



ACADEMIC
PRESS

J. Math. Anal. Appl. 270 (2002) 474–487

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.academicpress.com

On open-loop and feedback attainability of a closed set for nonlinear control systems [☆]

Paolo Nistri ^a and Marc Quincampoix ^{b,*}

^a *Dipartimento di Ingegneria dell'Informazione, University of Siena, via Roma 56, 53100 Siena, Italy*

^b *Département de Mathématiques, Université de Bretagne Occidentale, 6, avenue Victor le Gorgeu, BP 809, F-29285 Brest cedex, France*

Received 2 July 2001

Submitted by J.A. Ball

Abstract

In this note, we investigate the existence of controls which allow to reach a given closed set K through trajectories of a nonlinear control system. In the case where the set is sufficiently regular we give a condition allowing to find a feedback control law which ensures the existence of trajectories to reach the set. We also consider the case where all the trajectories reach K . When K is not necessarily attainable but only viable, we build a set-valued feedback for which the set is invariant. Our approach concerns continuous dynamics, possibly not C^1 , so our methods do not come from geometric control theory. Furthermore, we do not require any regularity of the set K in order to obtain our results, except when we want to establish the existence of a feedback control law to achieve our goals. © 2002 Elsevier Science (USA). All rights reserved.

Keywords: Attainability; Viability; Minimal time; Feedback control

[☆] Research partially supported by the short-term mobility program of the CNR (Italy) and the research project “Qualitative Theory and Control of Dynamical Systems” of the University of Siena.

* Corresponding author.

E-mail addresses: pnistri@dii.unisi.it (P. Nistri), marc.quincampoix@univ-brest.fr (M. Quincampoix).

1. Introduction

Throughout the paper we consider a control system

$$x'(t) = f(x(t), u(t)), \quad \text{for almost all } t \geq 0, \tag{1}$$

with initial condition

$$x(0) = x_0, \tag{2}$$

where $f : \mathbf{R}^n \times U \mapsto \mathbf{R}^n$ is continuous with linear growth, and U is a compact subset of \mathbf{R}^l .

Moreover, we assume

$$\forall x \in \mathbf{R}^n, \quad f(x, U) \text{ is a nonempty convex compact set.} \tag{3}$$

Let $K \subset \mathbf{R}^n$ be a nonempty closed set. The main question we address is the open-loop attainability of the set K by trajectories of (1). Namely, we pose the following question:

- Does there exist a neighborhood $I := K + B(0, r)$ of K such that for any point $x_0 \in I \setminus K$, there exists a measurable control such that a corresponding solution to (1) reaches the set K in finite small time?

More precisely, for any $x \in \partial K$ and $T > 0$, does there exist some $r > 0$ such that from any point $x_0 \in B(x, r) \setminus K$ starts a trajectory of (1), (2) reaching K in a time less than or equal to T ?

When the control is derived from a feedback law, the above property will be called *feedback attainability*.

Before answering this question, let us give two important consequences of a possible positive answer to the previous question:

- C1** The set K is viable for the control system (1). Namely, starting from any initial condition $x_0 \in K$ there exists a solution $x(\cdot)$ to (1) remaining in K ; that is:

$$\forall t \in [0, +\infty), \quad x(t) \in K. \tag{4}$$

- C2** The minimal time function

$$\Theta(x_0) := \inf \left\{ t \geq 0, \text{ there exists a solution } x(\cdot) \text{ to (1), (2) such that } x(t) \in K \right\}$$

takes finite values when $x_0 \in I$. Let us remark that in general Θ is only lower semicontinuous (cf. [1]).

The problem of attainability, or equivalently controllability for the backward dynamics, has been mainly studied in the case when K is a singleton (cf., for instance, [2]).

The set-attainability has been studied in [3–5] using various expansions of the trajectories of the dynamics. This approach requires the smoothness (at least C^1) of the dynamics (and also of the set for [3] and [5]) and it allows to prove the Hölder continuity of the minimal time in I .

A second type of approach for the set attainability [6–8] requires only the closedness of K and the Lipschitz continuity of the dynamics and as a consequence the Lipschitz continuity of the minimal time function can be derived.

A third approach [9] considers sufficiently regular sets, called proximate retracts, and by means of the construction of a feedback control law the set attainability can be obtained (cf. also [15]).

This approach implies also the Lipschitz continuity of the minimal time function.

Of course, each of the previous three approaches requires different sufficient conditions for obtaining the set attainability.

The main goal of the present note is to study the attainability of an *arbitrary closed set* for a continuous dynamics and to derive continuity properties of the minimal time function. Surprisingly, when neither the set nor the dynamics are smooth, our approach can cover cases where the minimal time is Hölder continuous. If the set is more regular, then our results permit us to derive the existence of a feedback control law providing the set attainability property. We shall also give sufficient conditions for the attainability of K by *all the trajectories* starting from a neighborhood of K .

When the set K is only viable, namely it satisfies **C1**, the following natural question arises:

- Is it possible to find a set-valued map $x \mapsto W(x) \subset U$ such that for any $x_0 \in K$, the set of solutions to

$$x'(t) \in f(x(t), W(x(t))), \quad x(0) = x_0, \quad t \geq 0, \quad (5)$$

coincides with the set of solutions to (1), (2) which are viable in K ?

This question was positively solved by Veliov [10] for Lipschitz dynamics by using proximal normals. Here we propose a different simple proof of this fact using viability theory.

We want also to stress that our results can be easily adapted to the case of nonautonomous dynamics.

2. Some preliminaries

We denote by d_K the Euclidean distance function to K and by $\Pi_K(x)$ the set of projection of x onto K :

$$\Pi_K(x) := \{z \in K, \|x - z\| = d_K(x)\}.$$

Under our assumptions on the dynamics f the multivalued map

$$x \mapsto F(x) := \{f(x, u), u \in U\}$$

is upper semicontinuous with nonempty compact convex values. So we can interpret the control system through the following differential inclusion:

$$x'(t) \in F(x(t)), \quad \text{for almost every } t \geq 0. \tag{6}$$

From **C1**, we know that the following condition is necessary for the attainability:

$$\forall x \in K, \quad F(x) \cap T_K(x) \neq \emptyset, \tag{7}$$

where

$$T_K(x) := \left\{ v \in \mathbf{R}^n, \liminf_{h \rightarrow 0^+} d_K(x + hv)/h = 0 \right\}$$

is the Bouligand contingent cone to K at x . Condition (7) can also be equivalently written in terms of proximal normals:

$$\forall x \in K, \forall p \in NP_K(x), \quad \min_{v \in F(x)} \langle v, p \rangle \leq 0, \tag{8}$$

where

$$NP_K(x) := \{p \in \mathbf{R}^n, d_K(x + p) = \|p\|\}$$

is the set of proximal normals (cf. [8]). Recall also that if $x \notin K$ and $z \in \Pi_K(x)$, then $x - z \in NP_K(z)$.

In fact, according to the celebrated Viability Theorem [11], the condition (7) is equivalent to the fact that the set K is viable for (1). Note that the condition (7) does not give any information for a point $x \notin K$ because in such a point $T_K(x)$ is reduced to \emptyset . So one can define the *external Bouligand cone*, still¹ denoted by $T_K(x)$, in the following way:

$$T_K(x) := \{v \in \mathbf{R}^n, D_{\uparrow}d_K(x)v \leq 0\},$$

where the contingent epiderivative (see [12]) of a function² $\phi : \mathbf{R}^n \rightarrow [0, +\infty]$ is defined by

$$D_{\uparrow}\phi(x)v = \liminf_{h \rightarrow 0^+, u \rightarrow v} \frac{\phi(x + hu) - \phi(x)}{h}.$$

Following [9], for $x \notin K$ we propose to define the following “affine external cone”³ associated with any $\gamma \geq 0$:

$$T_{K,\gamma}(x) := \{v \in \mathbf{R}^n, D_{\uparrow}d_K(x)v \leq -\gamma\}.$$

¹ Because obviously these two definitions coincide for any point $x \in K$.

² Note that for a Lipschitz function the contingent epiderivative coincide with the lower Dini derivative.

³ which is not a cone when $\gamma \neq 0$.

Also, for any $\alpha > 0$, we define

$$d_K^\alpha(x) := \inf\{\|y - x\|^\alpha, y \in K\}$$

and the set

$$T_{K,\gamma}^\alpha(x) := \{v \in \mathbf{R}^n, D_\uparrow d_K^\alpha(x)v \leq -\gamma\}$$

which reduces to $T_{K,\gamma}(x)$ when $\alpha = 1$. Note that if $\gamma > 0$ and $x \in K$, then $T_{K,\gamma}^\alpha(x) = \emptyset$, whenever $\alpha > 0$.

Recall also that a set K is said *sleek* (cf. [12, p. 101]) when the map $x \mapsto T_K(x)$ is lower semicontinuous at every point of K . In this case the contingent cone is convex.

3. Set attainability

We are now in the position to formulate our main result.

Theorem 1. *Let F be an upper semicontinuous map with nonempty compact convex values and linear growth. Let $I := K + r\hat{B}$ be an open neighborhood of the closed set $K \subset \mathbf{R}^n$. We suppose that there exists a Lipschitz positive function $\lambda = \lambda(\cdot)$ such that for any $y_0 \geq 0$ the solution $y(\cdot)$ to*

$$y'(t) = -\lambda(y(t)), \quad y(0) = y_0, \quad t \geq 0, \tag{9}$$

assumes the value 0 in a finite minimal time denoted by $\vartheta(y_0)$.

Suppose that there exists $\alpha > 0$ such that

$$\forall x \in I \setminus K, \quad F(x) \cap T_{K,\lambda(d_K^\alpha(x))}^\alpha(x) \neq \emptyset. \tag{10}$$

Then starting from any $x_0 \in I \setminus K$ there exists at least one solution to (6) reaching K in finite time. Moreover, Θ is continuous on $I \setminus K$ and

$$\Theta(x_0) \leq \vartheta(d_K^\alpha(x_0)). \tag{11}$$

Before proving this theorem, let us present some of its consequences.

Corollary 2. *Assume that F is a Lipschitz map with nonempty compact convex values and define the constant function $\lambda(\cdot) := -\delta$ with $\delta > 0$. Suppose that there exists some $\alpha > 0$ such that condition (10) holds true. Then from any $x_0 \in I \setminus K$ there exists at least a solution to (6) reaching K in finite time. Moreover, Θ is Hölder continuous on $I \setminus K$ with exponent α .*

Remark 1. In the particular case where $\alpha = 1$, we get the Lipschitz continuity of the minimal time function as obtained in [6–8].

As it can be easily seen in the following two-dimensional example with $K = \{(0, 0)\}$ and $U := [0, 1]$

$$\begin{cases} x' = -\frac{(x^2+y^2)^{1/2}}{x}(3+u), \\ y' = -\frac{(x^2+y^2)^{1/2}}{y}(3+u), \end{cases}$$

the minimal time $\Theta(x_0, y_0) = (x_0^2 + y_0^2)^{1/2}$ is Hölder continuous of exponent $1/2$.

Proof of Theorem 1. Denote by $t \mapsto y(t, y_0)$ the unique solution to (9) with initial condition $y(0) = y_0$. First, observe that because λ is positive and Lipschitz, then standard arguments based on the Gronwall inequality imply that $y_0 \mapsto \vartheta(y_0)$ is a Lipschitz function. Moreover, by [12, Proposition 6.1.4]

$$\forall x \in \mathbf{R}^n, \quad T_{\text{Epi}(d_K^\alpha)}(x, d_K^\alpha(x)) = \text{Epi}(D_\uparrow d_K^\alpha(x)),$$

where

$$\text{Epi}(\psi) := \{(x, y) \in \mathbf{R}^n \times \mathbf{R}, \psi(x) \leq y\}$$

denotes the epigraph of a function $\psi : \mathbf{R}^n \mapsto \mathbf{R} \cup \{+\infty\}$.

So condition (10) implies in particular

$$\forall x \in I, \exists v \in F(x), \quad (v, -\lambda(d_K^\alpha(x))) \in T_{\text{Epi}(d_K^\alpha)}(x, d_K^\alpha(x)). \tag{12}$$

Therefore, for any $x_0 \in I \setminus K$, according to the Viability Theorem [11, Theorem 3.3.2], there exists a solution $(x(\cdot), y(\cdot))$ to

$$\begin{cases} x'(t) \in F(x(t)), & x(0) = x_0, \\ y'(t) = -\lambda(y(t)), & y(0) = d_K^\alpha(x_0), \end{cases} \tag{13}$$

and a time $\tau > 0$ such that

$$\forall t \in [0, \tau], \quad d_K^\alpha(x(t)) \leq y(t);$$

that is, $(x(\cdot), y(\cdot))$ belongs to the epigraph of d_K^α for $t \in [0, \tau]$.

One can easily show that this solution is extendable into a maximal viable solution $(x(\cdot), y(\cdot))$ on $[0, \vartheta(d_K^\alpha(x_0))]$. Then, since $y(\vartheta(d_K^\alpha(x_0)), d_K^\alpha(x_0)) = 0$, we get

$$d_K^\alpha(x(\vartheta(d_K^\alpha(x_0)))) = 0.$$

So $x(\vartheta(d_K^\alpha(x_0))) \in K$. From this we obtain that $\Theta(x_0) \leq \vartheta(d_K^\alpha(x_0))$.

On the other hand, for any $x_0 \in I \setminus K$ there exists an optimal time control (cf. [1]). Fix x_1 and x_2 in $I \setminus K$ and suppose $\Theta(x_1) \leq \Theta(x_2)$. Denote by u_1 the optimal time control and by $x_1(\cdot)$ a corresponding trajectory, with associated $s_1 := \Theta(x_1)$, and by z the solution to the Cauchy problem

$$z'(t) = f(z(t), u_1(t)), \quad z(0) = x_2.$$

Clearly,

$$\Theta(x_2) \leq \Theta(x_1) + \Theta(z(s_1)).$$

But

$$\Theta(z(s_1)) \leq \vartheta(d_K^\alpha(z(s_1))) \leq \vartheta\{d_K(x_1(s_1)) + \|x_1(s_1) - z(s_1)\|^\alpha\}.$$

Since $d_K(x_1(s_1)) = 0$ we obtain

$$0 \leq \Theta(x_2) - \Theta(x_1) \leq \vartheta(\|x_1(s_1) - z(s_1)\|^\alpha). \tag{14}$$

Because

$$\|x_1(s_1) - z(s_1)\| \rightarrow 0 \quad \text{when } x_2 \rightarrow x_1,$$

we obtain

$$\Theta(x_2) - \Theta(x_1) \rightarrow 0 \quad \text{when } x_2 \rightarrow x_1$$

by the continuity of ϑ . Interchanging x_1 and x_2 , the proof is achieved. \square

Proof of Corollary 2. When λ is constant equal to $-\delta$, then $y(t, y_0) = y_0 - t\delta$. Equation (11) implies

$$\forall x_0 \in I \setminus K, \quad \Theta(x_0) \leq \vartheta(d_K^\alpha(x_0)) = \frac{1}{\delta} d_K^\alpha(x_0). \tag{15}$$

Then we can proceed as in the proof of Theorem 1.

Fix x_1 and x_2 in $I \setminus K$ with $\Theta(x_1) \leq \Theta(x_2)$ as above. Since F is Lipschitz, there exists a constant $C > 0$ such that

$$\|x_1(s_1) - z(s_1)\| \leq C\|x_1 - x_2\|.$$

This and Eqs. (14) and (15) imply

$$\Theta(x_2) - \Theta(x_1) \leq \frac{1}{\delta} C^\alpha \|x_1 - x_2\|^\alpha.$$

Interchanging x_1 and x_2 , the proof is complete. \square

Remark 2. In [6–8], the set attainability property is defined from a condition of the type

$$\forall x \in K, \quad \forall p \in NP_K(x), \quad \min_{z \in F(x)} \langle p, z \rangle \leq -a\|p\| \tag{16}$$

which is in fact equivalent to the Lipschitz continuity of Θ . Moreover, it is clear from the proof of [10, Theorem 5] that condition (16) implies the viability for (13) of $\text{Epi}(d_K)$ restricted to $I := K + (a/L)B(0, r)$, where L is a Lipschitz constant. So (16) implies (10) with $\alpha = 1$ and $\lambda = -(a/L)$.

Proposition 3. *Assume that conditions of Theorem 1 hold true. Moreover, assume that F is a continuous set-valued map with nonempty closed compact values and $\lambda(\cdot) = -\delta$ is constant.*

If $\text{Epi}(d_K^\alpha)$ is a sleek subset⁴ of \mathbf{R}^{n+1} then there exists a feedback control $\hat{u} : I \mapsto U$ such that

$$x \mapsto f(x, \hat{u}(x))$$

is continuous and for any $x_0 \in I \setminus K$ there exists a solution to

$$x'(t) = f(x(t), \hat{u}(x(t))), \quad x(0) = x_0, \tag{17}$$

which reaches K in finite time. Furthermore, all the solutions to (17) reach K in finite time.

Proof. If $\text{Epi}(d_K^\alpha)$ is a sleek subset of \mathbf{R}^{n+1} then, by (10) and (12), the map

$$(x, y) \in I \setminus K \times \mathbf{R}_+ \mapsto (F(x), -\delta) \cap T_{\text{Epi}(d_K^\alpha)}(x, y)$$

is closed convex nonempty valued. Then by Lemma 4 (postponed after the present proof), there exists a continuous selection of the above map which is necessarily of the form

$$(x, y) \mapsto (f(x, \bar{u}(x, y)), -\delta).$$

Define $\hat{u}(x) = \bar{u}(x, d_K^\alpha(x))$. Then the map

$$x \mapsto f(x, \hat{u}(x))$$

is continuous. Following the arguments in the proof of Theorem 1, one can prove that $\text{Epi}(d_K^\alpha)$ is a viable set for the differential equation

$$(x'(t), y'(t)) = (f(x(t), \hat{u}(x(t))), -\delta).$$

Thus starting from $(x_0, d_K^\alpha(x_0))$ there exists a viable solution $(x(\cdot), y(\cdot))$ to the above differential equation. Note that $x(\cdot)$ reaches K in a time not greater than $d_K^\alpha(x_0)/\delta$.

Now fix $x_0 \in I \setminus K$ and consider $x(\cdot)$ a solution to (17).

Define $\psi(t) := d_K^\alpha(x(t))$ which is absolutely continuous on every interval where $\psi(t)$ do not vanish. So for almost all t belonging to such an interval

$$\begin{aligned} \psi'(t) &= D_\uparrow d_K^\alpha(x(t))x'(t) \\ &= D_\uparrow d_K^\alpha(x(t))f(x(t), \bar{u}(x(t), d_K^\alpha(x(t)))) \leq -\delta \end{aligned}$$

by the definition of \bar{u} .

Thus on every interval were $d_K(x(t)) \neq 0$ we have

$$\psi(t) = d_K^\alpha(x(t)) \leq d_K^\alpha(x_0) - \delta t.$$

Hence $\psi(t)$ is equal to 0 before $d_K^\alpha(x_0)/\delta$.

This completes the proof. \square

⁴ It is enough to assume that $\text{Epi}(d_K^\alpha)$ is sleek at every point $(x, d_K^\alpha(x))$ where $x \in I \setminus K$.

Lemma 4. *Let $U \subset \mathbf{R}^N$ be an open set, G and H be two set-valued maps from U to \mathbf{R}^N . Assume that G and H have nonempty convex closed values, that G is continuous and H is lower semicontinuous. If*

$$\forall x \in U, \quad G(x) \cap H(x) \neq \emptyset,$$

then there exists a continuous function $f : U \mapsto \mathbf{R}^N$ with

$$\forall x \in U, \quad f(x) \in G(x) \cap H(x).$$

Proof. This is based mainly on an idea in [13].

Fix $\varepsilon > 0$. By Lemma 5.1 and Remark 5.2 in [13], there exists a continuous selection g_ε of H with

$$\forall x \in U, \quad g_\varepsilon(x) \in G(x + \varepsilon B) + \varepsilon B.$$

Using ideas developed in the proof of Michael’s Selection Theorem (cf. Lemma 9.1.4 and proof of Theorem 9.1.2 in [12]), one can obtain a continuous selection f of $G \cap H$. \square

Using the fact that $\text{Epi}(d_K^\alpha)$ is sleek on $I \setminus K$ when the map $x \mapsto d_K^\alpha(x)$ is $C_{\text{loc}}^{1,1}$ on $I \setminus K$ (see Proposition 9 in the Appendix), we can obtain the following:

Corollary 5. *Assume all the conditions of Proposition 3. Furthermore, assume that F is Lipschitz and that $x \mapsto d_K^\alpha(x)$ is locally of class $C^{1,1}$ on $I \setminus K$. Then there exists a feedback law \hat{u} which steers to K all the solutions to (17) starting from any $x_0 \in I \setminus K$. Moreover, the minimal time function associated with Eq. (17) is Hölder continuous on $I \setminus K$ of exponent α .*

We have also the following result.

Proposition 6. *Let F be an upper semicontinuous map with nonempty compact convex values and linear growth. Let $V : \mathbf{R}^n \mapsto \mathbf{R}$ be a continuous nonnegative function. Consider*

$$K := \{x \in \mathbf{R}^n, V(x) \leq 0\}$$

and $I := K + r\mathring{B} \subset \{x \in \mathbf{R}^n, V(x) < s\}$, $r > 0$. We assume that there exists a Lipschitz continuous positive function λ such that, for any $y_0 \geq 0$, the corresponding solution to (9) assumes the value 0 in a finite minimal time denoted $\vartheta(y_0)$.

Suppose that for any x in $I \setminus K$, there exists some $w \in F(x)$ such that

$$D_{\uparrow} V(x)w \leq -\lambda(V(x)). \tag{18}$$

Then the set K is attainable by trajectories of (6).

Furthermore, if we assume that V is of class $C^{1,1}$ on $I \setminus K$ with nonvanishing gradient on $I \setminus K$ (then $\text{Epi}(V)$ is sleek) and if $\lambda = -\delta$ is constant, then K is feedback attainable.

Proof. Fix $x_0 \in I \setminus K$. From (18) and [12, Proposition 6.1.4] we deduce

$$\forall x \in I \setminus K, \exists v \in F(x), \quad (v, -\lambda(V(x))) \in T_{\text{Epi}(V)}(x, V(x)). \tag{19}$$

By using similar arguments as in Theorem 1 and Proposition 3, one can easily complete the proof. Let us underline that when V is of class $C^{1,1}$ on $I \setminus K$ with nonvanishing gradient on $I \setminus K$, then $\text{Epi}(V)$ is sleek at any point $(x, V(x))$ with $x \in I \setminus K$ and $V(x) \neq 0$. \square

Set-attainability by all trajectories. We say that the set K is a global attainable set if starting from any point $x_0 \in I \setminus K$ of a neighborhood $I \setminus K$ of K all the solutions to the differential inclusion (6) reach K in finite time.

Proposition 7. *Assume that the set-valued map F is Lipschitz continuous with compact convex values. Suppose that there exists a function λ as in Theorem 1.*

If there exists $\alpha > 0$ such that

$$\forall x \in I \setminus K, \quad F(x) \subset T_{K, \lambda(d_K^\alpha(x))}^\alpha(x), \tag{20}$$

then starting from any $x_0 \in I$, every solution to (6) reaches K in finite time smaller or equal to $\vartheta(d_K^\alpha(x_0))$.

The idea is very similar to that of Theorem 1, the only difference being that we have to apply the Invariance Theorem [11, Theorem 5.3.4] instead the Viability Theorem. So we omit the proof.

4. Viability and invariance by set-valued selection

In this section, K is assumed to be compact and F Lipschitz continuous. Let us define the following function which was proposed by Veliov in [10]:

$$x \in \mathbf{R}^n \mapsto l(x) := \begin{cases} \max_{z \in \Pi_K(x)} \min_{v \in F(x)} \left\langle \frac{x-z}{\|x-z\|}, v \right\rangle & \text{if } x \notin K, \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

Proposition 8. *Let K be a compact set and F be a Lipschitz continuous map with Lipschitz constant $L > 0$ in $I = K + B(0, r)$.*

Assume K is viable, so it satisfies

$$\forall x \in K, \quad F(x) \cap T_K(x) \neq \emptyset.$$

Define on I the multivalued map⁵

$$x \mapsto G(x) := F(x) \cap T_{K, -l(x)}(x).$$

⁵ Observe that the “sub-map” G is equivalently defined by the formula $G(x) = \{v \in F(x), D_\uparrow d_K(x)v \leq l(x)\}$, and its restriction on K coincide with $F(x) \cap T_K(x)$.

Then the set of solutions to

$$x'(t) \in G(x(t)), \quad x(0) = x_0, \tag{22}$$

is nonempty and closed for any $x_0 \in I$. Furthermore, if $x_0 \in K$ then it coincides with the set of viable solutions to (6).

Proof. The first part of the proof is taken from [10] and it is given here for the reader’s convenience. Denote by M an upper bound of F on K .

Part I. Fix $x \in I \setminus K$. For any $z \in \Pi_K(x)$ and $v \in \mathbf{R}^n$ we have

$$D_{\uparrow}d_K(x)v = \min_{y \in \Pi_K(x)} \left\langle \frac{x - y}{\|x - y\|}, v \right\rangle \leq \left\langle \frac{x - z}{\|x - z\|}, v \right\rangle.$$

Thus there exists some $v \in F(x)$ such that

$$D_{\uparrow}d_K(x)v \leq \min_{v \in F(x)} \left\langle \frac{x - z}{\|x - z\|}, v \right\rangle \leq l(x). \tag{23}$$

Consequently, $G(x) \neq \emptyset$. For $x \in K$, G has also nonempty values by (8).

Fix $x \in I \setminus K$. We claim that

$$-M \leq l(x) \leq Ld_K(x). \tag{24}$$

The first inequality follows from the definitions of M and of the function l ; let us prove the second inequality. Let $z \in \Pi_K(x)$. Because K is viable so it satisfies (8). Hence, there exists $w \in F(z)$ such that

$$\left\langle \frac{x - z}{\|x - z\|}, w \right\rangle \leq 0.$$

The Lipschitz continuity of F implies that there exists $v \in F(x)$ with

$$\|v - w\| \leq Ld_K(x).$$

Thus

$$\left\langle \frac{x - z}{\|x - z\|}, v \right\rangle \leq \left\langle \frac{x - z}{\|x - z\|}, w \right\rangle + \left\langle \frac{x - z}{\|x - z\|}, v - w \right\rangle \leq Ld_K(x).$$

This proves our claim.

Part II. Let us remark that $v \in G(x)$, with $x \in I \setminus K$, if and only if

$$(v, l(x)) \in T_{\text{Epi}(d_K)}(x, d_K(x)), \tag{25}$$

and equivalently by (24)

$$(\{v\} \times [-M, l(x)]) \cap T_{\text{Epi}(d_K)}(x, d_K(x)) \neq \emptyset.$$

As G has nonempty values on I , we deduce that $\text{Epi}(d_K)$ is locally viable (in the interior of I) for

$$(x'(t), y'(t)) \in F(x(t)) \times [-M, l(x(t))]. \tag{26}$$

In fact, it is possible to use the Viability Theorem because

$$(x, y) \mapsto F(x) \times [-M, l(x)]$$

is an upper semicontinuous map since l is upper semicontinuous. Hence the set of solution to (22) is closed because it is the projection of the set of viable solutions to (26) which is compact.

Now let us choose a solution $x(\cdot)$ to (22). For $t \geq 0$ let define $g(t) = d_K(x(t))$ which is an absolutely continuous function. For all $t \geq 0$, for which $x(t) \notin K$, we have in virtue of (24)

$$g'(t) = D_{\uparrow}d_K(x)x'(t) \leq l(x) \leq Ld_K(x) = Lg(t).$$

From Gronwall’s Lemma we obtain

$$d_K(x(t)) \leq d_K(x_0)e^{Lt}, \quad \forall t \geq 0 \text{ such that } x(t) \in I \setminus K. \tag{27}$$

Taking $x_0 \in K$, we obtain that all the solutions to (22) are viable. \square

Remark 3. Observe that in general the map G is not convex-valued. However, G has convex values when $\text{Epi}(d_k)$ is sleek. This follows from [12, Theorem 4.1.8] and (25).

Remark 4. First, we want to stress that the map

$$x \mapsto H(x) := F(x) \cap T_K(x)$$

is not suitable to obtain the invariance property of G described in the above Proposition 8. In fact, the invariance property involves the behaviour of the map outside K .

Furthermore, when F is not Lipschitz, G is not suitable to obtain the Invariance Property. We illustrate this fact in the following elementary example in dimension 1.

Example. Consider $K := \{0\}$ and the ordinary differential equation $x' = 2x^{1/2}$. Clearly, for any $L > 0$, we have $G = H = x^{1/2}$, and obviously, $t \mapsto t^2$ is a solution of the above differential equation starting from 0 which is not viable.

Acknowledgments

This work was carried out during a stay of the second author at the University of Siena. He would like to thank the Faculty of Engineering of the University of Siena for its hospitality.

Appendix A. Sleek property of the epigraph of d_K

In this paper, we have often used the assumption that $\text{Epi}(d_K)$ is sleek. We shall now give an equivalent formulation of this condition.

Proposition 9. *Let K be a closed nonempty subset of \mathbf{R}^n . Then the following conditions are equivalent:*

- (i) *the set $\text{Epi}(d_K)$ is sleek on $I \setminus K$,*
- (ii) *the projection $x \mapsto \Pi_K(x)$ is single-valued on I ,*
- (iii) *the distance function $x \mapsto d_K(x)$ is locally of class $C^{1,1}$ on $I \setminus K$.*

Proof. Clearly (iii) \Rightarrow (i). The equivalence of (ii) and (iii) is due to Federer [14, Theorem 4.8]. Let us prove that (i) \Rightarrow (ii). For this, assume that $\text{Epi}(d_K)$ is sleek on $I \setminus K$. Assume by contradiction that for some $x \in I \setminus K$ there exist two elements $y_1 \neq y_2$ in $\Pi_K(x)$.

Taking into account that d_K is differentiable on the open segment (y_i, x) , $i = 1, 2$, we get

$$\forall i = 1, 2, \forall z_i \in (y_i, x),$$

$$T_{\text{Epi}(d_K)}(z_i, d_K(z_i)) = \left(\nabla d_K(z_i) \frac{z_i - y_i}{\|z_i - y_i\|}, -1 \right)^-, \tag{28}$$

where

$$A^- := \{p \in \mathbf{R}^n, \langle p, a \rangle \leq 0, \forall a \in A\}.$$

Since $\text{Epi}(d_K)$ is sleek its contingent cone is convex, so from [12, Theorem 4.1.9] we obtain⁶

$$T_{\text{Epi}(d_K)}(x, d_K(x)) \supset \text{co} \left(\left(\nabla d_K(z_1) \frac{z_1 - y_1}{\|z_1 - y_1\|}, -1 \right)^- \cup \left(\nabla d_K(z_2) \frac{z_2 - y_2}{\|z_2 - y_2\|}, -1 \right)^- \right). \tag{29}$$

We end the proof by considering the following two cases.

Case 1. The vectors $x - y_1$ and $x - y_2$ are parallel and consequently of opposite directions. On one hand, since

$$\xi \mapsto T_{\text{Epi}(d_K)}(\xi, d_K(\xi))$$

is lower semicontinuous, letting $\xi \in (y_i, x) \rightarrow x, i = 1, 2$, we obtain

$$T_{\text{Epi}(d_K)}(x, d_K(x)) \subset \{(v, w) \in \mathbf{R}^n \times \mathbf{R}, \langle v, x - y_1 \rangle = 0, w \geq 0\}.$$

On the other hand, because d_K is Lipschitz continuous with Lipschitz constant 1, we have

$$\forall v \in \mathbf{R}^n, (v, \|v\|) \in T_{\text{Epi}(d_K)}(x, d_K(x)).$$

A contradiction is obtained taking $v = x - y_1$.

⁶ where co denotes the closed convex hull of a set.

Case 2. The vectors $x - y_1$ and $x - y_2$ are not parallel. From (29), we obtain

$$T_{\text{Epi}(d_K)}(x, d_K(x)) = \mathbf{R}^n \times \mathbf{R}_-.$$

This is a contradiction with the lower semicontinuity of the tangent cone to the epigraph $\text{Epi}(d_K)$ together with (28). \square

References

- [1] P. Cardaliaguet, M. Quincampoix, P. Saint Pierre, Minimal time for constrained nonlinear control problems without controllability, *Appl. Math. Optim.* 36 (1997) 21–42.
- [2] H. Sussmann, A sufficient condition for local controllability, *SIAM J. Control Optim.* 16 (1978) 790–802.
- [3] R. Bianchini, G. Stefani, Time optimal problem and time optimal map, *Rend. Sem. Mat. Univ. Politec. Torino* 48 (1990) 401–429.
- [4] M. Krastanov, M. Quincampoix, Local small time controllability and attainability of a set for nonlinear control system, *ESAIM Control Optim. Calcul. Variations* 6 (2001) 499–516.
- [5] P. Soravia, Hölder continuity of the minimum-time function for C^1 -manifold targets, *JOTA* 75 (1992).
- [6] V. Veliov, Attractiveness and invariance: the case of uncertain measurement, in: Kurzhanski, Veliov (Eds.), *Modeling Techniques for Uncertain Systems*, PSCT, Vol. 18, Birkhäuser, Basel, 1994.
- [7] F.H. Clarke, P.R. Wolenski, Control of systems to sets and their interiors, *JOTA* 88 (1996) 3–23.
- [8] V. Veliov, On the Lipschitz continuity of the value function in optimal control, *JOTA* 94 (1997) 335–361.
- [9] L. Gorniewicz, P. Nistri, Feedback stability of closed sets for nonlinear control systems, in: *Progress in Nonlinear Differential Equations and Their Application*, Vol. 40, 1999, pp. 165–175.
- [10] V. Veliov, Stability-like properties of differential inclusions, *Set-Valued Analysis* 5 (1997) 73–88.
- [11] J.-P. Aubin, *Viability Theory*, Birkhäuser, Basel, 1991.
- [12] J.-P. Aubin, H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Basel, 1990.
- [13] H. Ben-el-Mechaiekh, W. Kryszewski, Equilibria of set-valued maps on non-convex domains, *Trans. Amer. Math. Soc.* 349 (1997) 4159–4179.
- [14] H. Federer, Curvature measures, *Trans. Amer. Math. Soc.* 93 (1959) 418–491.
- [15] L. Gorniewicz, P. Nistri, An invariance problem for control systems with deterministic uncertainty, in: *Topology in Nonlinear Analysis*, Banach Center Publications, Vol. 35, PWN, Warsaw, 1996, pp. 193–205.