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SET-VALUED PERTURBATIONS OF DIFFERENTIAL EQUATIONS AT RESONANCE

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0. INTRODUCTION

LET X, Y be Banach spaces, $L: D(L) \subset X \to Y$ be a Fredholm operator of nonnegative index and $F: X \multimap Y$ be a multivalued map. We shall be interested in finding sufficient conditions for the solvability of the abstract equation

$$Lu \in F(u). \tag{LF}$$

In Section 1, imposing assumptions concerning the kernel of L and the asymptotic behaviour of F we shall show, using the Liapunov–Schmidt method, that problem (LF) can be formulated in terms of a fixed-point problem in a suitable Banach space (cf. [2–4, 14] for single-valued nonlinearities as well as the extensive references in these papers).

Thus, applying topological degree arguments for upper-semicontinuous (u.s.c.) maps, the solvability of (LF) can be reduced to the study of the topological degree of an u.s.c. finite dimensional range map in a suitable ball (Theorem 1.1).

In Section 2 we give two examples of mappings L and F which satisfy the hypothesis of Theorem 1.1. The first one is a nonlinear elliptic boundary value problem of the form

$$\begin{aligned} \mathscr{L}u(x) &\in f(x, u(x)) && \text{in } \Omega \\ \mathscr{B}u(x) &= 0 && \text{on } \partial\Omega \end{aligned} \tag{I}$$

where $\mathscr{L}$ is a uniformly elliptic linear partial differential operator of order $2m$ in a bounded domain Ω, whose boundary $\partial\Omega$ is smooth. $\mathscr{B}$ represents a normal family of m smooth boundary operators of order less or equal to $2m - 1$ which cover $\mathscr{L}$ on $\partial\Omega$ (cf. [1, 9]) and $f: \Omega \times \mathbf{R} \multimap \mathbf{R}$ is a multivalued map with the following properties

f is a Borel-function;
$f(x, u)$ is a nonempty, compact and convex subset of $\mathbf{R}$ for almost all $x \in \Omega$ and all $u \in \mathbf{R}$; (f1)
$f(x, .)$ is an upper-semicontinuous map for almost all $x \in \Omega$;
f is uniformly bounded.

The second one is concerned with a nonlinear parabolic equation of the form

$$\begin{aligned} \frac{\partial}{\partial t}u(x, t) + \mathscr{M}u(x, t) &\in f(x, t, u(x, t)) && \text{in } \Omega \times (0, 1) \\ u(x, t) &= 0 && \text{on } \partial\Omega \times (0, 1) \\ u(x, 0) &= u(x, 1) && \text{in } \Omega \end{aligned} \tag{II}$$

where $\mathcal{M}$ is a formally self-adjoint uniformly elliptic operator of order $2m$ in a bounded domain Ω of $\mathbf{R}^n$ (with smooth boundary $\partial\Omega$) and

$$f : \Omega \times (0, 1) \times \mathbf{R} \multimap \mathbf{R}$$

is a multivalued map satisfying (f1) (where x is replaced by (x, t)).

Imposing an appropriate Landesman–Lazer type condition on the asymptotic behaviour of f we show that problem (I) has infinitely many solutions and problem (II) has at least one periodic solution. In the context of single-valued nonlinearities our results are contained in [2] and [3] respectively.

Finally in Section 3, Theorem 1.1 is applied to study the existence of periodic solutions of a multivalued differential equation of the form

$$\dot{u}(t) \in f(t, u(t)) \tag{III}$$

where $f : [0, 1] \times \mathbf{R}^n \multimap \mathbf{R}^n$ is a multivalued map satisfying the following conditions

$f(., u)$ is measurable for all $u \in \mathbf{R}^n$;

$f(t, u)$ is a nonempty, compact and convex subset of $\mathbf{R}^n$ for almost all $t \in [0, 1]$ and all $u \in \mathbf{R}^n$; (f2)

$f(t, .)$ is an upper-semicontinuous map for almost all $t \in [0, 1]$;

f is uniformly bounded.

We show that (III) has solutions if a Landesman–Lazer type condition is imposed on the nonlinear term f. This result contains in particular a result stated in [11] in the context of single-valued non-linearities.

1. AN ABSTRACT THEOREM

Let X, Y be Banach spaces and $\Gamma : X \multimap Y$ be a multivalued map. Firstly we recall some basic definitions. Γ is said to be *upper-semicontinuous* (u.s.c.) on X, if for every point $x \in X$, $\Gamma(x)$ is a nonempty 'convex' compact subset of Y, and for any open neighbourhood V of $\Gamma(x)$ there exists an open neighborhood U of x such that $\Gamma(U) \subset V$, where $\Gamma(U) = \bigcup_{z \in U} \Gamma(z)$. Moreover, Γ is said to be *compact* if it sends bounded sets into relatively compact sets. In the sequel we make use of the following characterization for u.s.c. maps. Γ is u.s.c. if and only if Γ is a non-empty, convex, compact valued map with closed graph and sends compact sets into relatively compact ones. Finally, Γ is said to be *uniformly bounded in* X if there exists $M \in \mathbf{R}$ such that $\|y\| \leqslant M$ for all $y \in \Gamma(x)$ and all $x \in X$, where $\|.\|$ denotes the norm in Y.

Let us consider the equation

$$Lu \in F(u), \tag{1.1}$$

where $L : D(L) \subset X \to Y$ is a closed linear operator with dense domain $D(L)$ and $F : D(F) \subset X \multimap Y$ is a multivalued closed, convex, bounded valued map with $D(L) \cap D(F) \neq \varnothing$.

Assume that L is a Fredholm operator of nonnegative index. We denote by $N(L)$ and $R(L)$ the kernel and the range of L respectively. Since L is a Fredholm operator, there exist a closed linear subspace $W \subset X$ such that $X = N(L) \oplus W$ and a closed linear subspace $Z \subset Y$ such that $Y = Z \oplus R(L)$. Let Q denote the projection onto $R(L)$ parallel to Z, that is $(I - Q)Y = Z$.

Recall that the restriction of L to $W \cap D(L)$ has a right continuous inverse $H: R(L) \to W$; Therefore it is obvious that (1.1) is equivalent to the following system

$$w \in HQF(v + w) \tag{1.2)'}$$

$$0 \in (I - Q)\, F(v + w) \tag{1.2)''}$$

with $u = v + w \in N(L) \oplus W$.

Moreover, since the index of L is nonnegative, there exists a one to one linear operator $T: Z \to N(L)$. Hence a solution of the following system

$$w \in HQF(Tz + w) \tag{1.3)'}$$

$$0 \in (I - Q)\, F(Tz + w) \tag{1.3)''}$$

where $z \in Z$ and $w \in W$, is a solution of (1.2)′–(1.2)″. Now it is obvious that a solution of (1.3)′–(1.3)″ is a fixed point of the multivalued map $R: Z \times W \multimap Z \times W$ defined by $R(z, w) = (z - (I - Q)\, F(Tz + w), HQF(Tz + w))$. Next we show that imposing some more assumptions on L and F problem (1.1) can be solved using the topological degree for u.s.c. maps introduced by Ma in [10].

Assume that

(L.F) (i) $(I - Q)F: X \multimap Z$ is u.s.c. on X; $HQF: X \multimap W$ is u.s.c. on X and compact;

(ii) for some $\xi \in \operatorname{Ker} L \backslash \{TZ \backslash \{0\}\}$ there exists a positive constant c_1 such that if

$$w \in tHQF(Tz + \xi + w) \text{ for } t \in [0, 1] \text{ and } z \in Z, \text{ then } \|w\| < c_1;$$

(iii) there exists a positive constant c_2 such that if $\|z\| \geq c_2$ and $\|w\| < c_1$ then $0 \notin (I - Q)\, F(Tz + \xi + w)$.

Then the following existence theorem can be stated

THEOREM 1.1. Suppose that the assumptions $(L.F)$, (i), (ii), (iii) are satisfied. Assume that $\deg((I - Q)\, F(Tz + \xi), B_r, 0) \neq 0$ for some $r > c_2$, where $\deg((I - Q)\, F(Tz + \xi), B_r, 0)$ denotes the topological degree of the map $(I - Q)\, F(Tz + \xi)$ with respect to 0 and the open ball B_r of radius r centered at zero. Then (1.1) has a solution of the form $u = Tz + \xi + w$.

Proof. Consider the map $G: Z \times W \times [0, 1] \to Z \times W$ defined by

$$G(z, w, \lambda) = ((I - Q)\, F(Tz + \xi + \lambda w), w - \lambda HQF(Tz + \xi + w)).$$

By the assumptions, it follows that G is an admissible homotopy in $B_r \times B_{c_1} \times [0, 1]$, with $r > c_2$, in the sense of Ma [10, p. 10]. Then by the homotopy invariance [10, Theorem 7.7] and the product of domains [10, Theorem 13.1] properties of the topological degree, we have that

$$\deg((I - Q)\, F(Tz + \xi + w), w - HQF(Tz + \xi + w), B_r \times B_{c_1}, 0)$$
$$= \deg((I - Q)\, F(Tz + \xi), B_r, 0).$$

Then the result follows from the solution property [10, Theorem 10.1].

It is our opinion that Theorem 1.1 can be proved with a different approach by using a suitably extended notion of coincidence degree [5] for multivalued maps.

2. PARTIAL DIFFERENTIAL BOUNDARY VALUE PROBLEMS

Throughout this section we denote by Ω a bounded domain of $\mathbf{R}^n$ whose boundary $\partial\Omega$ is an $(n-1)$ dimensional C^∞-manifold such that Ω lies locally on one side of $\partial\Omega$. Furthermore let $f:\Omega \times \mathbf{R} \multimap \mathbf{R}$ be a multivalued map satisfying (f1).

2.1. *A semilinear elliptic boundary value problem*

Let $\mathscr{L}$ be a linear uniformly elliptic partial differential operator of order $2m$, $m \geqslant 1$, with smooth real valued coefficient and let $\mathscr{B}$ be a normal family of m smooth boundary operators of order less or equal to $2m-1$ which covers $\mathscr{L}$ on $\partial\Omega$ (see [1, 9]).

We seek solutions of the multivalued elliptic boundary value problem

$$\begin{aligned} &\mathscr{L}u(x) \in f(x, u(x)) && \text{in } \Omega \\ &\mathscr{B}u(x) = 0 && \text{in } \partial\Omega \end{aligned} \tag{2.1}$$

where f satisfies (f1).

By a solution of (2.1) we mean a function $u \in W_p^{2m}(\Omega)$ for some $p > 1$ which satisfies the multivalued equation pointwise almost everywhere in Ω and the boundary conditions in the sense of the traces (see [1, 9]).

It is known (e.g. [1] for the case $p = 2$) that $(\mathscr{L}, \mathscr{B})$ induces a closed linear operator which is Fredholm of nonnegative index $L: L_p(\Omega) \to L_p(\Omega)$, the L_p-realization of $(\mathscr{L}, \mathscr{B})$, with domain $D(L) \subset W_p^{2m}(\Omega)$.

Then it is easily verified that the set

$$\left\{ w \in L_p(\Omega) : \int_\Omega wv = 0 \quad \text{for all } v \in N(L) \right\}$$

is a topological complement for $N(L)$ in $L_p(\Omega)$. Following the notation in Section 1, we can identify W with this subspace; analogously we can identify Z with the subspace $N(L^*)$ of $L_p(\Omega)$, where $N(L^*)$ is the kernel of the adjoint L^* of L.

Consider now the multivalued map F defined by $F(u) = \{z:\Omega \to \mathbf{R}, z \text{ measurable}; z(x) \in f(x, u(x)) \text{ a.e. in } \Omega\}$.

The following lemma state some properties of the multivalued map F which will be used in the sequel.

LEMMA 2.1. Let $p \in [1, +\infty)$ and let f satisfy (f1). Then:

(i) $F(u)$ is a nonempty, convex, closed, and bounded subset of $L_p(\Omega)$ for all $u \in L_p(\Omega)$;

(ii) if $u_n \to u$ in $L_p(\Omega)$ and if $z_n \in F(u_n)$ then $\lim_{n\to\infty} \operatorname{dist}(z_n, F(u)) = 0$ (where $\operatorname{dist}(z, F(u)) = \inf \|z - y\|_{L_p(\Omega)}: y \in F(u))$;

(iii) F is uniformly bounded.

For the proof see [8].

Let us rewrite problem (2.1) in the form of finding a solution

$$u = v + w \in N(L) \oplus W$$

such that

$$\begin{aligned} &w \in HQF(v + w), \\ &0 \in (I - Q)\, F(v + w). \end{aligned} \tag{2.2}$$

Let $p > 1$ from Lemma 2.1 and the reflexivity of $L_p(\Omega)$ we have that HQF and $(I - Q)F$ have closed graph since $H: R(L) \to W$ is compact (via the Sobolev imbedding theorem of $W_p^{2m}(\Omega)$ into $L_p(\Omega)$) and $I - Q)$ is a finite dimensional range map. Moreover HQF sends bounded sets into relatively compact ones. Since the maps have convex and compact values we have that $HQF: N(L) \oplus W \multimap W$ and $(I - Q)F: N(L) \oplus W \multimap Z$ are u.s.c. and compact.

To solve problem (2.2), we shall show that Theorem 1.1 can be applied under some additional conditions on f.

Indeed assume that $(\mathscr{L}, \mathscr{B})$ has 'the unique continuation property', that is

$$\text{if } v \in N(L) \text{ vanishes on a set of positive measure, then } v = 0 \text{ in } \Omega. \tag{UC}$$

Moreover for any $u \in \mathbf{R}$ let $s_u(x) = \{\sup y : y \in f(x, u)\}$ and $m_u(x) = \{\inf y : y \in f(x, u)\}$.

Since f is a closed map, by the Castaing representation we have that s_u and m_u are measurable functions for every $u \in \mathbf{R}$ (see, e.g. [13]).

We denote by

$$s_\pm(x) = \limsup_{u \to \pm\infty} s_u(x), \qquad m_\pm(x) = \liminf_{u \to \pm\infty} m_u(x),$$

and we assume that $s_\pm$ and $m_\pm$ are measurable functions.

Remark 2.1. Notice that the above assumption is satisfied if:

(i) f is a single-valued map;

(ii) $f(x, u) = g(u) - h(x)$ with $h \in L_p(\Omega)$ and $g: \mathbf{R} \multimap \mathbf{R}$ verifies (f1);

(iii) there exists a multivalued map $f: \bar{\Omega} \to R$ with compact and convex values such that

$$\lim_{u \to \pm\infty} \delta(f(x, u), f(x)) = 0$$

uniformly with respect to x, where δ denote the Hausdorff distance. In this case

$$s_\pm(x) = \lim_{u \to \pm\infty} s_u(x); \qquad m_\pm(x) = \lim_{u \to \pm\infty} m_u(x).$$

If $T: Z \to N(L)$ is a linear injective operator, $z^+(x) = \max\{z(x), 0\}$ and $z^-(x) = \min\{z(x), 0\}$, we denote by

$$\underline{M}_T(z) = \int_{Tz > 0} m_+ z^+ + s_+ z^- + \int_{Tz < 0} m_- z^+ + s_- z^-$$

and by

$$\overline{M}_T(z) = \int_{Tz > 0} s_+ z^+ + m_+ z^- + \int_{Tz < 0} s_- z^+ + m_- z^-.$$

Then the following result can be stated.

THEOREM 2.2. Suppose that the assumptions (fl) and (UC) are satisfied. Moreover assume that there exists a linear operator $T: Z \to N(L)$ such that $\underline{M}_T(z) > 0$ for all $z \in Z \setminus \{0\}$ or $\overline{M}_T(z) < 0$ for all $z \in Z\backslash\{0\}$. Then for every $\xi \in N(L)$ with $\int_\Omega \xi Tz = 0$, for all $z \in Z\backslash\{0\}$, there exists a solution u of (2.2) of the form $u = Tz + \xi + u$, with $z \in Z$ and $w \in W$.

Proof. Without loss of generality we can assume that $p = 2$.

Let us discuss the case $\underline{M}_T(z) > 0$ for all $z \in Z\backslash\{0\}$. Let $\xi \in N(L)$ with $\int \xi Tz = 0$, $\forall z \in Z$.

By the continuity of T it follows that the maps $(z, w) \multimap HQF(Tz + \xi + w)$ from $Z \times W$ into W and $(z, w) \mapsto\!\!\circ\, (I - Q)F(Tz + \xi + w)$ from $Z \times W$ into Z are u.s.c., compact and uniformly bounded. Hence $(L.F)_{(i)}$ is satisfied. Moreover the uniform boundedness of f and the L_p-estimates for linear elliptic equations, imply the existence of a constant $K > 0$ such that $\|w\|_{W_2^{2m}} \leqslant K$ for all $w \in \lambda\, HQF(Tz + \xi + w)$, $\lambda \in [0, 1]$, and so $(L.F)_{(ii)}$ is satisfied with $c_1 = K$.

To prove $(L.F)_{(iii)}$ assume the contrary, i.e. there exist sequences $\{r_n\} \subset R$, $\{z_n\} \subset Z$, $\{w_n\} \subset W$ such that

$$\lim_{n\to\infty} r_n = +\infty, \quad \|z_n\|_{L_2} = 1, \quad \|w_n\|_{W_2^{2m}} < c_1 \quad \text{and}$$

$$0 \in (I - Q)\, F(Tr_n z_n + \xi + w_n), \qquad \forall n \in \mathbf{N}$$

Therefore there exists a uniformly bounded sequence $\{y_n\}$ of measurable functions with $y_n(x) \in f(x, Tr_n z_n(x) + \xi(x) + w_n(x))$ a.e. in Ω such that $(I - Q)y_n = 0$. This implies that $\langle (I - Q)u_n, z_n \rangle = 0$ for every $n \in \mathbf{N}$; hence

$$\int_\Omega y_n z_n = 0 \qquad \text{for every } n \in \mathbf{N} \tag{2.3}$$

Since Z is finite dimensional we can assume that $z_n \to \bar{z}$ in $L_2(\bar{\Omega})$. Moreover by the compact imbedding of $W_2^{2m}(\Omega)$ in $L_2(\Omega)$ we can assume that $w_n \to w$ in $L_2(\Omega)$.

Hence, by passing to a subsequence, if necessary, we can assume that

$$Tz_n + (\xi + w_n)/r_n \to T\bar{z} \qquad \text{a.e.} \quad \text{in } \Omega.$$

Since $\|\bar{z}\|_{L_2(\Omega)} = 1$, we have that $T\bar{z} \in N(L)\backslash\{0\}$. Thus for almost all $x \in \Omega$ with $T\bar{z}(x) > 0$ it results that

$$\lim_{n\to\infty} (Tr_n z_n(x) + \xi(x) + w_n(x)) = +\infty;$$

while for almost all $x \in \Omega$ with $T\bar{z}(x) < 0$ it results that

$$\lim_{n\to\infty} (Tr_n z_n(x) + \xi(x) + w_n(x)) = -\infty.$$

Set

$$\Omega_n = \{x \in \Omega : y_n(x) \notin f(x, Tr_n z_n(x) + \xi(x) + w_n(x))\}.$$

Clearly $\text{meas}(\Omega_n) = 0$, $\forall n \in \mathbf{N}$.

Define $\Omega' = \Omega\backslash \bigcup \Omega_n$; let

$$\Omega_+ = \{x \in \Omega' : T\bar{z}(x) > 0\} \quad \text{and} \quad \Omega_- = \{x \in \Omega' : T\bar{z}(x) < 0\}$$

Notice that (UC) implies that $\text{meas}(\Omega\backslash(\Omega_+ \bigcup \Omega_-)) = 0$.

From (2.3) it follows that

$$0 = \liminf_{n\to\infty} \int_\Omega y_n z_n \geqslant \liminf_{n\to\infty} \int_{\Omega_+} y_n z_n + \liminf_{n\to\infty} \int_{\Omega_-} y_n z_n \tag{2.4}$$

Since $\{y_n\}$ is bounded and $z_n \to \bar{z}$ in $L_2(\Omega)$, if we denote by

$$\Omega_0 = \{x \in \Omega : \bar{z}(x) = 0\} \quad \text{we have that} \quad \lim_{n\to\infty} \int_{\Omega_0} y_n z_n = 0.$$

Hence by Fatou's Lemma and (2.4) we obtain

$$0 \geqslant \int_{\Omega_+} m_+ \bar{z}^+ + s_+ \bar{z}^- + \int_{\Omega_-} m_- \bar{z}^+ + s_- \bar{z}^-,$$

contradicting the hypotheses on $\underline{M}_T(z)$.

It remain to show that for some $r > c_1$, $\deg((I - Q)\, F(Tz + \xi), B_r, 0)$ is different from zero. In fact the following homotopy

$$(\lambda, z) \multimap (1 - \lambda)\, z + \lambda(I - Q)\, F(Tz + \xi), \qquad \lambda \in [0, 1]$$

is an admissible homotopy in the sense of Ma (see [10, p. 10]).

Assume the contrary. Then there exist sequences

$$\{r_n\} \subset \mathbf{R} \quad \text{with} \quad \lim_{n\to\infty} r_n = +\infty, \quad \{z_n\} \subset Z, \quad \|z_n\|_{L_2(\Omega)} = 1 \quad \text{and} \quad \{\lambda_n\} \subset [0, 1]$$

such that $0 \in (1 - \lambda_n)z_n + \lambda_n(I - Q)\, F(Tr_n z + \xi)$ for all $n \in \mathbf{N}$. Therefore there exists a sequence $\{y^n\}$, $y_n(x) \in f(x, Tr_n z_n(x) + \xi(x))$ for a.a. $x \in \Omega$, such that $(1 - \lambda_n)z_n + \lambda_n(I - Q)y_n = 0$.

Hence $\langle (1 - \lambda_n)z_n + \lambda_n(I - Q)y_n, z_n \rangle = 0$. This implies that

$$(1 - \lambda_n) \int_\Omega z_n^2 + \lambda_n \int_\Omega y_n z_n = 0; \qquad \text{then} \int_\Omega y_n z_n \leqslant 0$$

for all $n \in \mathbf{N}$.

Using the same argument as above, we get a contradiction with the assumption $\underline{M}_T(z) > 0$.

By the same argument it is possible to treat: $\overline{M}_T(z) < 0$ for all $z \in Z \backslash \{0\}$.

Remark 2.2. If $f(x, u) = g(u) - h(x)$ with $h \in L_p(\Omega)$ and $g : \mathbf{R} \multimap \mathbf{R}$ verifying (f1), then Theorem 2.2 contains a result due to McKenna [12] in the case where g is obtained from a discontinuous map when all the jumps are *filled in*.

2.2. *Periodic solutions for parabolic equations*

In this paragraph we denote by $\mathcal{M}$ a formally self-adjoint uniformly elliptic operator of order $2m$ with smooth coefficients.

We seek periodic solutions in time of the multivalued parabolic equation

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} + \mathcal{M}u(x,t) \in f(x,t,u(x,t)) && \text{in } \Omega \times (0,1) \\ &u(x,t) = 0 && \text{on } \partial\Omega \times (0,1) \\ &u(x,0) = u(x,1) && \text{in } \Omega \end{aligned} \tag{2.5}$$

with $f:\Omega \times (0,1) \times \mathbf{R} \multimap \mathbf{R}$ satisfying (f1) (where x is replaced by (x, t)).

THEOREM 2.3. Assume

$$\int_{\{(x,t)\in\Omega\times(0,1):v>0\}} m_+ v + \int_{\{(x,t)\in\Omega\times(0,1):v>0\}} m_- v > 0 \qquad \text{for every } v \in N(L), \quad v \neq 0.$$

Then (2.5) has a solution.

Proof. Together with homogeneus Dirichlet boundary conditions, $\mathcal{M}$ induces a self-adjoint differential operator

$$M:D(M) \subset L_2(\Omega) \to L_2(\Omega) \quad \text{where} \quad D(M) = W_2^{2m}(\Omega) \cap W_{2,0}^{m}(\Omega),$$
$$Mu = \mathcal{M}u \quad \text{for} \quad u \in D(M).$$

Following [3], set $X = Y = L_2(\Omega \times (0,1))$ *and* $Lu = (\partial u/\partial t) + Mu$ with

$$D(L) = \{u \in L_2(W_2^{2m}(\Omega) \cap W_{2,0}^{m}(\Omega), (0,1)), \partial u/\partial t \in L_2(\Omega \times (0,1)),$$
$$u(x,0) = u(x,1), \qquad x \in \Omega\}.$$

Clearly $N(L) = N(L^*) = N(M)$ (since M is self-adjoint). $R(L)$ is closed and L has a right-inverse H which is compact (e.g. [3]).

Here again we define the multivalued map $F: X \multimap X$ by $F(u) = \{z:\Omega \times (0,1) \to \mathbf{R}$, z measurable: $z(x,t) \in f(x,t,u(x,t))$ a.e. in $\Omega \times (0,1)\}$.

Hence by (f1) F has the properties as in Lemma 2.1 and problem (2.5) can be rewritten in the following form: Find $u = v + w \in N(L) \oplus R(L)$ such that:

$$\begin{aligned} &w \in HQF(v + w), \\ &0 \in (I - Q)F(v + w), \end{aligned} \tag{2.6}$$

where Q is the projection of X onto $R(L)$ parallel to $N(L)$. We may therefore proceed as in the proof of Theorem 2.2.

3. PERIODIC SOLUTIONS OF A MULTIVALUED DIFFERENTIAL EQUATION

Let $J = [0, 1]$ and $f:J \times \mathbf{R}^n \multimap \mathbf{R}^n$ satisfy condition (f2). We shall be interested in proving the existence of 1-periodic solutions for the multivalued differential equation

$$\dot{u}(t) \in f(t, u(t)) \tag{3.1}$$

over J. Solutions are understood in the Caratheodory sense, i.e. absolutely continuous functions $u: J \to \mathbf{R}^n$ verifying (3.1) for a.a. $t \in J$ together with

$$u(0) = u(1). \tag{3.2}$$

Let X be the space $AC(J, \mathbf{R}^n)$ of the absolutely continuous functions with the form $\|u\|_{AC} = \max_{t \in [0,1]} |u(t)| + \int_0^1 |\dot{u}(t)| \, dt$.

Let $Y = L_1(J, \mathbf{R}^n)$ with the norm $\|u\|_{L_1} = \int_0^1 |u(t)| \, dt$.

If we define $L: D(L) \subset Y \to Y$ and $F: Y \multimap Y$ by $D(L) = \{u \in X : u(0) = u(1)\}$, $(Lu)(t) = \dot{u}(t)$ for a.a. $t \in J$ and $F(u) = \{z: J \to \mathbf{R}^n,\ z \text{ measurable: } z(t) \in f(t, u(t)) \text{ a.e. in } J\}$, then our problem takes the abstract form

$$Lu \in F(u). \tag{3.3}$$

It is clear that L takes values in Y. Moreover, following [6] and [7], (f2) implies that F is a non-empty, closed, convex and bounded valued map from Y into itself which has closed graph. Furthermore F is uniformly bounded. By classical results on ordinary differential equations, we have that

$$N(L) = \{u \in D(L) : u(t) = u(0) \quad \text{for every } t \in J\},$$

$$R(L) = \left\{ y \in Y : \int_0^1 y = 0 \right\},$$

so that $L: D(L) \subset Y \to Y$ is Fredholm of index zero and Y can be decomposed in the topological direct sum $Y = N(L) \oplus R(L)$.

Let Q be the projection of Y onto $R(L)$ parallel to $N(L)$ defined by $(Qy)(t) = y(t) - \int_0^1 y(t) \, dt$. Therefore problem (3.3) is equivalent to the system

$$w \in HQF(c\mathbf{1} + w) \tag{3.4$'$}$$

$$0 \in (I - Q) F(c\mathbf{1} + w) \tag{3.4$''$}$$

where $\mathbf{1}(t) := 1$ for every $t \in J, c$ is a real constant, $c\mathbf{1} + w \in N(L) \oplus R(L)$ and H is the right inverse of L composed with the compact imbedding of $D(L)$ into Y. To prove that problem (3.3) has a solution, we shall apply Theorem 1.1 to (3.4)$'$–(3.4)$''$ in the particular case $X = Y$, $Z = N(L)$, $W = R(L)$, $T = I$ (the identity on $N(L)$) and $R(c, w) = (c\mathbf{1} - (I - Q) F(c\mathbf{1} + w), HQF(c\mathbf{1} + w))$. Now we are in a position of proving the following theorem.

THEOREM 3.1. Suppose that (f2) is satisfied. Moreover assume that

$$\int_0^1 m_+ > 0 \quad \text{and} \quad \int_0^1 s_- < 0.$$

Then there exists a solution of (3.3) of the form

$$u = c\mathbf{1} + w \in N(L) \oplus R(L).$$

Proof. The maps $(c, w) \multimap HQF(c\mathbf{1} + w)$ and $(c, w) \multimap (I - Q) F(c\mathbf{1} + w)$ from $N(L) \times R(L)$ into $\operatorname{Im} L$ and $N(L) \times R(L)$ into $N(L)$ respectively are u.s.c. and compact. Moreover the uniform boundedness of f implies the existence of a constant $K > 0$ such that $\|w\|_{AC} \leqslant K$ for all $w \in \lambda HQF(c\mathbf{1} + w)$, $\lambda \in [0, 1]$, and $c \in \mathbf{R}$. Therefore $(L.F)_{(i)}$ and $(L.F)_{(ii)}$, are satisfied.

To prove $(L.F)_{(iii)}$ assume the contrary. Then there exist sequences $\{r_n\} \subset \mathbf{R}$ and $\{w_n\} \subset W$ such that $\lim\limits_{n\to\infty} |r_n| = +\infty$ and $\|w_n\| \leqslant c_1$ with $0 \in (I - Q)F(r_n\mathbf{1} + w_n)$ for all $n \in \mathbf{N}$. Therefore there exists a sequence $\{y_n\}$ with $y_n(t) \in f(t, r_n + w_n(t))$ for a.a. $t \in J$ such that $(I - Q)y_n = 0$, that is

$$\int_0^1 y_n = 0 \quad \text{for all} \quad n \in \mathbf{N}. \tag{3.5}$$

Suppose now that $\lim\limits_{n\to\infty} r_n = +\infty$; then $\lim\limits_{n\to\infty} (r_n + w_n(t)) = +\infty$ for every $t \in J$. Denote by $J_n = \{t \in J : y_n(t) \notin f(t, r_n + w_n(t))\}$ and $J' = J \setminus \bigcup_n J_n$. Clearly meas $J_n = 0$ for all $n \in \mathbf{N}$.

We have $0 = \liminf\limits_{n\to\infty} \int_0^1 y_n \geqslant \int_0^1 \liminf\limits_{n\to\infty} y_n \geqslant \int_0^1 m_+$ contradicting the hypotheses. Analogously for the case where $\lim\limits_{n\to\infty} r_n = -\infty$.

Hence there exists a positive constant c_2 such that $0 \notin (I - Q)\, F(c\mathbf{1} + w)$ for all $\|w\| \leqslant c_1$ and $|c| > c_2$.

Finally, we prove that for some $c > c_2$ we have that $\deg((I - Q)\, F(c\mathbf{1}), B_c, 0)$ is different from zero. In fact the map

$$(\lambda, c\mathbf{1}) \multimap (1 - \lambda)c\mathbf{1} + \lambda(I - Q)\, F(c\mathbf{1})$$

is an admissible homotopy in the sense of Ma [10].

Assume the contrary; then there exist sequences $\{r_n\} \subset \mathbf{R}$ with $\lim\limits_{n\to\infty}|r_n| = +\infty$ and $\{\lambda_n\} \subset [0,1]$ such that

$$0 \in (1 - \lambda_n)r_n + \lambda_n(1 - Q)\, F(r_n\mathbf{1}) \quad \text{for all} \quad n \in \mathbf{N}.$$

Hence there exists a sequence $\{y_n\}$ with $y_n(x) \in f(x, r_n)$ for a.a. $t \in J$ such that $(1 - \lambda_n)r_n + \lambda_n(I - Q)y_n(t) = 0$ for all $n \in \mathbf{N}$. This implies that

$$(1 - \lambda_n)\int_0^1 r_n + \lambda_n \int_0^1 y_n = 0$$

Therefore if $\lim\limits_{n\to\infty} r_n = +\infty$, for n sufficiently large we have that $\int_0^1 y_n \leqslant 0$ and, using the same argument as above, we get a contradiction. Analogously if $\lim\limits_{n\to\infty} r_n = -\infty$.

Remark 3.1. The same result can be obtained with $\int_0^1 s_+ < 0$ and $\int_0^1 m_- > 0$.

Remark 3.2. If f is a single-valued continuous map and there exist $\lim\limits_{t\to+\infty} f(x, t)$ and $\lim\limits_{t\to-\infty} f(x, t)$, our result is contained in [11].

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