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# Sliding mode control of uncertain systems: a singular perturbation approach

MIKHAIL KAMENSKI†
*Department of Mathematics, Voronezh State University, Universitetskaja pl. 1, 394063 Voronezh, Russia*

PAOLO NISTRI‡
*Dipartimento di Ingegneria dell'Informazione, Università di Siena, Via Roma 56, 53100 Siena, Italy*

AND

MARC QUINCAMPOIX¶
*Département de Mathématiques, Université de Bretagne Occidentale, Avenue Victor le Gorgeu, n. 6, BP 809, F-29285 Brest cedex, France*

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In this paper we consider a nonlinear control system affected by deterministic uncertainty and described by a system of ordinary differential equations. The uncertainty is modelled by a multivalued map whose $t$-measurable and $x$-Lipschitz selections represent the possible system dynamics of the uncertain system. We propose a dynamical feedback control design, based on the singular perturbation theory, which allows all the possible system trajectories corresponding to the system dynamics to have the same prescribed behaviour. Specifically, given a manifold $K$ of the state space, defined as the zeros of a smooth map, the proposed control steers and then holds, during finite or infinite time intervals, any possible system trajectory to any prescribed neighbourhood of $K$. A result ensuring the exact attainability of $K$ is also provided. Some examples illustrating the obtained results are presented.

*Keywords*: attainability; uncertainty; singular perturbations; sliding manifolds.

## 1. Introduction

We consider a nonlinear control problem affected by deterministic uncertainty described by the following differential system:

$$\dot{x} = f(t, x, u, v), \tag{1.1}$$

where the map $(t, x, u, v) \to f(t, x, u, v)$ satisfies suitable conditions that will be specified later. The time variable $t \in [0, T]$ with $T \leqslant +\infty$, the state variable $x$ belongs

†Email: mikhail@kam.vsu.ru
‡Corresponding author. Email: pnistri@dii.unisi.it
¶Email: Marc.Quincampoix@univ-brest.fr

to $\mathbb{R}^n$, the control $u$ belongs to $\mathbb{R}^m$ and the parameter $v$, which represents the uncertainty, belongs to a set $V(t,x) \subset \mathbb{R}^p$.

The assumptions on the multivalued map $(t,x) \mapsto V(t,x)$ will be specified in Section 2. As we will see, under these conditions, the map $V$ has the Lipschitz selection property.

By a system trajectory of (1.1) corresponding to a control law $u(t) \in \mathbb{R}^m$, we mean a solution of the differential equation

$$\dot{x}(t) = f(t, x(t), u(t), v(t, x(t))) \tag{1.2}$$

where the function $v(t,x)$ is any $t$-measurable and $x$-Lipschitz selection of $V(t,x)$ which is called *modelled uncertainty* or simply *uncertainty*. The single-valued function

$$f(t,x,u,v(t,x)) \in F(t,x,u) := \{\, f(t,x,u,v),\ v \in V(t,x)\,\}$$

is called a system dynamics of the uncertain system (1.1). In Bartolini & Zolezzi (1991) and Gorniewicz & Nistri (1996) a system dynamics is defined as a single-valued Carathéodory selection of the uncertain dynamics. Here, we assume more regularity with respect to $x$ (Lipschitz condition) in order to prove our main result. Together with (1.1) we consider a 'sliding' manifold

$$K = \{\,(t,x) \in [0,T] \times \mathbb{R}^n :\ k(t,x) = 0\,\}, \tag{1.3}$$

with the function $k :\ [0,T] \times \mathbb{R}^n \to \mathbb{R}^l$, $l \leqslant n$.

We pose the following control problem:

$$\text{(CP)} \quad \begin{cases} \text{For any neighbourhood } I \text{ of } K \text{ and any initial state } x_0 \in \mathbb{R}^n\text{, we want, by} \\ \text{means of a suitably defined dynamical feedback control to steer and then} \\ \text{hold any possible system trajectory } x = x(t),\ t \in [0,T] \text{ of (1.1) starting from} \\ x_0 \text{ to } I. \end{cases}$$

To solve this problem, we introduce a suitable $C^1$ function $s : [0,T] \times \mathbb{R}^n \to \mathbb{R}^m$, $m \leqslant n$, defined by means of $K$, and depending on $x_0$ in such a way that the set

$$S := \{\,(t,x) \in [0,T] \times \mathbb{R}^n :\ s(t,x) = 0\,\} \tag{1.4}$$

is as near as we like to $K$ for $t$ bounded away from zero.

For instance, as we will see in Section 4, in tracking control problems $K$ is a reference trajectory $K = \{x \in \mathbb{R}^n :\ x = x_{\mathrm{ref}}(t),\ t \in [0,T]\}$ and we can choose

$$s(t,x) := \mathrm{e}^{Ct}(x_0 - x_{\mathrm{ref}}(0)) + x - x_{\mathrm{ref}}(t). \tag{1.5}$$

where $C$ is a $n \times n$ matrix whose eigenvalues have negative real parts.
Therefore, to solve (CP) we can solve the problem of steering trajectories as near as we like to $S$. To do this, we introduce smooth controls by means of the function $s$, that is, for fixed $\varepsilon \geqslant 0$, we consider the following system of differential equations

$$\begin{cases} \dot{x} = f(t,x,u,v(t,x)) & x(0) = x_0 \in S \\ \varepsilon \dot{u} = \frac{\partial s}{\partial t}(t,x) + \frac{\partial s}{\partial x}(t,x)\, f(t,x,u,v(t,x)) & u(0) = u_0. \end{cases} \tag{1.6}$$

with $t \in [0,T]$.

We first consider the case when $T < +\infty$ and we define on $C([0, T], \mathbb{R}^n)$ the cost functional $G(x) = \int_0^T |s(t, x(t))| \mathrm{d}t$ to be maximized on the set $\Sigma_\varepsilon$ of all the possible system trajectories $x_\varepsilon$ of (1.6) associated with the uncertainties $v(t, x)$. We denote by $x_\varepsilon^*$ a trajectory where $G$ attains its maximum on $\Sigma_\varepsilon$ and by $v_\varepsilon^*(t, x)$ the corresponding uncertainty. As we will see, this maximum exists since it is possible to prove the compactness of $\Sigma_\varepsilon,\ \varepsilon > 0$ in $C([0, T], \mathbb{R}^n)$. Denoting by $\Sigma_0$ the set of system trajectories of (1.6) when $\varepsilon = 0$ (the reduced system), we obtain that $\max_{\Sigma_0} G(x) = 0$. Roughly speaking, for any $\varepsilon > 0$, the system dynamics corresponding to the uncertainty $v_\varepsilon^*(t, x)$ represents the worst possible case, in the sense that if we solve our problem (CP) for the trajectory $x_\varepsilon^*$, then we have solved the problem for any other system dynamics of the uncertain system (1.1). The main tool to do this is represented by the results and methods of Bensoussan (1988) and Quincampoix & Zhang (1995) which allows us to show for the singularly perturbed system

$$\begin{cases} \dot{x} = f(t, x, u, v_\varepsilon^*(t, x)) & x(0) = x_0 \in S \\ \varepsilon \dot{u} = \frac{\partial s}{\partial t}(t, x) + \frac{\partial s}{\partial x}(t, x) f(t, x, u, v_\varepsilon^*(t, x)) & u(0) = u_0, \end{cases} \tag{1.7}$$

the convergence as $\varepsilon \to 0$ both of the maximum and of the corresponding solution pair $(x_\varepsilon^*, u_\varepsilon^*)$ to (1.7) respectively to zero and to a solution pair $(x_0^*, u_0^*)$ of the reduced system. The convergence of the state $x_\varepsilon^*$ to $x_0^*$ is in the uniform topology, while $u_\varepsilon^*$ converges to $u_0^*$ in $L_2((0, T), \mathbb{R}^m)$. Indeed, as we will see, in our case from the proof of Theorem 1 we can derive the convergence, as $\varepsilon \to 0$, in the uniform topology of any system trajectory $x_\varepsilon$ to a system trajectory $\hat{x}_0$ of the reduced system.

Then we consider the case when $T = +\infty$: under more restrictive assumptions than those of Theorem 1 we obtain in Theorem 2 the uniform convergence of $x_\varepsilon$ to $\hat{x}_0$ in $[0, +\infty)$.

As a consequence, for any neighbourhood $I$ of $K$, there exists $\varepsilon_0^*$ such that for $0 < \varepsilon < \varepsilon_0^*$, all the possible trajectories starting from $x_0$ of system (1.6) reach $I$ and remain therein for all the future times. Observe that no matching conditions for the uncertainty are imposed here. The control technique employed in this paper (without any associated cost functional $G$) was introduced first in Johnson & Nistri (1992) and then successfully applied to concrete control problems in finite-dimensional spaces in Cavallo *et al.* (1993a, 1993b, 1996). Several refinements of this technique have been also proposed, in particular to avoid an excessive initial rate of the control law, see Cavallo *et al.* (1999). Finally, a second-order manifold approach has been recently adopted in Cavallo *et al.* (2000) for the reduction of the vibration of a flexible structure: this paper constitutes a first attempt in the direction of extending the previous approach to infinite-dimensional control problems, namely for a control system described by partial differential equations.

A different approach to solve similar control problems is based on set-valued analysis. Specifically, we pose the following control problem: given a nonempty closed set $K$ of the state-space, we want, starting from a point near enough to $K$, to reach $K$ in finite time and then stay there.

To solve this problem, we need stronger assumptions on $K$ than those used for problem (CP). The main feature of this second approach is that we obtain an 'exact' attainability in finite time, but this is a local result (i.e. we start near enough to $K$). On the other hand, by means of the first approach, we solve only approximatively the attainability of $K$, but we can start from any initial condition $x_0 \in \mathbb{R}^n$. The suitable assumption on $K$ for solving the previous problem is that $K$ is a proximal retract: that is, there is a neighbourhood $\mathcal{N}$ of $K$ such that there exists a single-valued continuous metric retraction $r : \mathcal{N} \mapsto K$. In Gorniewicz & Nistri (1999) and Nistri & Quincampoix (2000), the notion of proximal retracts allowed the derivation of a feedback control law, which turns out to be a selection of a suitably defined regulation map which makes $K$ attainable. In this paper, we consider a particular proximal retract $K$ defined by (1.3) where $k$ is of class $C^{1,1}$ with nonvanishing gradient on $K$. Various characterizations of proximal retracts in Hilbert spaces can be found in Colombo & Goncharov (1999).

Finally, we would like to point out that we can use both the previous approaches to solve exactly the problem (CP) whenever $x_0 \in \mathbb{R}^n$. Indeed, given $x_0 \in \mathbb{R}^n$, we can design, by the singular perturbation method, a control law which steers the state in a prescribed neighbourhood of $K$ and then we can change the control law according to the exact attainability method of the second approach.

The paper is organized as follows. In Section 2, we discuss the singular perturbation method. For finite time intervals we provide a preliminary result (Proposition 1) and the conditions under which we will establish the main result of the paper (Theorem 1). This result states the convergence as $\varepsilon \to 0$ of the pair $(x^*_\varepsilon, u^*_\varepsilon)$ to a solution $(x^*_0, u^*_0)$ of the reduced problem (corresponding to a system dynamics) and also the convergence as $\varepsilon \to 0$ of the values $G(x^*_\varepsilon)$ to $G(x^*_0)$. We shall show that $x^*_\varepsilon \to x^*_0$ in $C([0, T], \mathbb{R}^n)$ and $u^*_\varepsilon \to u^*_0$ in the $L_2((0, T), \mathbb{R}^m)$-topology. Furthermore, in the second part of Section 2 we prove the uniform convergence of $x_\varepsilon$ to $x_0$ in $[0, +\infty)$. In Section 3 we state an exact attainability result and in Section 4 we provide some examples to illustrate how our results can be applied.

## 2. Control design via a singular perturbation approach

We first consider the case when $T < +\infty$.

### 2.1 *Convergence on finite time intervals*

In this section we assume the following conditions on the dynamics $f$ and the functions $s, V$:

- ($f_1$) For any $(x, u, v) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ the map $t \to f(t, x, u, v)$ is Lebesgue measurable on $[0, T]$. Furthermore, for almost all (a.a.) $t \in [0, T]$, the map $(x, u, v) \to f(t, x, u, v)$ is Lipschitz.
- ($f_2$) For any $\rho > 0$ there exists $c > 0$ such that

$$|f(t, x, u, v)| \leqslant c(1 + |x| + |u|),$$

for a.a. $t \in [0, T]$ and $|v| \leqslant \rho$.
- ($f_3$) The multivalued map $(t, x, u) \to F(t, x, u)$ given by

$$F(t, x, u) = \{f(t, x, u, v) : v \in V(t, x)\},$$

has nonempty, compact, convex values.

($s_1$) The map $s : [0, T] \times \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable. Moreover, for any $t \in [0, T]$ there exists $x \in \mathbb{R}^n$ such that

$$(t, x) \in S = \{(t, x) \in [0, T] \times \mathbb{R}^n : s(t, x) = 0\}.$$

($s_2$) The maps $x \to \frac{\partial s}{\partial t}(t, x)$ and $x \to \frac{\partial s}{\partial x}(t, x)$ are Lipschitz and there exists $\hat{c} > 0$ such that

$$\left|\frac{\partial s}{\partial t}(t, x)\right| \leqslant \hat{c}(1 + |x|) \qquad \text{for a.a. } t \in [0, T].$$

Moreover, there exists $N > 0$ such that

$$\left|\frac{\partial s}{\partial x}(t, x)\right| \leqslant N,$$

for a.a. $t \in [0, T]$ and any $x \in \mathbb{R}^n$.

(V) The multivalued map $(t, x) \to V(t, x)$ is bounded, i.e. $|V(t, x)| \leqslant M$ for some $M > 0$ and $(t, x) \in [0, T] \times \mathbb{R}^n$, Lebesgue measurable on $[0, T]$ for any $x \in \mathbb{R}^n$ and Lipschitz with respect to $x$ for a.a. $t \in [0, T]$. Moreover, we assume that $V(t, x)$ is a nonempty closed convex set of $\mathbb{R}^p$ for a.a. $t \in [0, T]$ and any $x \in \mathbb{R}^n$.

Observe that ($f_1$) and (V) imply that the multivalued map $(x, u) \to F(t, x, u)$ is Lipschitz for a.a. $t \in [0, T]$.

Finally, we make the following crucial assumption.

(H) There exists $\nu > 0$ such that

$$-\nu|u_1 - u_2|^2 \geqslant \left\langle \frac{\partial s}{\partial x}(t, x)(f(t, x, u_1 \cdot v) - f(t, x, u_2, v)), u_1 - u_2 \right\rangle,$$

for a.a. $t \in [0, T]$, any $u_1, u_2 \in \mathbb{R}^m$, any $x \in \mathbb{R}^n$ and any $v \in \mathbb{R}^p$.

REMARK 1 If the initial states $x_0$ for system (1) belong to a specified bounded set of $\mathbb{R}^n$, then all the previous assumptions can be reformulated for $x$ belonging to a bounded set of $\mathbb{R}^n$. Moreover, note that (V) implies that the map $(t, x) \to V(t, x)$ has the Lipschitz selection property. That is, for $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$ and any $y_0 \in V(t_0, x_0)$ there exists a $t$-measurable and $x$-Lipschitz selection $v(t, x) \in V(t, x)$ such that $y_0 = v(t_0, x_0)$ (see Aubin & Frankowska (1990)).

For $\varepsilon \geqslant 0$, we consider now the following system of differential equations:

$$\begin{cases} \dot{x} = f(t, x, u, v(t, x)), \quad x(0) = x_0 \\ \varepsilon \dot{u} = \frac{\partial s}{\partial t}(t, x) + \frac{\partial s}{\partial x}(t, x) f(t, x, u, v(t, x)), \quad u(0) = u_0, \end{cases} \tag{2.1}$$

where $t \in [0, T]$. Together with (2.1) we consider a continuous cost functional $G(\cdot)$ defined on the set $\Sigma_\varepsilon$ of all the possible system trajectories $x_\varepsilon(t)$ of (1), where $(x_\varepsilon(t), u_\varepsilon(t))$, $t \in [0, T]$, is the solution of (2.1) corresponding to the uncertainty $v(x, t)$ and $v_\varepsilon(t) = v(t, x_\varepsilon(t))$ for $t \in [0, T]$. The cost functional $G(\cdot)$ is defined as follows:

$$G(x_\varepsilon) = \int_0^T |s(t, x_\varepsilon(t))| \, \mathrm{d}t.$$

REMARK 2 Note that for our purposes we can take any continuous cost functional $G$ on $C([0,T],\mathbb{R}^n)$ having the property that $G(x)=0$, with $x\in C([0,T],\mathbb{R}^n)$, implies that $(t,x(t))\in S$ for any $t\in[0,T]$.

We have the following proposition.

PROPOSITION 1 Under assumptions ($f_1$)–($f_3$), ($s_1$)–($s_2$) and (V), for any $\varepsilon>0$ the solution set given by

$$S_\varepsilon=\{(x,u)\in C([0,T],\mathbb{R}^n)\times C([0,T],\mathbb{R}^m):(x,u)\quad\text{satisfies}\\ \text{(2.1) for some uncertainty } v(t,x)\in V(t,x)\}$$

is compact in $C([0,T],\mathbb{R}^n)$.

*Proof.* Let $\varepsilon>0$ be fixed and consider a sequence $\{(x_n,u_n)\}\subset S_\varepsilon$. Obviously, $\{(x_n,u_n)\}\subset\hat{S}_\varepsilon$, where $(x,u)\in\hat{S}_\varepsilon$ if and only if $(x,u)\in C([0,T],\mathbb{R}^n)\times C([0,T],\mathbb{R}^m)$ satisfies the differential system

$$\begin{cases}\dot{x}(t)\in F(t,x(t),u(t)), & x(0)=x_0\\ \varepsilon\dot{u}(t)=\dfrac{\partial s}{\partial t}(t,x(t))+\dfrac{\partial s}{\partial x}(t,x(t))\,\dot{x}(t), & u(0)=u_0,\end{cases}$$

for a.a. $t\in[0,T]$. Or equivalently,

$$\begin{cases}\dot{x}(t)\in F(t,x(t),u(t)), & x(0)=x_0\\ u(t)=\dfrac{1}{\varepsilon}[s(t,x(t))-s(0,x(0))]+u_0,\end{cases}$$

where $F(t,x,u)=f(t,x,u,V(t,x))$. It turns out that, under our assumptions on $F$, $\hat{S}_\varepsilon$ is a compact set, see for instance Theorem 10.1.9 of Aubin & Frankowska (1990). Therefore, by passing to a subsequence, if necessary, we have that $(x_n,u_n)\to(x_0,u_0)$ in $C([0,T],\mathbb{R}^n)\times C([0,T],\mathbb{R}^m)$ and $(x_0,u_0)$ satisfies for a.a. $t\in[0,T]$ the system of differential inclusions

$$\begin{cases}\dot{x}_0(t)\in F(t,x_0(t),u_0(t)), & x(0)=x_0\\ \varepsilon\dot{u}_0(t)\in\dfrac{\partial s}{\partial t}(t,x_0(t))+\dfrac{\partial s}{\partial x}(t,x_0(t))\,\dot{x}_0(t), & u(0)=u_0,\end{cases}$$

where $F(t,x_0(t),u_0(t))=f(t,x_0(t),u_0(t),V(t,x_0(t)))$. To prove this, let $\{(x_n,u_n)\}\subset C([0,T],\mathbb{R}^n)\times C([0,T],\mathbb{R}^m)$ be a sequence of solutions of (2.1) corresponding to uncertainties $v_n(t,x)\in V(t,x)$. Assume that $(x_n,u_n)\to(x_0,u_0)$ in $C([0,T],\mathbb{R}^n)\times C([0,T],\mathbb{R}^m)$ and consider

$$x_n(t)=x_0+\int_0^t f(s,x_n(s),u_n(s),v_n(s,x_n(s)))\,\mathrm{d}s.$$

Put $\varphi_n(t)=f(t,x_n(t),u_n(t),v_n(t,x_n(t)))$, then $\varphi_n$ converges weakly to some $\varphi_0$ in $L_1([0,T],\mathbb{R}^n)$ such that

$$x_0(t)=x_0+\int_0^t\varphi_0(s)\,\mathrm{d}s.$$

From $\varphi_n(t) \in f(t, x_n(t), u_n(t), V(t, x_n(t)))$ for a.a. $t \in [0, T]$ and the weak convergence of $\varphi_n$ to $\varphi_0$; for a.a. $t \in [0, T]$ and any $y \in \mathbb{R}^n$ we have that

$$\limsup_{n\to\infty}\left[\sup_{v\in V(t,x_n(t))}\langle y,\ f(t, x_n(t), u_n(t), v)\rangle\right]$$
$$\geqslant \langle y, \varphi_0(t)\rangle \geqslant \liminf_{n\to\infty}\left[\inf_{v\in V(t,x_n(t))}\langle y,\ f(t, x_n(t), u_n(t), v)\rangle\right].$$

By ($f_1$), (V) and using Theorem 6, p. 53, of Aubin & Cellina (1984) we obtain

$$\sup_{v\in V(t,x_0(t))}\langle y,\ f(t, x_0(t), u_0(t), v)\rangle \geqslant \langle y, \varphi_0(t)\rangle$$
$$\geqslant \inf_{v\in V(t,x_0(t))}\langle y,\ f(t, x_0(t), u_0(t), v)\rangle.$$

Then, by ($f_3$) we get

$$\varphi_0(t) \in f(t, x_0(t), u_0(t), V(t, x_0(t)))$$

for a.a. $t \in [0, T]$ and so by Filippov's selection theorem we have the existence of a measurable function $v_0(t) \in V(t, x_0(t))$ such that

$$\varphi_0(t) = f(t, x_0(t), u_0(t), v_0(t))$$

for a.a. $t \in [0, T]$. In conclusion, $\dot{x}_0(t) \in F(t, x_0(t), u_0(t))$ for a.a. $t \in [0, T]$.

Therefore there exists a measurable selection $v_0(t) \in V(t, x_0(t))$ such that $\dot{x}_0(t) = f(t, x_0(t), u_0(t), v_0(t))$ for a.a. $t \in [0, T]$. On the other hand, by (V) and Theorem 9.5.3 of Aubin & Frankowska (1990), there exists a (modelled) uncertainty $v_0(t, x) \in V(t, x)$ such that $v_0(t) = v_0(t, x_0(t))$. □

By virtue of Proposition 1, for any $\varepsilon > 0$, the cost functional $G(\cdot)$ attains its maximum on $\Sigma_\varepsilon := P_1 S_\varepsilon$, where $P_1$ is the projection on the first component.

Let $x^*_\varepsilon(t) = x^*_\varepsilon(t, u^*_\varepsilon(t), v^*_\varepsilon(t))$ such that $G(x^*_\varepsilon) = \max_{\Sigma_\varepsilon} G(x_\varepsilon)$, where $(x^*_\varepsilon, u^*_\varepsilon)$ is a solution of (2.1) and $v^*_\varepsilon(t) = v^*_\varepsilon(t, x^*_\varepsilon(t))$, $t \in [0, T]$, with $v^*_\varepsilon(t, x) \in V(t, x)$.

For $\varepsilon = 0$ and for any system dynamics $f(t, x, u, v(t, x))$ we obtain the reduced system

$$\begin{cases} \dot{x} = f(t, x, u, v(t, x)), \quad x(0) = x_0 \\ 0 = \dfrac{\partial s}{\partial t}(t, x) + \dfrac{\partial s}{\partial x}(t, x) f(t, x, u, v(t, x)), \quad u(0) = u_0. \end{cases} \tag{2.2}$$

Now we pose the following condition on (2.2).

($R_0$) We assume that there exists an uncertainty $v_0(t, x)$ such that the set

$$R_0(t, x) = \left\{u \in \mathbb{R}^m : \frac{\partial s}{\partial t}(t, x) + \frac{\partial s}{\partial x}(t, x) f(t, x, u, v_0(t, x)) = 0\right\}$$

is nonempty for a.a. $t \in [0, T]$ and any $x \in \mathbb{R}^n$.

Observe that, by virtue of assumptions (H), ($s_1$)–($s_2$) and ($f_1$)–($f_2$) condition ($R_0$) implies that

$$(t, x) \to R_0(t, x)$$

is a locally $x$-Lipschitz single-valued map with linear growth. Therefore, there is defined a $t$-measurable and locally $x$-Lipschitz single-valued map $(t, x) \to u_0(t, x)$ such that

$$\frac{\partial s}{\partial t}(t, x) + \frac{\partial s}{\partial x}(t, x) f(t, x, u_0(t, x), v_0(t, x)) = 0$$

for a.a. $t \in [0, T]$ and any $x \in \mathbb{R}^n$. The control law $u_0(t, x)$ is called the 'equivalent control' in the classical theory of the sliding modes, see Utkin (1978).

Observe that $t \to u_0(t, x)$ is an $L_\infty([0, T], \mathbb{R}^m)$ function for any $x$. Furthermore, if $(x_0^*, u_0^*)$ with $u_0^*(t) = u_0(t, x_0^*(t))$ is the solution of (2.2) corresponding to $v_0(t, x) \in V(t, x)$ with $x_0^*(0) \in S$ then $s(t, x_0^*(t)) = 0$ for any $t \in [0, T]$ and so $G(x_0^*) = 0$. In fact, from

$$\begin{cases} \dot{x}_0^*(t) = f(t, x_0^*(t), u_0^*(t), v_0^*(t)), \quad x_0^*(0) \in S \\ 0 = \dfrac{\partial s}{\partial t}(t, x_0^*(t)) + \dfrac{\partial s}{\partial x}(t, x_0^*(t))\, \dot{x}_0^*(t) \end{cases}$$

for a.a. $t \in [0, T]$, where $v_0^*(t) = v_0(t, x_0^*(t))$, it follows that $\dfrac{\mathrm{d}}{\mathrm{d}t} s(t, x_0^*(t)) = 0$ for any $t \in [0, T]$ with $s(0, x_0^*(0)) = 0$ and so $s(t, x_0^*(t)) = 0$ for any $t \in [0, T]$.

REMARK 3 Observe that for any system dynamics $f(t, x, u, \hat{v}_0(t, x))$, with $\hat{v}_0(t, x) \in V(t, x)$, which satisfies ($R_0$) the previous considerations apply. Namely, $s(t, \hat{x}_0^*(t)) = 0$ for any $t \in [0, T]$, where $(\hat{x}_0^*, \hat{u}_0^*)$ is the solution of (2.2) with $v(t, x) = \hat{v}_0(t, x)$. In what follows we will refer to any such solution of (2.2) as an optimal pair of the reduced system (2.2).

We are now in the position to formulate our main result.

THEOREM 1 Assume ($f_1$)–($f_3$), ($s_1$)–($s_2$), (V), (H) and ($R_0$). Let $(0, x_0) \in S$, then $x_\varepsilon^* \to x_0^*$ weakly in $H_1([0, T], \mathbb{R}^n)$ and $u_\varepsilon^* \to u_0^*$ in $L_2((0, T), \mathbb{R}^n)$ as $\varepsilon \to 0$, where $(x_0^*, u_0^*)$ is an optimal pair of (2.2). Furthermore, $x_\varepsilon^* \to x_0^*$ in $C([0, T], \mathbb{R}^n)$ and so $G(x_\varepsilon^*) \to G(x_0^*) = 0$ as $\varepsilon \to 0$.

*Proof.* We consider $(x_\varepsilon^*, u_\varepsilon^*, v_\varepsilon^*)$ and $(\hat{x}_0^*, \hat{u}_0^*, \hat{v}_0^*)$, where $(\hat{x}_0^*, \hat{u}_0^*)$ is an optimal pair of (2.2) corresponding to $\hat{v}_0(t, x)$ and $\hat{v}_0^*(t) = \hat{v}_0(t, \hat{x}_0^*(t))$. We put $\widetilde{x}_\varepsilon^* = x_\varepsilon^* - \hat{x}_0^*$ and $\widetilde{u}_\varepsilon^* = u_\varepsilon^* - \hat{u}_0^*$. Consider the differential system

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\widetilde{x}_\varepsilon^*(t) = f(t, x_\varepsilon^*(t), u_\varepsilon^*(t), v_\varepsilon^*(t)) - f(t, \hat{x}_0^*(t), \hat{u}_0^*(t), \hat{v}_0^*(t)), \quad \widetilde{x}_\varepsilon^*(0) = 0 \\ \varepsilon \dfrac{\mathrm{d}}{\mathrm{d}t} u_\varepsilon^*(t) = g(t, x_\varepsilon^*(t), u_\varepsilon^*(t), v_\varepsilon^*(t)) - g(t, \hat{x}_0^*(t), \hat{u}_0^*(t), \hat{v}_0^*(t)), \quad x_0^*(0) = u_0, \end{cases} \tag{2.3}$$

where we have put $g(t, x, u, v) = \dfrac{\partial s}{\partial t}(t, x) + \dfrac{\partial s}{\partial x}(t, x) f(t, x, u, v)$. Rewrite the second equation in the form

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} u_\varepsilon^*(t) = g(t, x_\varepsilon^*(t), u_\varepsilon^*(t), v_\varepsilon^*(t)) - g(t, \hat{x}_0^*(t), u_\varepsilon^*(t), \hat{v}_0^*(t)) \\ + g(t, \hat{x}_0^*(t), u_\varepsilon^*(t), \hat{v}_0^*(t)) - g(t, \hat{x}_0^*(t), \hat{u}_0^*(t), \hat{v}_0^*(t)).$$

Multiplying this equation by $u_\varepsilon^*(t)$ we obtain from ($f_1$), ($s_2$) and (H)

$$\varepsilon \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} |u_\varepsilon^*(t)|^2 \leqslant L(|x_\varepsilon^*(t) - \hat{x}_0^*(t)| + |v_\varepsilon^*(t) - \hat{v}_0^*(t)|)\, |u_\varepsilon^*(t)| \\ - \nu |\widetilde{u}_\varepsilon^*(t)|^2 + \langle g(t, \hat{x}_0^*(t), u_\varepsilon^*(t), \hat{v}_0^*(t)), \hat{u}_0^*(t) \rangle$$

for some $L > 0$.

By ($f_1$), ($f_2$) and ($s_2$) integrating from 0 and $t > 0$ we get

$$\nu \|\widetilde{u}_\varepsilon^*\|^2_{L^2(0,t)} \leqslant L \int_0^t |\widetilde{x}_\varepsilon^*(s)|\, |u_\varepsilon^*(s)| \,\mathrm{d}s + 2LM \int_0^t |u_\varepsilon^*(s)| \,\mathrm{d}s \\ + (\hat{c} + Nc) \int_0^t (1 + |\hat{x}_0^*(s)|)\, |\hat{u}_0^*(s)| \,\mathrm{d}s \\ + Nc \int_0^t |u_\varepsilon^*(s)|\, |\hat{u}_0^*(s)| \,\mathrm{d}s + \frac{\varepsilon}{2} |u_0|^2.$$

By the Cauchy–Schwartz inequality we obtain

$$\nu \|\widetilde{u}_\varepsilon^*\|^2_{L_2(0,t)} \leqslant L \|\widetilde{x}_\varepsilon^*\|_{L_2(0,t)} \|u_\varepsilon^*\|_{L_2(0,t)} + 2LM\sqrt{t} \|u_\varepsilon^*\|_{L_2(0,t)} \\ + Nc \|\hat{u}_0^*\|_{L_2(0,t)} \|u_\varepsilon^*\|_{L_2(0,t)} + c_0,$$

where $c_0 = (\hat{c} + Nc) \int_0^t (1 + |\hat{x}_0^*(s)|)\, |\hat{u}_0^*(s)| \,\mathrm{d}s + \frac{\varepsilon}{2} |u_0|^2$.

Since $\|u_\varepsilon^*\|_{L_2(0,t)} \leqslant \|\widetilde{u}_\varepsilon^*\|_{L_2(0,t)} + \|\hat{u}_0^*\|_{L_2(0,t)}$ we have

$$\nu \|\widetilde{u}_\varepsilon^*\|^2_{L_2(0,t)} \leqslant \mu(\|\widetilde{x}_\varepsilon^*\|_{L_2(0,t)}) \|\widetilde{u}_\varepsilon^*\|_{L_2(0,t)} + \hat{c}_0, \tag{2.4}$$

where $0 < \hat{c}_0 = c_0 + \mu(\|\widetilde{x}_\varepsilon^*\|_{L_2(0,t)})\, \|\hat{u}_0^*\|_{L_2(0,t)}$ and

$$\mu(\|\widetilde{x}_\varepsilon^*\|_{L_2(0,t)}) = L \|\hat{x}_\varepsilon^*\|_{L_2(0,t)} + 2LM\sqrt{t} + Nc \|\hat{u}_0^*\|_{L_2(0,t)}.$$

Using the fact that $\|\widetilde{x}_\varepsilon^*\|_{L_2(0,t)} \leqslant 1 + \|\widetilde{x}_\varepsilon^*\|^2_{L_2(0,t)}$ from (2.4) we obtain

$$0 \leqslant \|\widetilde{u}_\varepsilon^*\|_{L_2(0,t)} \leqslant k_1 \|\widetilde{x}_\varepsilon^*\|_{L_2(0,t)} + k_2, \tag{2.5}$$

with $k_1, k_2 > 0$.

We prove now that there exists $R > 0$ such that $\|\widetilde{x}_\varepsilon^*\|_{L_2(0,t)} \leqslant R$ for any $\varepsilon > 0$ and $t \in [0, T]$. From the first equation of (2.3) by means of ($f_2$) and (V) we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\widetilde{x}_\varepsilon^*(t)|^2 \leqslant k|\widetilde{x}_\varepsilon^*(t)|(|\widetilde{x}_\varepsilon^*(t)| + |\widetilde{u}_\varepsilon^*(t)| + |v_\varepsilon^*(t) - \hat{v}_0^*(t)|)$$
$$\leqslant k|\widetilde{x}_\varepsilon^*(t)|(|\widetilde{x}_\varepsilon^*(t)| + |\widetilde{u}_\varepsilon^*(t)| + 2M). \tag{2.6}$$

Integrating from 0 and $t > 0$ we have

$$\tfrac{1}{2}|\widetilde{x}_\varepsilon^*(t)|^2 \leqslant 2kM \int_0^t |\widetilde{x}_\varepsilon^*(s)|\,\mathrm{d}s + k\int_0^t |\widetilde{x}_\varepsilon^*(s)|^2\,\mathrm{d}s$$
$$+ k\int_0^t |\widetilde{x}_\varepsilon^*(s)|\,|\widetilde{u}_\varepsilon^*(s)|\,\mathrm{d}s$$
$$\leqslant 2kMt + 2kM\|\widetilde{x}_\varepsilon^*\|_{L_2(0,t)}^2 + k\|\widetilde{x}_\varepsilon^*\|_{L_2(0,t)}^2 + k\|\widetilde{x}_\varepsilon^*\|_{L_2(0,t)}\|\widetilde{u}_\varepsilon^*\|_{L_2(0,t)}$$
$$\leqslant 2kMt + (2kM + k)\|\widetilde{x}_\varepsilon^*\|_{L_2(0,t)} + c_1\|\widetilde{x}_\varepsilon^*\|_{L_2(0,t)}^2 + c_2\|\widetilde{x}_\varepsilon^*\|_{L_2(0,t)},$$

and using once again the inequality $\|\widetilde{x}_\varepsilon^*\|_{L_2(0,t)} \leqslant 1 + \|\widetilde{x}_\varepsilon^*\|_{L_2(0,t)}^2$ we get

$$\tfrac{1}{2}|\widetilde{x}_\varepsilon^*(t)|^2 \leqslant (2kMt + c_2) + (2kM + k + c_1 + c_2)\|\widetilde{x}_\varepsilon^*\|_{L_2(0,t)}^2.$$

Integrating from 0 and $t > 0$ and using the Grönwall inequality we derive

$$\|\widetilde{x}_\varepsilon^*\|_{L_2(0,t)} \leqslant R \tag{2.7}$$

for any $\varepsilon > 0$ sufficiently small and any $t \in [0, T]$. From (2.5)–(2.7) we conclude the existence of a subsequence such that

$$x_\varepsilon^* \to x_0^* \text{ weakly in } H_1([0, T], \mathbb{R}^n)$$
$$u_\varepsilon^* \to u_0^* \text{ weakly in } L_2((0, T), \mathbb{R}^m).$$

Observe that, since $\{x_\varepsilon^*\}_{\varepsilon>0}$ is bounded in $H_1([0, T], \mathbb{R}^n)$, by using (H), ($f_2$) and ($s_2$) we easily obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_\varepsilon^*(t)|^2 \leqslant -\nu|u_\varepsilon^*(t)|^2 + c|u_\varepsilon^*(t)|\,|x_\varepsilon^*(t)| + c|x_\varepsilon^*(t)|^2 + c$$
$$\leqslant -\frac{\nu}{2}|u_\varepsilon^*(t)|^2 + \hat{c}^2|x_\varepsilon^*(t)|^2 + \widetilde{c},$$

for some positive constants $c, \hat{c}$.

By the Grönwall inequality we get

$$|u_\varepsilon^*(t)| \leqslant M \tag{2.8}$$

for some $M > 0$ and all $t \in [0, T]$.

We can then prove that $u_\varepsilon^* \to u_0^*$ in $L_2((0, T), \mathbb{R}^m)$,. For this, observe that by (H) we have that the applications defined by $A_\varepsilon : z(\cdot) \to -g(\cdot, x_\varepsilon^*(\cdot), z(\cdot), v_\varepsilon^*(\cdot))$ for $\varepsilon > 0$, and by $A_0 : z(\cdot) \to -g(\cdot, x_0^*(\cdot), z(\cdot), v_0^*(\cdot))$ for $\varepsilon = 0$, are monotone from $L_2((0,T), \mathbb{R}^m)$ into themselves. Furthermore, $\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} u_\varepsilon^*(t) = -A_\varepsilon(u_\varepsilon^*(t))$ for $t \in [0,T]$. Thanks to the monotonicity we have

$$\langle A_\varepsilon(u_\varepsilon^*) - A_\varepsilon(z), u_\varepsilon^* - z\rangle_{L_2(0,T)} \geqslant 0,$$

for any $z \in L_2((0,T), \mathbb{R}^m)$.

In particular, taking $z = u_0^*$ by (H) we have

$$\langle A_\varepsilon(u_\varepsilon^*) - A_\varepsilon(u_0^*), u_\varepsilon^* - u_0^*\rangle \geqslant \nu \|u_\varepsilon^* - u_0^*\|^2_{L_2(0,t)} \geqslant 0. \tag{2.9}$$

On the other hand $\langle A_\varepsilon(u_\varepsilon^*), u_0^*\rangle_{L_2(0,T)}$ tends to zero as $\varepsilon \to 0$ since $A_\varepsilon(u_\varepsilon^*) \to 0$ weakly as $\varepsilon \to 0$. In fact, for any $\eta \in C^\infty([0,T])$ with support contained in $[0,T]$ we have by integrating by parts

$$\langle A_\varepsilon(u_\varepsilon^*), \eta\rangle_{L_2(0,T)} = -\varepsilon\langle u_\varepsilon^*, \eta'\rangle$$

which converges to zero, since $u_\varepsilon^* \to u_0^*$ weakly in $L_2(0,T)$. Thus $A_\varepsilon(u_\varepsilon^*)$ converges weakly to zero. To show that $\langle A_\varepsilon(u_\varepsilon^*), u_\varepsilon^*\rangle_{L_2(0,T)} \to 0$ as $\varepsilon \to 0$ we simply observe that

$$\langle A_\varepsilon(u_\varepsilon^*), u_\varepsilon^*\rangle_{L_2(0,T)} = -\frac{\varepsilon}{2}(|u_\varepsilon^*(T)|^2 - |u_0|^2)$$

and by using (2.8) the right-hand side tends to zero. Finally, passing to the limit as $\varepsilon \to 0$ we obtain that $u_\varepsilon^* \to u_0^*$ in $L_2(0,T)$.

Consider now the limit as $\varepsilon \to 0$ in the system of differential inclusions

$$\begin{cases} \dot{x}_\varepsilon^*(t) \in F(t, x_\varepsilon^*(t), u_\varepsilon^*(t)) \\ \varepsilon\dot{u}_\varepsilon^*(t) \in L(t, x_\varepsilon^*(t), u_\varepsilon^*(t)) \end{cases}$$

for a.a. $t \in [0,T]$, where

$$L(t, x_\varepsilon^*(t), u_\varepsilon^*(t)) = \frac{\partial s}{\partial t}(t, x_\varepsilon^*(t)) + \frac{\partial s}{\partial x}(t, x_\varepsilon^*(t)) F(t, x_\varepsilon^*(t), u_\varepsilon^*(t)).$$

We obtain

$$\begin{cases} \dot{x}_0^*(t) \in F(t, x_0^*(t), u_0^*(t)) \\ 0 \in L(t, x_0^*(t), u_0^*(t)). \end{cases}$$

In fact, since $\varepsilon\dot{u}_\varepsilon$ is bounded in $L_2(0,T)$ it converges weakly to some $w \in L_2(0,T)$. We show now that $w = 0$; for this observe that

$$\langle \varepsilon\dot{u}_\varepsilon, \eta\rangle = -\varepsilon\langle u_\varepsilon, \eta'\rangle \to 0$$

as $\varepsilon \to 0$ for any $\eta \in C^\infty([0,T], \mathbb{R}^m)$. Therefore, $\langle w, \eta\rangle = 0$ for any $\eta$ thus $w = 0$.

Therefore, there exists a measurable selection $\hat{v}_0^*(t) \in V(t, \hat{x}_0^*(t))$ such that

$$\begin{cases} \dot{\hat{x}}_0^*(t) = f(t, \hat{x}_0^*(t), \hat{u}_0^*(t), \hat{v}_0^*(t)) \\ 0 = g(t, \hat{x}_0^*(t), \hat{u}_0^*(t), \hat{v}_0^*(t)), \end{cases}$$

for a.a. $t \in [0, T]$. By (V) and Theorem 9.5.3 of Aubin & Frankowska (1990) there exists a (modelled) uncertainty $\hat{v}_0(t, x) \in V(t, x)$ such that $\hat{v}_0^*(t) = \hat{v}_0(t, \hat{x}_0^*(t))$ for a.a. $t \in [0, T]$. Thus $(\hat{x}_0^*, \hat{u}_0^*)$ is an optimal solution of (2.2) and so $G(\hat{x}_0^*) = 0$, since $s(t, \hat{x}_0^*(t)) = 0$ for a.a. $t \in [0, T]$.

Finally, $u_\varepsilon^*(t) \in U$ for any $t \in [0, T]$ and $\varepsilon > 0$ sufficiently small, $U \subset \mathbb{R}^m$ bounded set. Thus, by assumptions ($f_2$) and (V) we have that $\{x_\varepsilon^*\}_{\varepsilon>0} \subset C([0, T], \mathbb{R}^n)$ is a set of equibounded and equicontinuous functions. Therefore, by passing to a subsequence, if necessary, we have that $x_\varepsilon^* \to \hat{x}_0^*$ in $C([0, T], \mathbb{R}^n)$ and so $G(x_\varepsilon^*) \to G_0(\hat{x}_0^*) = 0$. This concludes the proof. □

A direct consequence of the previous result is the following corollary.

COROLLARY 1 If $(x_\varepsilon, u_\varepsilon)$, $\varepsilon > 0$, is the solution of (2.1), with $(0, x_\varepsilon(0)) = (0, x_0) \in S$, corresponding to any uncertainty $v_\varepsilon(t, x) \in V(t, x)$. Then $\{x_\varepsilon\}_{\varepsilon>0}$ converges as $\varepsilon \to 0$ to some function $\hat{x}_0$ in $C([0, T], \mathbb{R}^n)$ such that $s(t, \hat{x}_0(t)) = 0$ for any $t \in [0, T]$.

*Proof.* Let $(x_\varepsilon, u_\varepsilon)$, $\varepsilon > 0$, be the solution of (2.1) corresponding to a modelled uncertainty $v_\varepsilon(t, x) \in V(t, x)$. By repeating the arguments of Theorem 1 with $(x_\varepsilon, u_\varepsilon)$ instead of $(x_\varepsilon^*, x_\varepsilon^*)$ we can prove that $x_\varepsilon \to \hat{x}_0$ in $C([0, T], \mathbb{R}^n)$ and $u_\varepsilon \to \hat{u}_0$ in $L_2((0, T), \mathbb{R}^m)$ and $(\hat{x}_0, \hat{u}_0)$ is an optimal pair, that is $(\hat{x}_0, \hat{u}_0)$ is the solution of (2.2) corresponding to some uncertainty $\hat{v}_0(t, x)$ for which ($R_0$) is satisfied. Thus, in particular $G(\hat{x}_0) = 0$ and so $s(t, \hat{x}_0(t)) = 0$ for any $t \in [0, T]$. □

REMARK 4 Observe that if there exists a unique uncertainty $v_0(t, x)$ such that ($R_0$) is satisfied then for any fixed initial state $x_0 \in \mathbb{R}^n$ there is a unique equivalent control $u_0(t, x)$ and then a unique pair $(x_0, u_0)$ satisfying (2.2).

### 2.2 *Convergence in infinite time intervals*

As is easy to see, by means of the arguments employed in the proof of Theorem 1 we cannot extend the uniform convergence of $x_\varepsilon$ to $x_0$ to the infinite time interval $[0, +\infty)$ as $\varepsilon \to 0$ without any other assumptions on the system dynamics $f(t, x, u, v(t, x))$. Indeed, even in the classical singular perturbation theory, i.e. in the absence of the uncertainty parameter $v$, as shown in Hoppensteadt (1966) the zero solution $x = 0$ of the reduced system is required to be asymptotically stable uniformly with respect to the other parameters involved.

These considerations lead us to attempt to obtain the uniform convergence of $x_\varepsilon$ to $x_0$ in $[0, +\infty)$ as $\varepsilon \to 0$, by means of the introduction of appropriate Liapunov functions for the reduced system. This property is of relevance in many control problems, in particular the tracking problems. We will do this in the following, assuming that the conditions of the previous section are reformulated for $t \in [0, +\infty)$ and requiring extra conditions in Theorem 2.

First, observe that for any system dynamics $f(t, x, u, v(t, x))$ for which the equivalent control $u_v(t, x)$ exists, namely $g(t, x, u_v(t, x), v(t, x)) = 0$ for any $t \geqslant 0$ and $x \in \mathbb{R}^n$, condition (H) provides the uniform exponential stability of the boundary layer

$$\begin{cases} \dot{z} = g(\alpha, \beta, z, v(\alpha, \beta)) \\ z(0) = u_0. \end{cases}$$

In fact, introducing as Liapunov function

$$W_v(t,x,u) = \frac{|u - u_v(t,x)|^2}{2},$$

it turns out that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\tau} W_v(\alpha,\beta,z(\tau)) = \frac{\partial}{\partial z} W_v(\alpha,\beta,z(\tau)) \cdot g(\alpha,\beta,z(\tau),v(\alpha,\beta))\\ &= \langle z(\tau) - u_v(\alpha,\beta), g(\alpha,\beta,z(\tau),v(\alpha,\beta)) - g(\alpha,\beta,u_v(\alpha,\beta),v(\alpha,\beta))\rangle\\ &\leqslant -\nu|z(\tau) - u_v(\alpha,\beta)|^2.\end{aligned}$$

Then, for the corresponding reduced problem

$$\begin{cases}\dot{x} = f(t,x,u_v(t,x),v(t,x))\\ x(0) = x_0,\end{cases}$$

where $t \in [0,+\infty)$, we assume that $Z(x) = \dfrac{|x|^2}{2}$ is a Liapunov function: precisely, we assume the existence of a positive constant $\alpha_v > 0$ such that

$$\frac{\partial}{\partial x} Z(x) \cdot f(t,x,u_v(t,x),v(t,x)) \leqslant -\alpha_v |x|^2,$$

for $t$ sufficiently large.

Observe that the existence of such a Liapunov function implies that $x = 0$ is an asymptotically stable equilibrium point for the reduced system, and so $f(t,0,u_v(t,0),v(t,0)) = 0$, for any $t \geqslant 0$.

Consider now for $\lambda \in [0,1]$ the candidate Liapunov function

$$\lambda W_v(t,x,u) + (1-\lambda) Z(x) := \Psi_v^{\lambda}(t,x,u)$$

for the system dynamics

$$\begin{cases}\dot{x} = f(t,x,u,v(t,x))\\ \varepsilon\dot{u} = g(t,x,u,v(t,x)),\end{cases} \tag{2.10}$$

with $\varepsilon > 0$ and $t \in [0,+\infty)$ and compute $\dfrac{\mathrm{d}}{\mathrm{d}t}\Psi_v^{\lambda}(t,x,u)$ with respect to (2.10). We obtain

$$\begin{cases}\lambda\left[\dfrac{\partial}{\partial t} W_v(t,x,u) + \dfrac{\partial}{\partial x} W_v(t,x,u)\cdot f(t,x,u,v(t,x))\right.\\ \left.+\dfrac{1}{\varepsilon}\dfrac{\partial}{\partial u} W_v(t,x,u)\cdot f(t,x,u,v(t,x))\right] + (1-\lambda)\left[\dfrac{\partial}{\partial x} Z(x)\cdot f(t,x,u_v(t,x),v(t,x))\right.\\ \left.+\dfrac{\partial}{\partial x} Z(x)\cdot (f(t,x,u,v(t,x)) - f(t,x,u_v(t,x),v(t,x))\right].\end{cases} \tag{2.11}$$

We are now in the position to prove the following theorem.

THEOREM 2 Assume ($f_2$), ($f_3$), ($s_2$), (V), and (H) for $t \in [0, +\infty)$. Moreover we assume:

($\hat{f}_1$) $f \in C^1([0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p)$;
($\hat{s}_1$) $s \in C^2([0, +\infty) \times \mathbb{R}^n)$;
($\hat{R}_0$) for any uncertainty $v(t, x) \in V(t, x)$ there is $u_v(t, x)$ such that $g(t, x, u_v(t, x), v(t, x)) = 0$ for any $t \in [0, +\infty)$ and any $x \in \mathbb{R}^n$. In other words, we assume that for any system dynamics $f(t, x, u, v(t, x))$ there exists the equivalent control;
(Z) there exists $\alpha > 0$ and $t_0 > 0$ such that

$$\frac{\partial}{\partial x} Z(x) \cdot f(t, x, u_v(t, x), v(t, x)) \leqslant -\alpha |x|^2,$$

for any uncertainty $v(t, x) \in V(t, x)$ and $t \geqslant t_0$.

Then for any sequence $(x_\varepsilon, u_\varepsilon)$ of solutions to (2.10), such that $(0, x_\varepsilon(0)) = (0, x_0) \in S$ and $u_\varepsilon(0) = u_0$, for $\varepsilon \geqslant 0$, we have that $x_\varepsilon \to x_0$ uniformly in $[0, +\infty)$.

*Proof.* From (2.11) under our assumptions we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \Psi_v^\lambda(t, x, u) \leqslant \lambda d_\eta |u - u_v(t, x)| (L|u - u_v(t, x)| + \gamma |x|)$$
$$- \frac{\lambda}{\varepsilon} \nu |u - u_v(t, x)|^2 - (1 - \lambda)\alpha |x|^2 + (1 - \lambda) L |x| |u - u_v(t, x)|$$

for any $t \geqslant t_0$ and any $(x, u)$ such that $|x| + |u| \leqslant \eta$.
Here $d_\eta \geqslant \max\left\{\sup \frac{\partial}{\partial t} W_v(t, x, u), \sup \frac{\partial}{\partial x} W_v(t, x, u)\right\}$, where the suprema are considered on $[0, +\infty) \times \{(x, u) : |x| + |u| \leqslant \eta\}$. Note that our assumptions ensure that $d_\eta$ is finite for any $\eta > 0$.

Equivalently, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t} \Psi_v^\lambda(t, x, u)$$
$$\leqslant -(|x|,\ |u - u_v(t, x)|) \cdot \begin{pmatrix} (1 - \lambda)\alpha & -1/2[(1 - \lambda)L - \lambda d_\eta \gamma] \\ -1/2[(1 - \lambda)L - \lambda d_\eta \gamma] & \frac{\lambda}{\varepsilon} L \nu - \lambda d_\eta \end{pmatrix} \cdot \begin{pmatrix} |x| \\ |u - u_v(t, x)| \end{pmatrix}.$$

It is easy to see that for any $\varepsilon$ satisfying the inequality

$$0 < \varepsilon < \varepsilon^* := \frac{\alpha \nu}{\alpha L d_\eta + \frac{1}{4(1 - \lambda)\lambda} [(1 - \lambda) L + \lambda d_\eta \gamma]^2},$$

we have that for any $\lambda \in (0, 1)$, any uncertainty $v(t, x) \in V(t, x)$ and any $t \geqslant t_0$, the function $\Psi_v^\lambda(t, x, u)$ is a Liapunov function for (2.10). Therefore, the origin $x = 0$ is an asymptotically stable equilibrium point for (2.10) for any $\varepsilon \in (0, \varepsilon^*)$ and any uncertainty $v(t, x) \in V(t, x)$.

Consider now any sequence $(x_\varepsilon, u_\varepsilon)$ of solutions to (2.10), from the previous considerations we have that for any $\varepsilon \in (0, \varepsilon^*)$ there exist two continuous, non-negative functions $\rho_\varepsilon(\cdot)$ and $\sigma_\varepsilon(\cdot)$ defined in $[0, +\infty)$, with $\rho_\varepsilon(\cdot)$ strictly increasing, $\rho_\varepsilon(0) = 0$, $\sigma_\varepsilon(\cdot)$ strictly decreasing, $\sigma_\varepsilon(s) \to 0$ as $s \to \infty$, and such that

$$|x_\varepsilon(t)| \leqslant \rho_\varepsilon(x_\varepsilon(t_0))\sigma_\varepsilon(t),$$

for any $t \geqslant t_0$, see Hoppensteadt (1966).

Let $r > 0$ and $t \geqslant t_0$, consider

$$\sup_{\varepsilon \in [0,\varepsilon^*)} \rho_\varepsilon(r)\, \sigma_\varepsilon(t) = \Sigma(r, t).$$

It turns out that $\Sigma(r_1, t) < \Sigma(r_2, t)$ for any $r_1 < r_2$ and $t \geqslant t_0$, and $\Sigma(r, t_1) > \Sigma(r, t_2)$ for any $r > 0$ and $t_0 < t_1 < t_2$. Furthermore, $\Sigma(r, t) \to 0$ as $r \to 0$ for any $t \geqslant t_0$ and $\Sigma(r, t) \to 0$ as $t \to \infty$ for any $r > 0$.

Therefore, for any $r > 0$ there exists $t_r \geqslant t_0$ such that $\Sigma(r, t) < r/2$ for any $t \geqslant t_r$ and any $\varepsilon \in [0, \varepsilon^*)$, and so $|x_\varepsilon(t) - x_0(t)| < r$ for any $t \geqslant t_r$ and any $\varepsilon \in [0, \varepsilon^*)$. On the other hand, by Corollary 1 we have that for any $r > 0$ there exists $\varepsilon_r > 0$ such that $|x_\varepsilon(t) - x_0(t)| < r$, for any $\varepsilon \in [0, \varepsilon_r)$ and any $t \in [0, t_r]$.

In conclusion, for any $r > 0$ there exists $\varepsilon_0 > 0$, with $\varepsilon_0 = \min\{\varepsilon^*, \varepsilon_r\}$, such that $|x_\varepsilon(t) - x_0(t)| < r$ for any $\varepsilon \in [0, \varepsilon_0)$ and $t \geqslant 0$. That is, $x_\varepsilon$ converges to $x_0$ uniformly in $[0, +\infty)$ as $\varepsilon \to 0$. Furthermore, as far as concerns the control $u_\varepsilon$ we have that $|u_\varepsilon(t) - u_v(t, x_\varepsilon(t))| \to 0$ as $t \to +\infty$. □

## 3. Exact attainability

Our goal in this short section is to present a method extensively studied in Krastanov & Quincampoix (2001), Nistri & Quincampoix (2000) and Veliov (1994, 1997) for obtaining local attainability in small time of the set $K$.

PROPOSITION 2 Assume (V). Assume that $u \in U$, $U$ compact set of $\mathbb{R}^m$, and that $f$, $k$ are Lipschitz continuous, where $k : [0, T] \times \mathbb{R}^n \to \mathbb{R}$. Assume that

$$K = \{\, (t, x) \in [0, T] \times \mathbb{R}^n,\ k(t, x) = 0 \,\}$$

is a $C^{1,1}$ manifold. Moreover, assume that there exists $a > 0$ such that for any $(t, x) \in [0, T) \times K$

$$\begin{cases} (i) & \min_{u \in U} < f(t, x, u, v), \frac{\partial k}{\partial x}(t, x) > + \frac{\partial k}{\partial t}(t, x) \leqslant -a \\ (ii) & a \leqslant \max_{u \in U} < f(t, x, u, v), \frac{\partial k}{\partial x}(t, x) > + \frac{\partial k}{\partial t}(t, x) \end{cases} \tag{3.1}$$

for all $v \in V(t, x)$. Then there exists a neighbourhood $I$ of $K$ such that starting from any point of $I \backslash K$, and for any uncertainty $v(t, x)$, there exists a control for which the corresponding solution to (1.2) reaches $K$ in finite time and remains in $K$ for all the future time $t < T$.

*Proof.* We only sketch the proof: for the details we refer the reader to Proposition 3.3 of Nistri & Quincampoix (2000). Let $v(t, x)$ be any Lipschitz selection of $V(t, x)$. For any $(t, x)$, the vectors $+\nabla k(t, x)$ and $-\nabla k(t, x)$ are (proximal) normals to $K$ at $(t, x)$. By the viability theorem expressed in terms of proximal normals (see for instance Veliov (1994)) condition (3.1)(i) implies that the set

$$\{ (t, x, z) \in [0, T) \times \mathbb{R}^n \times \mathbb{R},\ k(t, x) \leqslant z \}$$

is locally viable for the differential systems

$$(\dot{t}, \dot{x}, \dot{z}) \in (1,\ f(t, x, U, v(t, x)), -a). \tag{3.2}$$

Thus starting from any point $(t_0, x_0)$ with $k(t_0, x_0) > 0$ small enough, there exists a measurable control $u(t)$ such that the corresponding solution $t \to (t, x(t), k(t_0, x_0) - at)$ of (3.2) satisfies the inequality

$$k(t, x(t)) \leqslant k(t_0, x_0) - at,$$

and so for $t = k(t_0, x_0)/a$ the trajectory $x$ reaches $K$. Using (3.1)(ii), a similar argument shows the attainability of $K$ in finite time starting from any point of

$$\{ (t, x, z) \in [0, T) \times \mathbb{R}^n \times \mathbb{R},\ k(t, x) \geqslant z \}.$$

Because of the continuity of $f$, there exists $u$ such that

$$\left\langle f(t, x, u, v), \frac{\partial k}{\partial x}(t, x) \right\rangle + \frac{\partial k}{\partial t}(t, x) = 0.$$

Therefore, $K$ is locally viable. □

## 4. Applications

In this section we present some illustrative applications of the results obtained in this paper.

### 4.1 *A tracking problem*

In order to clarify the meaning of our assumptions and of the obtained results we first consider a simple tracking problem for a nonlinear, dynamical system affected by deterministic uncertainty which can be modelled by the differential system (1.1). The dynamics $f$ will be detailed later.

Specifically, given a smooth reference trajectory $x_{\text{ref}}(t)$, $t \in [0, T]$, with $T > 0$ large, and given any initial condition $x_0 \in \mathbb{R}^n$ we want, by means of the control design proposed in Section 2, to solve problem (CP). For this, we introduce a smooth function $s : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$s(t, x) := \mathrm{e}^{Ct}(x_0 - x_{\text{ref}}(0)) + x_{\text{ref}}(t) - x. \tag{4.1}$$

where $\lambda(C) \leqslant -\alpha < 0$. Here $\lambda(C)$ denotes the eigenvalues of the symmetric $n \times n$ matrix $C$.

Note that $(0, x_0) \in S = \{(t, x) \in [0, T] \times \mathbb{R}^n : s(t, x) = 0\}$. We consider here the case when $f$ is of the form

$$\dot{x} = \varphi(t, x, v) + Bu, \qquad x(0) = x_0, \tag{4.2}$$

where $\varphi(t, x, v)$ is a function depending on the time $t \in [0, T]$, the state $x \in \mathbb{R}^n$ and it is affected by the uncertainty $v \in V(t, x)$. In other words, the considered real dynamics is among the possible system dynamics

$$\dot{x} = \varphi(t, x, v(t, x)) + Bu, \tag{4.3}$$

where $(t, x) \in [0, T] \times \mathbb{R}^n$ and the function $(t, x) \to v(t, x) \in V(t, x)$ is $t$-measurable and $x$-Lipschitz. Finally, $B$ is a $n \times n$ matrix and $u \in U \subseteq \mathbb{R}^n$ is the control variable. On the map $(t, x, v) \to \varphi(t, x, v)$ we assume conditions ($f_1$)–($f_3$) and on the multivalued map $(t, x) \to V(t, x)$ we assume condition (V). It is also easy to verify that the function $s$ defined in (4.1) satisfies ($s_1$)–($s_2$). In the present case assumption (H) takes the form

$$-\nu|u_1 - u_2|^2 \geqslant \langle B(u_1 - u_2), u_1 - u_2\rangle,$$

which means that the matrix $B$ is negative defined.

Under the previous assumptions it is easy to see that ($R_0$) is satisfied for any modelled uncertainty $v(t, x) \in V(t, x)$ and so we can apply our main result to conclude that problem (CP) is solved for system (4.2).

In fact, let $x_\varepsilon$ be the solution of (4.3) corresponding to some uncertainty $v_\varepsilon(t, x)$ and fix a $2\delta$-neighbourhood of the trajectory $x_{\text{ref}}$, then we have that

$$|x_\varepsilon(t) - x_{\text{ref}}(t)| \leqslant |x_{\text{ref}}(t) - x_0(t)| + |x_0(t) - x_\varepsilon(t)|, \quad t \in [0, T],$$

where $x_0$ is the limit trajectory of $\{x_\varepsilon\}_{\varepsilon>0}$ in the uniform topology (Corollary 1). Therefore, since $s(t, x_0(t)) = 0$ for any $t \in [0, T]$, we have for $\varepsilon > 0$ sufficiently small

$$|x_\varepsilon(t) - x_{\text{ref}}(t)| \leqslant |e^{Ct}(x_{\text{ref}}(0) - x_0)| + |x_0(t) - x_\varepsilon(t)| \leqslant e^{-\alpha t}|x_{\text{ref}}(0) - x_0| + \delta. \tag{4.4}$$

Note that the larger is $\alpha > 0$, the smaller is the time $\hat{t}_0 > 0$, and the following property holds:

$$e^{-\alpha\hat{t}_0}|x_{\text{ref}}(0) - x_0| = \delta \quad \text{and} \quad e^{-\alpha\hat{t}_0}|x_{\text{ref}}(0) - x_0| > \delta \quad \text{for} \quad 0 \leqslant t < \hat{t}_0.$$

Observe that $\alpha > 0$ can be arbitrarily chosen for defining the function $s$ and hence for designing the control law.

Furthermore, after replacing $[0, T]$ with $[0, +\infty)$ in all the previous assumptions, it is easy to see that condition ($\hat{R}_0$) is also verified. Moreover, if we solve the second equation of the reduced system with respect to $u$ whenever $v(t, x) \in V(t, x)$ and we replace it in the first equation we obtain

$$\dot{x}(t) = \dot{x}_{\text{ref}}(t) + Ce^{Ct}(x_0 - x_{\text{ref}}(0))$$

which is independent of $v(t, x)$ and represents the dynamics of $(x_0(t) - x_{\text{ref}}(t))$ in $[0, +\infty)$. It is also clear that the change of variable $X = x - x_{\text{ref}}$ makes the equilibrium point $X = 0$ of this dynamics asymptotically stable: in fact

$$\langle X, \dot{X}\rangle = \langle X, C X\rangle \leqslant -\alpha|X|^2.$$

Letting $Z(X) = \frac{|X|^2}{2}$ we have that condition $(Z)$ is also satisfied and then, assuming ($\hat{f}_1$), Theorem 2 applies to conclude that (4.4) holds on $[0, +\infty)$.

### 4.2 *Control of a rigid robot manipulator*

We now present an application of our results to a tracking control problem of a rigid robot manipulator. To introduce the problem we closely follow Cavallo *et al.* (1996), where the tracking problem was considered in the deterministic case. In this case the manipulator dynamical model can be written in the form

$$B(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + e(\theta) = \tau, \tag{4.5}$$

where $\theta$ is the vector of generalized coordinates, $\tau$ is the vector of generalized forces applied at various joints, $B(\theta)$ is the inertia matrix, the vector $c(\theta, \dot{\theta})$ takes into account Coriolis and centrifugal torques, and $e(\theta)$ is the vector of gravitational torques. We rewrite (4.5) in state variable form for a manipulator with $m$ degrees of freedom. For this, it is convenient to introduce the column state vector $x \in \mathbb{R}^{2m}$ as follows:

$$x_i = \theta_i, \quad x_{i+m} = \dot{\theta}_i, \quad u_i = \tau_i, \quad \text{where} \quad i = 1, 2, \ldots, m;$$

and so $x = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}$ and $u = (u_i)_{i=1}^m$. Furthermore, define

$$\overline{B}(x) = B^{-1}(\theta).$$

Therefore (4.5) assumes the form

$$\dot{x} = A(x)x + B(x)u,$$

where

$$A(x) = \begin{pmatrix} 0 & I_m \\ A_{21}(x) & A_{22}(x) \end{pmatrix}$$

and

$$B(x) = \begin{pmatrix} 0 \\ \overline{B}(x) \end{pmatrix}$$

with $\overline{B}(x)$ symmetric and positive defined. We consider the case when $B(\theta)$, $C(\theta, \dot{\theta})$ and $e(\theta)$ and so $A_{21}(x)$, $A_{22}(x)$ and $\overline{B}(x)$ are affected by deterministic uncertainty. We assume that for any system dynamics the matrix $B(\cdot)$ is a inertia matrix. In fact, the matrix coefficient of $\ddot{\theta}$ must be an inertia matrix and so for any system dynamics we consider only symmetric and positive definite matrices for such a role.

Moreover, we assume that the uncertainty on $A_{21}(x)$, $A_{22}(x)$ and $\overline{B}(x)$ can be modelled respectively by means of multivalued maps $x \to V_i(x) \subset \mathbb{R}^m$, $i = 1, 2, 3$, such that all possible system dynamics are obtained as Lipschitz selections $v_i$ of $V_i$ as follows:

$$\dot{x} = A(x, v(x))x + B(x, w(x))u, \tag{4.6}$$

where $v(x) = (v_1(x), v_2(x))$ and $w(x) = v_3(x)$. For instance, this is the case when, as in Bartolini & Zolezzi (1991), we have explicit knowledge of the upper and lower bounds of each component of the $m \times m$ matrices $A_{21}(x)$, $A_{22}(x)$ and $\overline{B}(x)$. We assume condition (V) on the maps $V_i$, $i = 1, 2, 3$, and condition ($f_3$) on the dynamics $A(x, v)x + B(x, w)u$.

Furthermore, we assume that the maps $(x, v) \to A(x, v)$ and $(x, w) \to B(x, w)$ are Lipschitz and we also assume ($f_3$).

Since $x \to A(x, v)$ and $x \to B(x, v)$ are bounded functions on $\mathbb{R}^{2m}$, when $v$ belongs to a bounded set, condition ($f_2$) is satisfied.

Now, given a reference state trajectory $x_{\text{ref}}(t)$ which the state $x(t)$ of system (4.6) is required to track in $[0, T]$, with $T > 0$ large, and given an initial state $x_0$ for $x(t)$ we define a smooth function $s : [0, T] \times \mathbb{R}^{2m} \to \mathbb{R}^m$ as follows:

$$s(t, x) = S[\mathrm{e}^{Ct}(x_0 - x_{\text{ref}}(0)) + x_{\text{ref}}(t) - x]$$

where $S = (S_1, S_2)$, and $S_1, S_2$ $m \times m$ are matrices to be specified later. Moreover, $C$ is a $2m \times 2m$ symmetric matrix with $\lambda(C) \leqslant -\alpha$ for some $\alpha > 0$.

Clearly $s$ satisfies ($s_1$) and ($s_2$). For any system dynamics (4.6) condition (H) takes the form

$$\langle S_2\overline{B}(x, w(x))(u_1 - u_2), u_1 - u_2\rangle \leqslant -\nu|u_1 - u_2|^2$$

for some $\nu > 0$.

Since we have assumed that $\overline{B}(x, w(x))$ is an inertia matrix for any $w(x) \in V_3(x)$, in order to satisfy (H) in any compact set $Q$ of $\mathbb{R}^{2m}$ it is sufficient to choose $S_2$ in such a way that $S_2\overline{B}(x, w(x))$ is symmetric and negative defined on $\mathbb{R}^{2m}$ and so $S_2$, in particular, is invertible. Finally, if $Q$ is any compact set containing the reference trajectory $x_{\text{ref}}$ and if we take $S_1$ and $S_2$ in such a way that $-S_2^{-1}S_1$ is a symmetric matrix with $\lambda(S_2^{-1}S_1) > \dfrac{\alpha}{2}$, then we have solved problem (CP) for any initial condition $x_0 \in Q$.

Indeed, if we put $e(t) = x_0(t) - x_{\text{ref}}(t)$ we have that

$$Se(t) = S\mathrm{e}^{Ct}e(0),$$

or equivalently, $S_1e_1 + S_2e_2 = S\mathrm{e}^{Ct}e(0)$, where $e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$, $e_i \in \mathbb{R}^n$, $i = 1, 2$ and $e_2 = \dot{e}_1$. Therefore

$$\dot{e}_1 = -S_2^{-1}S_1e_1 + S_2^{-1}S\mathrm{e}^{Ct}e(0)$$

and

$$\dot{e}_2 = -S_2^{-1}S_1e_2 + S_2^{-1}SC\mathrm{e}^{Ct}e(0).$$

Observe that the error dynamics does not depend on the uncertainty.

Now, from the condition $\lambda(S_2^{-1}S_1) > \dfrac{\alpha}{2}$ it can be proved as in Cavallo *et al.* (1996) that $e_1$ and so $e_2$ have exponential decay. This argument together with Corollary 1 leads to the conclusion that problem (CP) is solved on any large interval time $[0, T]$.

Furthermore, a simple calculation shows that (Z) is satisfied with $Z(e) = \dfrac{|e|^2}{2}$ together ($\hat{s}_1$) and ($\hat{R}_0$) and thus Theorem 2 applies to conclude that (CP) is solved in $[0, +\infty)$ for any $x_0 \in Q$. Finally, we would like to point out that other relevant control problems can be represented in the form (4.6) as in the case of the attitude control of a satellite, see Cavallo *et al.* (1993b).

### 4.3 *A direct Liapunov function approach*

Consider the nonlinear uncertain system

$$\begin{cases} \dot{x} = f(x, u, v) \\ x(0) = x_0 \end{cases} \tag{4.7}$$

with $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ Lipschitz with respect to the variables $(x, u, v) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p$, $f(0, u, v) = 0$ for any $u \in \mathbb{R}^n$ and any $v \in V(x) \subset \mathbb{R}^p$. Assume ($f_2$)–($f_3$) and condition $(V)$ on the multivalued map $x \to V(x)$.

We assume that $x = 0$ is the isolated equilibrium point for the reduced system

$$\begin{cases} \dot{x} = f(x, u_0(x), v_0(x)) \\ x(0) = x_0 \end{cases} \tag{4.8}$$

Let $Z : \mathbb{R}^n \to \mathbb{R}$ be a Liapunov function for (4.8) and we assume that

(i) $\dfrac{\partial Z}{\partial x}(x) = 0$ if and only if $x = 0$;

(ii) $\det \dfrac{\partial^2 Z}{\partial x^2}(0) \neq 0$.

Given $x_0 \in \mathbb{R}^n$, let us define $s : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$s(t, x) = \frac{\partial Z}{\partial x}(x) - \frac{\partial Z}{\partial x}(x_0)\mathrm{e}^{-\alpha t}$$

where $\alpha > 0$ and we assume ($s_1$)–($s_2$).

We want to design a dynamical feedback control by means of the approach presented in Section 2 in order to solve problem (CP) in any prescribed neighbourhood of $x = 0$ of (4.7). For this, we form the singularly perturbed system

$$\begin{cases} \dot{x} = f(x, u, v(x)), \quad x(0) = x_0 \\ \varepsilon \dot{u} = \dfrac{\partial^2 Z}{\partial x^2}(x) f(x, u, v(x)) + \alpha \dfrac{\partial Z}{\partial x}(x_0)\mathrm{e}^{-\alpha t}, \quad u(0) = u_0. \end{cases} \tag{4.9}$$

We assume condition (H), that is

$$\left\langle \frac{\partial^2 Z}{\partial x^2}(f(x, u_1, v) - f(x, u_2, v)), u_1 - u_2 \right\rangle \leqslant -\nu |u_1 - u_2|^2$$

for some $\nu > 0$, any $x \in \mathbb{R}^n$ and any $v \in R^p$.

We also assume ($R_0$). Therefore we can apply Corollary 1 to deduce that $\{x_\varepsilon\}_{\varepsilon > 0}$ converges uniformly on $[0, T]$, whenever $T > 0$, to some solution $x_0$ of the reduced problem. Furthermore, we have $s(t, x_0(t)) = 0$ for any $t \in [0, T]$, namely

$$\frac{\partial Z}{\partial x}(x_0(t)) - \frac{\partial Z}{\partial x}(x_0)\mathrm{e}^{-\alpha t} = 0$$

for any $t \in [0, T]$. Under assumptions (i) and (ii), for any neighbourhood $I$ of the origin we have that

$$x_0(t) = \left(\frac{\partial Z}{\partial x}\right)^{-1} \left(\frac{\partial Z}{\partial x}(x_0)\mathrm{e}^{-\alpha t}\right) \in I$$

for any $t \geqslant t_0$, where $t_0$ depends only on the data, in particular, from $\alpha > 0$ and $I$.

In conclusion, for any neighbourhood of the origin $x = 0$ the trajectory $x_\varepsilon$ corresponding to any system dynamics (4.9) belongs to that neighbourhood for $\varepsilon > 0$ sufficiently small and for $t \in [0, T]$. Indeed,

$$|x_\varepsilon(t)| \leqslant |x_\varepsilon(t) - x_0(t)| + |x_0(t)|.$$

Observe that if $Z(x) = x^t C x$, $\det C \neq 0$, then conditions (i) and (ii) are satisfied. Moreover, if $f(x, u, v) = g(x, v) + B(x)u$ with $g(0, v) = B(0) = 0$ for any $v \in \mathbb{R}^p$, the condition (H) becomes

$$\langle C B(x)(u_1 - u_2), u_1 - u_2\rangle \leqslant -\nu|u_1 - u_2|^2$$

and ($\mathrm{R}_0$) is satisfied for any modelled dynamics, that is ($\hat{R}_0$) is satisfied. If the other assumptions of Theorem 2 are also satisfied then $x_\varepsilon$ converges uniformly to $x_0$ in $[0, +\infty)$ as $\varepsilon \to 0$.

## 5. Conclusions

We have presented an approach for the control of nonlinear dynamical systems affected by deterministic uncertainty. We have considered as possible system dynamics any $t$-measurable and $x$-Lipschitz selections of the multivalued map which models the uncertainty. Our approach is based on the theory of singular perturbation for dynamical systems depending on a parameter (in our case the uncertainty) as presented in Bensoussan (1988) and Quincampoix & Zhang (1995).

First, it is shown that we can solve the proposed problem (CP) in any time interval $[0, T]$ by choosing in a convenient way a smooth function $s$ depending on the control objective which in turn permits us to define the dynamical feedback control law which solves the problem.

Then, under more restrictive assumptions, the uniform convergence of the states can be extended to all of $[0, +\infty)$ and so problem (CP) is solved in infinite time intervals. Some applications of our method show that a large class of relevant control problems can be solved by means of our approach.

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