LINEAR TRACKING PROBLEMS BY A SLIDING MANIFOLD APPROACH

A. Cavallo G. De Maria
Dipartimento di Informatica e Sistemistica
Università degli Studi di Napoli
via Claudio 21 - 80125 Napoli
Italy
e-mail: DE_MARIA@DISNA.CRIAI.IT

P. Nistri
Dipartimento di Sistemi e Informatica
Università degli Studi di Firenze
via di S. Marta 3 - 50139 Firenze
Italy
e-mail: PNISTRI@IFIIDG.BITNET

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1 Introduction

In this paper we consider a linear, time-invariant, completely controllable system of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \tag{1.1}$$

where the state $\boldsymbol{x} \in \mathbf{R}^n$ and the control $\boldsymbol{u} \in \mathbf{R}^m$. A control technique for system (1.1) which guarantees robust performances and insensitivity with respect to disturbances and parameter variations is that based on high gain feedback controls.
In [5] and [9] different control problems for system (1.1), as transmission of zeros, pole placement, cheap controls and sliding modes, have been treated in an unified way via the previous technique. Roughly speaking, the method consists in introducing the high-gain feedback $\boldsymbol{u} = \boldsymbol{H}\boldsymbol{x}/\varepsilon$, where $\boldsymbol{H}$ is an $m \times n$ constant matrix, in the control system obtaining consequently in the state dynamics two different time scale, that is a fast and a slow mode. The use of the theory of singularly perturbed systems permits to unify the approach to treat different control problems.
One of the main drawbacks of this method is the peaking phenomenon, namely the state or one part of it peaks to very large values. The phenomenon occurs, for instance, when the high-gain feedback control is used to produce eigenvalues with very negative real parts. An interesting study of this phenomenon and its consequences on the global stabilization for a class of cascade systems is carried out in [6].
In this paper we propose a different approach to deal with a tracking problem for system (1.1). It is based on the definition, for a given initial state $\boldsymbol{x}_0 \in \mathbf{R}^n$, of a suitable sliding manifold and on the use of the theory of singularly perturbed systems. More specifically, given $\boldsymbol{x}_0 \in \mathbf{R}^n$, by means of a suitable function, which depends only on the data of the considered problem, we define a sliding manifold and we introduce a dynamical feedback control which turns out to be the solution of a differential equation containing a small parameter $\varepsilon > 0$, the control after a fast transient, depending only on its initial value, in which it approaches the well-defined *equivalent control* remains close to the latter in the uniform topology. This approach has been introduced in [4] with the purpose of eliminating the chattering phenomenon in variable structure control systems. Then it has been furtherly investigated and successfully employed in [2] to control rigid robotic manipulator. The assumptions under which we solve our tracking problem, even if expressed in a different way, turn out to be equivalent to that of [5] and [9]. We would like to point out that, using this approach, the only fast dynamics is that of the control law, while all the state variables follow the slow dynamics. Furthermore, the relationship between the sliding mode obtained for $\varepsilon = 0$ and the states corresponding to $\varepsilon > 0$, which is prescribed by the singular perturbation theory, turns out to be different from that presented in [9], in fact in this case any state close to the sliding mode corresponds to some control law discontinuous along the sliding manifold.

2 The tracking problem

Consider the linear, time invariant dynamic system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u},\ \boldsymbol{x} \in \mathbf{R}^n,\ \boldsymbol{u} \in \mathbf{R}^m. \tag{2.1}$$

Assume that $\boldsymbol{B}$ is a matrix of full rank m. Therefore, there is a nonsingular $n \times n$ matrix $\boldsymbol{T}$ such that

$$\boldsymbol{T}\boldsymbol{B} = \begin{pmatrix} 0 \\ \boldsymbol{B}_2 \end{pmatrix};\ \boldsymbol{B}_2 \in \mathbf{R}^{m \times m} \tag{2.2}$$

Furthermore, if we put

$$\boldsymbol{z} = \boldsymbol{T}\boldsymbol{x} \tag{2.3}$$

then we can rewrite system (2.1) in the form

$$\dot{\boldsymbol{z}} = \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{pmatrix} \boldsymbol{z} + \begin{pmatrix} 0 \\ \boldsymbol{B}_2 \end{pmatrix} \boldsymbol{u} \tag{2.4}$$

where

$$A_{11} \in \mathbf{R}^{(n-m)\times(n-m)}; \; A_{22} \in \mathbf{R}^{m\times m}$$

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}; \; z_1 \in \mathbf{R}^{n-m}, \; z_2 \in \mathbf{R}^{m}. \tag{2.5}$$

The following is a well known result of linear system theory [5].

Lemma 2.1 *If system (2.1) is completely controllable then the pair* (A_{11}, A_{12}) *is completely controllable.*

Let $\hat{z}(t)$ be a desired state trajectory differentiable and bounded in $[0,\infty)$ which the state $z(t)$ of system (2.4), corresponding to a given initial condition z_0, is required to follow in $[0,\infty)$. To solve this tracking problem we propose a sliding manifold approach based on the singular perturbation theory. For this, define a function $s : \mathbf{R}^n \times [0,\infty) \to \mathbf{R}^m$ as follows

$$s(z,t) = H\left(\hat{z}(t) - z - e^{\mathbf{C}t}(\hat{z}_0 - z_0)\right), \tag{2.6}$$

where $\hat{z}_0 = \hat{z}(0)$ and $z_0 = z(0), H = (H_1 \quad H_2)$ with H_1, H_2 $m\times(n-m)$ and $m\times m$ matrices respectively and $\mathbf{C}$ is a $n \times n$ symmetric matrix. All these matrices will be chosen later in a suitable way. Observe that $s(z_0, 0) = \mathbf{0}$. Define the related sliding manifold $\mathcal{S}$ as follows

$$\mathcal{S} = \{(z,t) \in \mathbf{R}^n \times \mathbf{R}_+ : s(z,t) = \mathbf{0}\}. \tag{2.7}$$

Thus $(z_0, 0) \in \mathcal{S}$.
For any $\varepsilon > 0$, we form the system of differential equations

$$\dot{z} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} z + \begin{pmatrix} \mathbf{0} \\ B_2 \end{pmatrix} u \tag{2.8}$$

$$\varepsilon\dot{u} = (H_1 \quad H_2)\left(\dot{\hat{z}} - \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} z - \begin{pmatrix} \mathbf{0} \\ B_2 \end{pmatrix} u - \mathbf{C}e^{\mathbf{C}t}(\hat{z}_0 - z_0)\right) \tag{2.9}$$

We have the following

Theorem 2.1 *Let δ, β and γ be given positive numbers. Assume that*

(*i*) $Re\lambda(H_2 B_2) \geq \beta$;

(*ii*) $Re\lambda(A_{12}H_2^{-1}H_1 - A_{11}) \geq \beta + \gamma$;

(*iii*) $-\lambda_{\max}(\mathbf{C}) \geq \beta$.

Moreover, assume that the following matching condition is satisfied

(*iv*) $\dot{\hat{z}}_1 = A_{11}\hat{z}_1 + A_{12}\hat{z}_2$.

Then there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0]$, the solution $(z(t,\varepsilon), u(t,\varepsilon))$ to (2.8) – (2.9) satisfying $(z(0,\varepsilon), u(0,\varepsilon)) = (z_0, u_0)$ whenever $u_0 \in \mathbf{R}^m$ is such that

$$|\, \hat{z}(t) - z(t,\varepsilon) \,| \leq \delta + a_1 e^{\lambda_{\max}(\mathbf{C}t)} + a_2 e^{-\beta t} \tag{2.10}$$

$$u(t,\varepsilon) = \frac{1}{\varepsilon} H\left(\hat{z}(t) - z(t,\varepsilon) - e^{\mathbf{C}t}(\hat{z}_0 - z_0)\right) + u_0 \tag{2.11}$$

with $t \in [0,\infty)$ and a_1, a_2 positive constants depending on the data.

Proof. Assumption (*i*) guarantees that the algebraic equation

$$\mathbf{g}(z,u,t) := (H_1 \quad H_2)\left(\dot{\hat{z}} - \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} z - \begin{pmatrix} \mathbf{0} \\ B_2 \end{pmatrix} u - \mathbf{C}e^{\mathbf{C}t}(\hat{z}_0 - z_0)\right) = \mathbf{0} \tag{2.12}$$

has an unique solution $u^*(z,t)$ for any $(z,t) \in \mathbf{R}^n \times [0,\infty)$ called the *equivalent control.*
Furthermore, for any $(z,t) \in \mathbf{R}^n \times [0,\infty)$ the equilibrium point $u^*(z,t)$ of (2.9) turns out to be globally exponentially stable. In other words, the solution $\mathbf{v} = \mathbf{v}(\tau)$ of the equation

$$\begin{aligned} \dot{\mathbf{v}} &= \mathbf{g}(z, \mathbf{v}, t) \\ \mathbf{v}(0) &= \mathbf{v}_0, \end{aligned} \tag{2.13}$$

whenever $\mathbf{v}_0 \in \mathbf{R}^m$, tends exponentially to $u^*(z,t)$ as $\tau \to \infty$ and assumption (*i*) assures that the exponential stability is uniform in $\mathbf{R}^n \times [0,\infty)$.
Let $(\tilde{z}(t), \tilde{u}(t))$ be the solution of system (2.8)–(2.9) corresponding to $\varepsilon = 0$. Define

$$\tilde{e}(t) = \hat{z}(t) - \tilde{z}(t). \tag{2.14}$$

We can easily prove that $\tilde{e} = 0$ is an equilibrim point of the equation

$$\dot{\tilde{e}} = \phi(t, \tilde{e}) \tag{2.15}$$

and is exponentially stable. Here ϕ is given by

$$\phi(t,\tilde{e}) = \dot{\hat{z}}(t) - A(\hat{z}(t) - \tilde{e}) + Bu^*(t, \hat{z}(t) - \tilde{e}) \tag{2.16}$$

Therefore all the assumptions of Theorem 8.4 of [8] are satisfied for the pair $(\hat{z}(t) - z(t,\varepsilon), u(t,\varepsilon))$ and then we have that the solution $(z(t,\varepsilon), u(t,\varepsilon))$ of (2.8)–(2.9) with $(z(0,\varepsilon), u(0,\varepsilon)) = (z_0, u_0)$ is such that

$$\begin{aligned} \lim_{\varepsilon\to 0} z(t,\varepsilon) &= \tilde{z}(t) \quad \text{uniformly in } [0,\infty) \\ \lim_{\varepsilon\to 0} u(t,\varepsilon) &= \tilde{u}(t) \quad \text{uniformly in } [t_1,\infty) \end{aligned} \tag{2.17}$$

whenever $t_1 > 0$. Here $(\tilde{z}(t), \tilde{u}(t))$ is the solution of the reduced system (2.8) – (2.12), with $\tilde{u}(t) = u^*(t, \tilde{z}(t))$, $t \in [0,\infty)$.
Hence, given $\delta > 0$ there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0]$ we have that

$$|\, z(t,\varepsilon) - \tilde{z}(t) \,| \leq \delta \tag{2.18}$$

for any $t \in [0, \infty)$.
Now we want to estimate $| \tilde{z}(t) - \hat{z}(t) |$, $t \in [0, \infty)$.
Equation (2.12) is equivalent to

$$0 = \frac{d}{dt} H \left(\hat{z}(t) - \tilde{z}(t) - e^{\mathbf{C}t}(\tilde{z}_0 - z_0) \right). \tag{2.19}$$

Since for $t = 0$ the term in the bracket vanishes, we have that

$$H \left(\hat{z}(t) - \tilde{z}(t) - e^{\mathbf{C}t}(\hat{z}_0 - z_0) \right) = \mathbf{0},\ t \in [0, \infty), \tag{2.20}$$

or equivalently

$$H_1 (\hat{z}_1(t) - \tilde{z}_1(t)) + H_2 (\hat{z}_2(t) - \tilde{z}_2(t)) = H e^{\mathbf{C}t}(\hat{z}_0 - z_0), \tag{2.21}$$

for any $t \in [0, \infty)$.
Using the matching condition (iv) we obtain

$$\begin{aligned} \dot{\tilde{z}}_1 - \dot{\hat{z}}_1 &= (A_{11} - A_{12} H_2^{-1} H_1)(\tilde{z}_1(t) - \hat{z}_1(t)) - \\ & A_{12} H_2^{-1} H e^{\mathbf{C}t}(\hat{z}_0 - z_0). \end{aligned} \tag{2.22}$$

Therefore by assumptions (ii) and (iii) we get

$$| \tilde{z}_1(t) - \hat{z}_1(t) | \leq L e^{-\beta t} \left(| \tilde{z}_1(0) - \hat{z}_1(0) | + \frac{\| A_{12} H_2^{-1} H_1 \| \, | \hat{z}_0 - z_0 |}{\gamma} \right). \tag{2.23}$$

Here we use the estimate

$$\| e^{\mathbf{Q}t} \| \leq L e^{(-a+\gamma)t} \tag{2.24}$$

where $Re\lambda(\mathbf{Q}) \leq -a$, $L = L(\gamma)$ and $\gamma > 0$ is sufficiently small ([3], proposition 4 p.4).
From (2.21) and (2.23) and assumptions (ii) and (iii) we obtain

$$\begin{aligned} | z(t, \varepsilon) - \hat{z}(t) | \leq \delta + L \left(1 + \| H_2^{-1} H_1 \|\right) \\ \left(| \tilde{z}_1(0) - \hat{z}(0) | + \frac{\| A_{12} H_2^{-1} H_1 \| \, | \hat{z}_0 - z_0 |}{\gamma} \right) e^{\beta t} \\ + \| H_2^{-1} H \| \, | \hat{z}_0 - z_0 | \, e^{\lambda_{max}(\mathbf{C}t)} \end{aligned} \tag{2.25}$$

$t \in [0, \infty)$, which is the assertion. ∎

Remark 2.1 Observe that, in virtue of Lemma 2.1 it is possible to choose H_1 in such a way that assumption (ii) is satisfied. Furthermore, since $\det(B_2) \neq 0$ we can also choose H_2 to satisfy assumption (i).

Remark 2.2 Since the eigenvalues of the matrices $H_2 B_2$ and $B_2 H_2$ are the same, our assumptions $(i) - (ii)$ are equivalent to the assumptions of [5] and [9]. In these papers system (2.4) was controlled by the high gain feedback control

$$u = \frac{1}{\varepsilon} H x, \quad \varepsilon > 0. \tag{2.26}$$

The substitution of this control in (2.4) produces a two time scale system for which, under assumptions $(i) - (ii)$ the singular perturbation theory applies. This allows to solve several control problems for system (2.4).
We want to point out that, even if the assumptions are the same, our approach is different from that based on the high-gain feedback controls (2.26). In fact, in our case the state and the control do not present, in general, the peaking phenomenon.
Indeed, for $\varepsilon > 0$ sufficiently small, the state is confined in a prescribed neighbourhood of the manifold $\mathcal{S}$ for any time $t > 0$ and the control, except for a very fast transient depending on the initial value u_0, remains whithin a neighbourhood of the equivalent control.

Remark 2.3 If the reference trajectory is $\hat{z}(t) \equiv \mathbf{0}$, then the tracking problem reduces to the regulation of system (2.1) up to a prescribed error $\delta > 0$. Observe that in this case the matching condition (iv) is satisfied.
As a special application of the theory presented we consider the problem of bringing the state of an LTI system from an arbitrary initial condition to zero minimizing an LQ performance index. Due to space limitations this will be illustrated via an example.

Example 2.1. Consider the simplified model of a position servo system ([10], Example 2.4)

$$\begin{aligned} \dot{z}(t) &= \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} z(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix} u(t) \\ &= A z(t) + \mathbf{b} u(t),\ z(0) = z_0 \end{aligned} \tag{2.27}$$

where $z(t) = (\theta(t), \dot{\theta}(t))^T$, $\alpha = \beta/J$, $\kappa = k/J$, J is the moment of inertia of all the rotating parts, β the coefficient of viscous friction, k the motor constant, $u(t)$ the input voltage to the motor.
The nominal values are:

$$\alpha = 4.6 s^{-1},\ \kappa = 0.787 \text{rad}/(\text{V} \cdot s^2),\ z_0 = (0.1\ 0)^T \tag{2.28}$$

The optimization criterion is

$$\min \int_0^\infty (\theta^2(t) + \rho u^2(t)) dt \tag{2.29}$$

with $\rho = 0.00002$ rad^2/V^2. The classical LQ solution gives

$$u(t) = -\mathbf{F} z(t),\ \mathbf{F} = (223.6\ 18.69) \tag{2.30}$$

Now let

$$\mathbf{C} = A - B\mathbf{F} \tag{2.31}$$

and the related function $s(t, z)$ (2.6)

$$s(z, t) = H(\exp(\mathbf{C}t) z_0 - z) \tag{2.32}$$

It is easy to verify that assumption (iv) of Theorem 2.1 is satisfied, then we have just to choose the matrix H as to fullfill assumptions (i) and (ii). A possible selection is $H = (3 \quad 1)$.

In Fig. 1 the behaviour of the angular position and the input voltage are depicted for the two cases $\varepsilon = 0.01$ and $\varepsilon = 0.002$, and compared to the classical LQ solution. Note that the smaller ε is the quicker is the transient bringing the control close to the *equivalent control* (a). The main feature of the proposed approach is its ability to reject disturbances and to compensate for parameter variations. Then the combined inertia of load and armature of the motor has been changed to 2/3 and 3/2 of its nominal value and further simulations have been carried out. In Fig. 2 the effect of the perturbation on $\theta(t)$ and $u(t)$ are shown for the LQR design, while the behaviour of the same variables with the sliding manifold approach is illustrated in Fig. 3. Note that in the latter case the effect of the parameter variations is utterly absorbed by the control, while the variation of the state is practically negligible. Finally, as one can easily verify, the performance index retains the same values in the two approaches.

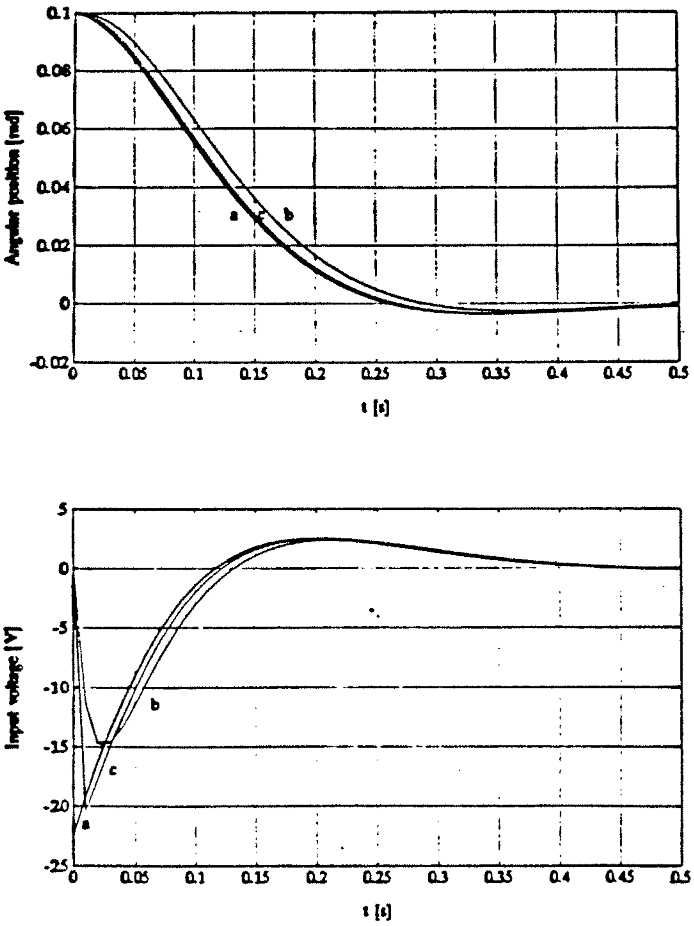

Figure 1: (a) LQR design; (b) sliding manifold approach, $\varepsilon = 0.01$; (c) $\varepsilon = 0.002$.

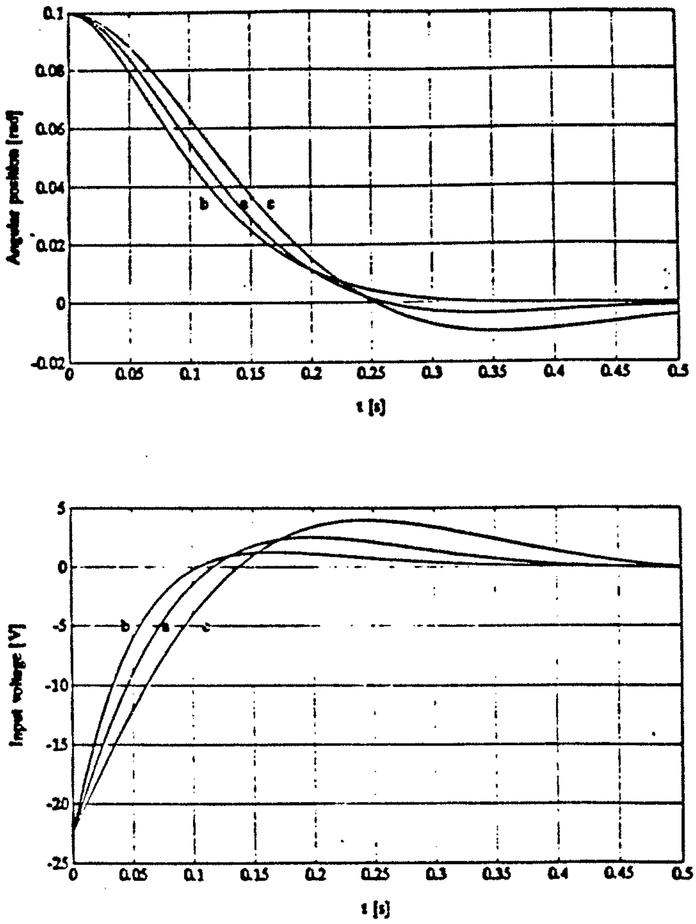

Figure 2: LQR design: effect of parameter variation; (a) nominal load; (b) inertial load 2/3 of nominal; (c) inertial load 3/2 of nominal.

3 Approximability property

We conclude by introducing the definition of approximability property.
To formulate it, given $M \in \mathbf{L}^1([0,\infty), \mathbf{R}^+)$, let $\mathcal{H}_1$ be the set of all one–parameter families $\{\mathbf{p}_\eta : \eta > 0\}$ of functions $\mathbf{p}_\eta \in \mathbf{L}^1([0,\infty), \mathbf{R}^m)$, $\eta > 0$ such that

$$|\, \mathbf{p}_\eta(t) \,| \leq M(t) \tag{3.1}$$

and

$$\sup \left\{ \left| \int_0^t \mathbf{p}_\eta(s) ds \right| : t \in [0,\infty) \right\} \to 0 \text{ as } \eta \to 0_+ . \tag{3.2}$$

We can now give the following

Definition 3.1. We say that system (2.8) – (2.9) fulfills the *approximability property* if and only if the following conditions hold
(a) the equivalent control is well defined;
(b) the set $\mathcal{H}_1$ is not empty;
(c) if $\{\mathbf{p}_\eta\} \subset \mathcal{H}_1$, if $\varepsilon > 0$, $\eta > 0$ are given numbers, if

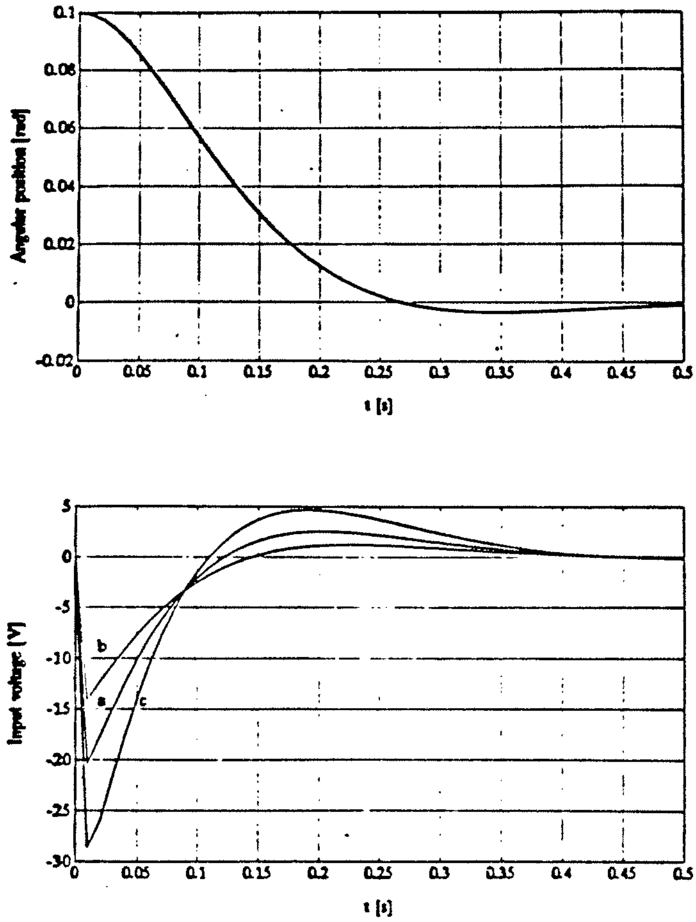

Figure 3: Sliding manifold approach: effect of parameter variation; (a) nominal load; (b) inertial load 2/3 of nominal; (c) inertial load 3/2 of nominal.

$(x(t,\varepsilon,\eta),\, u(t,\varepsilon,\eta))$ is a solution of the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} x + \begin{pmatrix} 0 \\ B_2 \end{pmatrix} u \\ \varepsilon \dot{u} &= \mathrm{g}(x,u,t) + \mathrm{p}_\eta(t) \end{aligned} \tag{3.3}$$

such that $s(x(0,\varepsilon,\eta),0) \to 0$ as $\eta \to 0_+$ and if $\tilde{x}(t)$ is the solution of

$$\begin{aligned} \dot{x} &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} x + \begin{pmatrix} 0 \\ B_2 \end{pmatrix} u \\ 0 &= \mathrm{g}(x,u,t) \end{aligned} \tag{3.4}$$

satisfying $s(\tilde{x}(0),0) = 0$, then the condition $\lim_{\eta\to 0} x(0,\varepsilon,\eta) = \tilde{x}(0)$ implies that

$$\lim_{\varepsilon\to 0} \lim_{\eta\to 0} x(t,\varepsilon,\eta) = \tilde{x}(t) \tag{3.5}$$

uniformly in $[0,\infty)$.

It is a matter of fact that, under assumptions of Theorem 2.1, system (2.8) – (2.9) fulfills the *approximability property* and so it is robust with respect to the class of perturbations $\mathcal{H}_1$. Such a property has been introduced for variable structure systems by [1], [7] and by [4], [2] in this context.

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