambda \in [0, 1], \quad (23)$$

where  $f_2(t) \in F_2(t, x(t), y(t))$  for a.a.  $t \in [0, 1]$ . This homotopy does not have fixed points on  $\partial V$ . In fact, if

$$x^* \in \tilde{\mathcal{J}}(x^*(0)), \quad y^* = \lambda^*(\mathcal{G}_2 f_2^*),$$

with  $(x^*, y^*) \in \partial V$  and  $f_2^*(t) \in F_2(t, x^*(t), y^*(t))$  for a.a.  $t \in [0, 1]$ , then  $x^*$  is a constant function and  $y^*$  is a solution of the boundary value problem (10) with  $\bar{x} = x^*$ . Therefore  $\|y^*\| \leq M$  and  $x^* \in \partial U$ , which is a contradiction with our assumption  $\text{Fix}(\mathcal{J}) \cap \partial U = \emptyset$ . In conclusion  $\Phi_0$  is homotopic to the map  $H_0 = (\tilde{\mathcal{J}}(x(0)), 0)$  and so,

$$\deg(I - \Phi_0, V, 0) = \deg(I - H_0, V, 0).$$

By the reduction property of the topological degree we have

$$\deg(I - H_0, V, 0) = \deg(I - \mathcal{J}, U, 0).$$

Finally, since  $\deg(I - \Phi_\varepsilon, V, 0) = \deg(I - \Phi_0, V, 0) \neq 0$ , the solution property of the topological degree implies the existence, for  $\varepsilon > 0$  sufficiently small, of a pair  $(x_\varepsilon, y_\varepsilon) \in \tilde{U} \times B_{2M}$ , which solves  $(P_\varepsilon)$ . The convergence of  $(x_\varepsilon, y_\varepsilon)$  to a solution of the limit system and thus belonging to  $\Sigma_0$  can be proved by means of the same arguments used in proving that the linear homotopy between  $\Phi_\varepsilon$  and  $\Phi_0$  is admissible. Indeed, it is sufficient to put  $\lambda_n = 1$  for any integer  $n$  in this proof.  $\square$

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