PERGAMON

Nonlinear Analysis 47 (2001) 1003–1015

Nonlinear Analysis

www.elsevier.nl/locate/na

Two nonlinear feedback control problems on proximate retracts of Hilbert spaces

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Abstract

Given a nonempty, closed set K in a Hilbert space H, we consider the following two control problems involving the set K.

1. For a nonlinear control problem with state $x \in H$ we study the existence of a state feedback control, taking values in a Hilbert space H_1, which "stabilizes" the set K in a precise sense, which is related to the viability of K.

2. For a nonlinear control problem affected by deterministic uncertainty we look for a state feedback control, taking values in H_1, such that K is invariant under any system dynamics f, which is represented by a selection of the uncertain controlled dynamics.

In this paper, by using methods from multivalued analysis, we will solve these problems for a suitable class of sets $K \subset H$. that of proximate retracts. Indeed, this class will provide us the necessary continuity property for the external contingent Bouligand cone to K and for the metric projection on K, which permit to use recent results on the existence of selections for multivalued maps. These selections will represent the required feedback control laws.

Key words: Proximate retracts, φ-convex sets, sleek sets, Bouligand cones, regulation maps, feedback controls.

1 INTRODUCTION

In this paper we extend to Hilbert spaces the results that the authors have already established in [8] and [9] for Euclidean spaces. In order to formulate

0362-546X/01/$ - see front matter
PII: S0362-546X(01)00241-3

the problems that we will treat throughout the paper, we consider here two Hilbert spaces H, H_1 of infinite dimension; a nonempty closed subset K of H and two neighbourhoods U, U_0 of K in H such that $U_0 \subset U$. Moreover, in the sequel $f: X \multimap Y$ will denote a multivalued map from the topological space X to the topological space Y. For all the relevant concepts concerning multivalued maps we refer the reader, for instance, to [2]. The aim of this paper is to solve the following two control problems, (compare [8] and [9] respectively for the finite dimensional case). In the sequel we assume that the solutions of the considered Cauchy problems exist on the time interval of interest. For classical existence and prolongability criteria for ordinary differential equations in infinite dimensional spaces we refer the reader to [11] and for differential inclusions to [10].

1.1 FEEDBACK STABILITY PROBLEM (FSP)

Let $f: H \times H_1 \to H$ be a continuous compact map and $\Phi: H \multimap H_1$ an upper semicontinuous (u.s.c.) bounded map with compact convex values. We shall consider a nonlinear control problem described by the differential system:

$$x'(t) = f(x(t), u(t)), \quad t \geq 0, \tag{1.1}$$

where $u(t) \in \Phi(x(t))$.

(FSP) Given $x_0 \in U$ we want to show that there exists a trajectory $x = x(t)$ and a corresponding control $u = u(t) \in \Phi(x(t))$ satisfying (1.1), with $x(0) = x_0$, and that there exists $t_0 \geq 0$ such that the following condition is verified:

$$x(t) \in U_0, \quad \text{for every } t \geq t_0. \tag{1.2}$$

Our second problem is the following.

1.2 FEEDBACK INVARIANCE PROBLEM (FIP)

We shall consider a nonlinear control system with deterministic uncertain dynamics modelled by a differential inclusion of the form

$$x'(t) \in F(t, x(t), u(t)), \tag{1.3}$$

where $F: [0,T] \times H \times A \multimap H$ is compact and (t, x, u) – Hausdorff continuous with nonempty closed convex values and A is a compact convex subset of H_1. Then by a system dynamics f of (1.3) we shall understand a continuous the problems that we will treat throughout the paper, we consider here two Hilbert spaces H, H_1 of infinite dimension; a nonempty closed subset K of H and two neighbourhoods U, U_0 of K in H such that $U_0 \subset U$. Moreover, in the sequel $f: X \multimap Y$ will denote a multivalued map from the topological space X to the topological space Y. For all the relevant concepts concerning multivalued maps we refer the reader, for instance, to [2]. The aim of this paper is to solve the following two control problems, (compare [8] and [9] respectively for the finite dimensional case). In the sequel we assume that the solutions of the considered Cauchy problems exist on the time interval of interest. For classical existence and prolongability criteria for ordinary differential equations in infinite dimensional spaces we refer the reader to [11] and for differential inclusions to [10].

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(FIP) The problem is to find a state feedback control $\overline{u} = \overline{u}(t,x)$, $\overline{u}\colon [0,T] \times U_0 \to H$ such that for every dynamics f we have

$$f(t,x,\overline{u}(t,x)) \in F(t,x,\overline{u}(t,x))$$

and for every $x_0 \in K$, any solution $x = x(t)$ of the Cauchy problem

$$\begin{cases} x'(t) = f(t,x(t),\overline{u}(t,x(t))), \\ x(0) = x_0 \end{cases}$$

satisfies the condition

$$x(t) \in K \quad \text{for every } t \in [0,T].$$

In other words we want to find a state feedback control $\overline{u} = \overline{u}(t,x)$, $(t,x) \in [0,T] \times U$ which makes K invariant under every system dynamics f.

Observe that under the conditions imposed on the functions involved in both the previous problems we have the local existence of a solution of any Cauchy problem. A sufficient condition for the prolongability is the sublinearity of the dynamics f and F with respect to x, (compare [11] and [10]).

In order to solve (FSP) and (FIP) we first introduce an appropriate class of nonempty closed sets K for which the external contingent Bouligand cone is proved to be lower semicontinuous with nonempty convex values in a suitable neighbourhood of K. This class is that of proximate retracts: a nonempty closed set K is said to be a proximate retract if there exists an open neighbourhood U of K in H such that the set $\Pi_K(x) = \{y \in K \mid \|x-y\| = \inf_{z\in K} \|x-z\|\}$ is a singleton $\pi_K(x)$ for any $x \in U$, and the map $x \mapsto \pi_K(x)$ is continuous in U.

The class of proximate retracts can be also characterized in terms of φ-convexity (see [6] and [5]).

The methods employed for solving the two proposed control problems are based on the application of different selection theorems to a multivalued map suitably defined for each control problem. Such a map is called regulation map. Specifically, for the problem (FSP) it is defined as the intersection of the set of velocities $F(x) = \{f(x,u),\ u \in \Phi(x)\}$, $x \in U$, with a suitable lower semicontinuous cone-valued selection $\widetilde{T}_{K,\gamma}(x)$, $\gamma \geq 0$, of the external contingent Bouligand cone $\widetilde{T}_K(x)$, $x \in U$. Then, by using both a selection result of [3] and a result for the approximation in graph of multivalued maps analogous to that in [7] for finite dimensional spaces to conclude the existence of a trajectory

$x = x(t)$, $t \geq 0$, $x(0) = x_0 \in U$, corresponding to a measurable control $u(t) \in \Phi(x(t))$ which reaches in finite time the prescribed neighbourhood U_0 of K and remains in it for all the future times.

Finally, for the second problem we define the regulation map as $R(t,x) = \{u \in A \mid F(t,x,u) \subset T_K(\pi_K(x))\}$ for $(t,x) \in [0,T] \times U$ and by a selection theorem due to Bressan ([4], Theorem 1) we will obtain a directionally continuous feedback control $\overline{u}(t,x)$ which guarantees that any Krasovskiĭ solution $x = x(t)$ of any Cauchy problem relative to any system dynamics $f(t,x,\overline{u}(t,x)) \in F(t,x,\overline{u}(t,x))$ with $x(0) \in K$ remains in K for all times.

The important role of multivalued analysis for solving control problems is well known and it is documented by the wide literature dedicated to this argument. In this paper we have considered two control problems which require viability and invariance properties of a given subset K of the state space represented by an infinite dimensional Hilbert space. Due to the limitation on the number of pages, this paper should be considered as a preliminary contribution to the study in infinite dimensional spaces of control problems requiring a prescribed behaviour of the trajectories with respect to a nonempty closed set. Moreover, for the same reason, the proofs of the presented results will be omitted or simply sketched, they will be presented elsewhere.

2 THE CLASS OF PROXIMATE RETRACTS

Let $K \subset H$ be a nonempty closed set, we recall that the function $d_K \colon H \to \mathbb{R}_+$ defined by

$$d_K(x) = \inf\{\|x-y\| \mid y \in K\}$$

is called the distance function to K.

For given $x \in K$ we let

$$T_K(x) = \left\{ y \in H \,\middle|\, \liminf_{\tau \to 0+} \frac{d_K(x+\tau y)}{\tau} = 0 \right\}.$$

$T_K(x)$ is called the Bouligand contingent cone to K at x. We will denote by $N_K(x)$ the polar cone of $T_K(x)$. Recall also that the multivalued map $\Pi_K \colon H \multimap K$, given by

$$\Pi_K(x) = \{y \in K \mid \|x-y\| = d_K(x)\} \tag{2.1}$$

is called the metric retraction.

We recall the following definition.

Definition 2.2 (compare [2], [6], [12]) (2.2.1) K is called *sleek* provided that the map $T: K \multimap H$ given by $T(x) = T_K(x)$, $x \in K$, is lower semicontinuous (l.s.c.);

(2.2.2) K is called *φ-convex* provided that there exists a continuous map $\varphi: K \times K \to \mathbb{R}_+$ such that

$$\langle v, y - x\rangle \leq \varphi(x, y) \, \|v\| \, \|y - x\|^2$$

for every x, $y \in K$ and $v \in N_K(x)$;

(2.2.3) K is called *a proximate retract* provided that there exists an open neighbourhood U of K in H such that $\Pi_K(x)$ is a singleton $\pi_K(x)$ for every $x \in U$, and the map $\pi_K: U \to K$ is a continuous function.

In view of ([1], Lemma 5.1.2) one can show the following result.

Proclaim 2.3 *We have that*

$$T_K(\Pi_K(x)) \subset \widetilde{T}_K(x) \quad \text{for any } x \in H,$$

where $\widetilde{T}_K(x) = \{y \in H \mid \liminf_{\tau \to 0+} (d_K(x + \tau y) - d_K(x))/\tau \leq 0\}$ *is the external Bouligand contingent cone.*

If K is a proximate retract, then the distance function d_K is a C^1-function on $U \setminus K$. Moreover, the gradient $\operatorname{grad} d_K$ *of the distance function d_K can be expressed through the (unique) projection π_K as follows*

$$\operatorname{grad} d_K(x) = \frac{x - \pi_K(x)}{d_K(x)}, \quad x \in U \setminus K,$$

and the metric projection as well as $\operatorname{grad} d_K$ *are locally Lipschitzean on $U \setminus K$.*

The following theorem plays an important role in our considerations.

Theorem 2.4 ([6], Theorem 6.1)) *The following statements are equivalent*

(2.4.1) *K is φ-convex;*
(2.4.2) *K is a proximate retract.*

Since any φ-convex set is sleek (see [6]) we have as a consequence the following result.

Corollary 2.5 *Any proximate retract is sleek.*

Now, for any $x \in U$ we let

$$K(x) = K + d_K(x) \cdot B_1 = \{y + d_K(x)b \mid y \in K, \quad \|b\| \leq 1\}.$$

Clearly, $K(x) = K$ for every $x \in K$.

Following [8] we will define for $\gamma \geq 0$ the external Bouligand contingent cone $\widetilde{T}_{K,\gamma}(x)$ to K at x as follows:

$$\text{(2.6)} \qquad \widetilde{T}_{K,\gamma}(x) = \{y \in H \mid \liminf_{\tau \to 0+} \frac{d_K(x + \tau y) - d_K(x)}{\tau} \leq -\gamma\}.$$

For simplicity we let $\widetilde{T}_K(x) = \widetilde{T}_{K,0}(x)$. Observe that $\widetilde{T}_K(x) = T_K(x)$ provided $x \in K$.

We are able to prove the following.

Theorem 2.7 *Let K be a proximate retract and let U be choosen according to* (2.2.1). *Then the map $T: U \multimap H$ defined as follows:*

$$T(x) = T_{K(x)}(x) \quad \textit{for any } x \in U$$

is lower semicontinuous, i.e. the set $K(x)$ is sleek at any $x \in U$.

Proof: The proof is rather technical and it consists of proving the following three facts (compare [9] where the case when H is of finite dimension is considered):

(2.7.1) $T_{K(x)} = T_K(\pi_K(x)) + T_{d_K(x) \cdot B_1}(x - \pi_K(x)), \quad x \in U;$
(2.7.2) $\widetilde{T}_K(x) = T_{K(x)}(x), \quad x \in U;$
(2.7.3) The map $\widetilde{T}: U \multimap H$, given by $\widetilde{T}(x) = \widetilde{T}_K(x)$, $x \in U$, is lower semicontinuous.

3 THE FEEDBACK STABILITY PROBLEM (FSP)

Let $K \subset H$ be a proximate retract and $U \supset K$ a neighborhood given by (2.2.3). First, we will consider the set of velocities

$$V(x) = \{f(x, u) \mid u \in \Phi(x)\}, \quad x \in U.$$

From our assumptions on the dynamics f and the feedback map Φ, it follows that the map $V: U \multimap H$ is u.s.c. compact with nonempty compact values. Furthermore, we will assume that

(3.1) the set $V(x)$ is convex for any $x \in U$;
(3.2) the map $\widetilde{T}_\gamma: U \setminus K \multimap H$ defined by $\widetilde{T}_\gamma(x) = \widetilde{T}_{K,\gamma}(x)$ is nonempty for sufficiently small $\gamma \geq 0$.

By using the arguments of ([8], Proposition 2) we can prove the following result.

Theorem 3.3 *For sufficiently small $\gamma \geq 0$, the map $\widetilde{T}_\gamma: U \setminus K \multimap H$ is l.s.c. with closed convex values.*

We give now the following definition.

Definition 3.4 For $x \in U$ we define *the regulation map* associated to (1.1) as follows

$$R_{K,\gamma}(x) = V(x) \cap \widetilde{T}_{K,\gamma}(x).$$

Our last assumption is the following:

(3.4.1) $R_{K,\gamma}(x)$ is nonempty for any $x \in U$ and $\gamma \geq 0$ sufficiently small.

By using ([3], Lemma 5.1) one can show:

Proclaim 3.5 *For any $\delta > 0$ there exists a continuous map $g: U \to H$ such that*

(3.5.1) *$g(x) \in \widetilde{T}_{K,\gamma}(x)$ for any $x \in U$;*
(3.5.2) *g is a δ-approximation of F in the graph, i.e.* Graph(g) $\subset$ Graph(F) $+ \delta \, \mathrm{B}_1$.

Now for solving (FSP) it is enough to prove the following two results.

Theorem 3.6 *Let g be as in* (3.5). *Then any solution of the Cauchy problem*

$$\text{(3.6.1)} \qquad \begin{cases} x'(t) = g(x(t)), \\ x(0) = x_0 \qquad x_0 \in U \setminus K, \end{cases}$$

reaches the set K in finite time $t_0 > 0$ and remains in it for all $t \geq t_0$. Moreover, $t_0 \leq d_K(x_0)/\gamma$.

Proof: Consider any solution $x = x(t)$ of (3.6.1) and let $d(t) = d_K(x(t))$, $t \geq 0$. One sees that $\dot{d}(t) \leq -\gamma$ for $t \geq 0$, thus there exists $t_0 > 0$ such that

$d(t_0) = 0$ and $d(t) > 0$ for $0 \le t < t_0$. Moreover, $t_0 \le d_K(x_0)/\gamma$.

Theorem 3.7 *For any $\varepsilon > 0$ there exists $\delta > 0$ such that if $g: U \to H$ is a continuous δ-approximation on the graph of V, then for any solution $y = y(t)$ of (3.6.1) there exists a solution $x = x(t)$, $t \ge 0$, of the initial value problem*

$$\begin{cases} x'(t) = V(x(t)), \\ x(0) = x_0, \end{cases} \tag{3.7.1}$$

such that for any time interval $[0, T]$, $T > 0$, we have

$$\max_{t \in [0,T]} \|x(t) - y(t)\| \le \varepsilon.$$

Moreover, there exists a measurable control $u(t) \in \Phi(x(t))$, such that $x'(t) = f(x(t), u(t))$ for a.a. $t \ge 0$.

Proof: The proof is a consequence of the infinite dimensional version of ([7], Theorem 1, p. 87) and of ([2], Theorem 8.2.10). (Compare also [13]).

It is now evident that from Theorems (3.6) and (3.7), that the trajectory $x = x(t)$ reaches in finite time the ε-neighbourhood of K and remains in it for all the future time. Hence the problem (FSP) is solved.

4 THE FEEDBACK INVARIANCE PROBLEM (FIP)

Let us assume the following conditions:

(4.1) there exists $r > 0$ such that for any $(t, x) \in [0, T] \times U$ there is $u \in A$ for which

$$F(t, x, u) + r \cdot B_1 \subset T_K(\pi_K(x));$$

(4.2) for any $(t, x) \in [0, T] \times U$, u, $v \in A$ and $\theta \in [0, 1]$ there exists $\widetilde{u} \in A$ such that

$$\theta F(t, x, u) + (1 - \theta) F(t, x, v) = F(t, x, \widetilde{u}).$$

Furthermore, there are $a, b > 0$ such that

(4.3) $a\,\|u - v\| \le d_H(F(t, x, u), F(t, x, v)) \le b\,\|u - v\|$, where d_H stands for the Hausdorff distance.

To solve (FIP) we consider a regulation map R as given in the following definition.

Definition 4.4 $R\colon [0,T] \times U \multimap A$ is the map defined by

$$R(t,x) = \{u \in A \mid F(t,x,u) \subset T_K(\pi_K(x))\}.$$

We are able to prove the following result.

Theorem 4.5 *Assume that K is a proximate retract and U is given by* (2.2.3). *Assume that all the above assumptions are satisfied, then the regulation map $R\colon [0,T] \times U \multimap A$ is l.s.c. with compact values.*

Proof: The proof follows closely that of ([9], Theorem 1).

Since our map F is compact, if U is bounded then there exists a constant $M > 0$ such that $\|y\| \leq M$ for every $y \in F(t,x,u)$, where $(t,x,u) \in [0,T] \times U \times A$.

Definition 4.6 Let Γ^M be the cone in $\mathbb{R} \times H$ defined as follows:

$$\Gamma^M = \{(t,x) \in \mathbb{R} \times H \mid \|x\| \leq M\,t\}.$$

We say that $f\colon [0,T] \times U \to H$ is *directionally* Γ^M*-continuous* at a point $(\overline{t},\overline{x}) \in [0,T] \times U$ if and only if $\lim_{n\to\infty} f(t_n,x_n) = f(\overline{t},\overline{x})$ provided that $\lim_{n\to\infty}(t_n,x_n) = (\overline{t},\overline{x})$ and $(t_n - \overline{t}, x_n - \overline{x}) \in \Gamma^M$.

Now in view of [4] we have the following.

Theorem 4.7 *Under the same assumptions of Theorem* 4.5 *together with the locally compactness of U the regulation map $R\colon [0,T] \times U \multimap A$ admits a Γ^M-continuous selection $\overline{u} = u(t,x)$ defined in $[0,T] \times U$.*

Remark 4.8 Theorem 4.7 remains true if we replace U by a neighbourhood (not necessary open) W of K such that: $W \subset U$ and W is locally compact.

Observe that for any system dynamics, i.e. for any continuous selection f of the multivalued map F, we have that the map $f(\cdot,\cdot,\overline{u}(\cdot,\cdot))$ is Γ^M-continuous in $[0,T] \times U$. Furthermore, under our assumption on F the set of system dynamics is nonempty in virtue of the Michael selection theorem.

Consider now the map $g\colon [0,T] \times U \to H$ defined by

$$g(t,x) = f(t,x,\overline{u}(t,x))$$

and let $K(g)\colon [0,T] \times U \to U$ given by

$$K(g)\,(t,x) = \bigcap_{\delta>0} \overline{co}\, g(B((t,x),\delta))\,,$$

where $B((t,x),\delta)$ is the closed ball centered at (t,x) of radius δ. The map $K(g)$ is called the Krasovskiĭ regularization of g.

Definition 4.9 An absolutely continuous function x is called *a Krasovskiĭ solution* of the Cauchy problem

$$\begin{cases} x'(t) = f(t, x(t), \overline{u}(t, x(t))), \\ x(0) = x_0, \end{cases}$$

provided that it is a solution of the following Cauchy problem for differential inclusions

$$\begin{cases} x'(t) = K(g)\,(t, x(t)), \\ x(0) = x_0. \end{cases}$$

Now, keeping all the assumptions of Theorem (4.5) we are able to prove the following result which solves the proposed problem (FIP).

Theorem 4.10 *Any Krasovskiĭ solution of the Cauchy problem*

$$\begin{cases} x'(t) = f(t, x(t), \overline{u}(t, x(t))), \\ x(0) = x_0 \in K, \end{cases}$$

satisfies $x(t) \in K$ *for any* $t \in [0,T]$.

5 CONCLUDING REMARKS AND COMMENTS

We have shown that the class of proximate retracts in Hilbert spaces is very useful for solving viability and invariance control problems. As we have illustrated in the previous Sections this is mainly due to the fact that for this class of sets K the external contingent Bouligand cone is lower semicontinuous (with nonempty, convex values) in a suitable neighbourhood of K and it has there a nice representation like that for convex sets. Furthermore, this class is quite large and in Hilbert spaces it can be characterized by means of an analytic condition which defines the class of the so-called φ-convex sets (see [6] and [5]).

We would like also to point out that if in the first problem (FSP) we assume that the set of velocities $F(x)$ satisfies $F(x) \subseteq \widetilde{T}_{K,\gamma}(x)$, for $x \in U$, and that it is compact lower semicontinuous with nonempty, convex values then any trajectory of the control system starting from any point $x_0 \in U$ reaches K in finite time and remains there for all future time. Furthermore, if in the second problem (FIP) the uncertain dynamics $F(t,x,u)$ satisfies $F(t,x,u) \subseteq \widetilde{T}_{K,\gamma}(x)$, for any $x \in U$, and the corresponding regulation map $\widetilde{R}(t,x)$, where $(t,x) \in \mathbb{R}_+ \times U$, is lower semicontinuous with nonempty closed values then it can be easily derived from the results presented in the previous Sections that any Krasovskiĭ solution of any possible system dynamics $f(t,x,\overline{u}(t,x)) \in F(t,x,\overline{u}(t,x))$ starting from any point $x_0 \in U$ reaches K in finite time and it remains there for all future times.

In conclusion, under similar conditions on the set of velocities, the behaviour of the trajectories in the two considered control problems is the same in the neighbourhood U of K. Moreover, this behaviour is obtained by means of different selection theorems for the corresponding regulation maps.

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