15. Periodic Solutions of a Singularly Perturbed System of Differential Inclusions in Banach Spaces*

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Abstract: By means of the Hausdorff measure of noncompactness and the topological degree theory for condensing operator in locally convex spaces we show the existence of periodic solutions of a singularly perturbed system of differential inclusions in infinite dimensional Banach spaces. Moreover, the behaviour of such periodic solutions when the parameter ϵ tends to zero is also investigated.

Keywords: Differential inclusions, condensing operators, periodic solutions, singularly perturbed system

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INTRODUCTION

The aim of this chapter is to investigate the existence of periodic solutions for a system of differential inclusions in infinite dimensional spaces, depending on a small parameter $\epsilon > 0$, which has the form

$$\begin{aligned} x'(t) &\in A\,x(t) + f_1(t, x(t), y(t)), \quad t \geq 0 \\ \epsilon y'(t) &\in B\,y(t) + f_2(t, x(t), y(t)), \end{aligned} \tag{1}$$

where A and B are infinitesimal generators of C^0-semigroups of linear operators e^{At} and $e^{Bt}, t \geq 0, x \in E_1$ and $y \in E_2$ with E_1, E_2 infinite dimensional Banach spaces. The nonlinear multivalued operators $f_i,\ i = 1, 2,$ are T-periodic in time with nonempty, convex and compact values and satisfying suitable conditions expressed in terms of the Hausdorff measure of noncompactness. All the assumptions will be precised in the next section.

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In [2] we provide conditions which guarantee the upper semicontinuity at $\epsilon = 0$ of the solution map $\epsilon \to S_\epsilon$ of the Cauchy problem for (1). The considered topology for the solution pair (x, y) is that of the uniform topology with respect to the x-variable and that of the $L^1(E_2)$-weak convergence with respect to the y-variable. For singularly perturbed systems of differential inclusions in finite dimensional spaces in [10] and [15] the same result is proved by using completely different methods. In this case, if the uniform convergence is also considered for the y-variable then the upper semicontinuity can be obtained for a suitable subset of S_ϵ (see [8,18,19]). In fact, in general, the map $\epsilon \to S_\epsilon$ is not upper semicontinuous at $\epsilon = 0$ (see [9]). Finally, In ([11] and [12], see also the references therein), an approach in order to approximate the slow motions of a singularly perturbed control system in finite dimension by a limit differential inclusion was proposed. This approach is based on the averaging method applied to the fast dynamics, as result the uniform convergence of the slow motions to a solution of the limit differential inclusion is obtained. Furthermore, any such solution is the uniform limit of slow motions. Singular perturbation methods for partial differential equations are also intensively studied (see e.g., [16,17]).

In this chapter our attention is devoted to the existence of periodic solutions for small $\epsilon > 0$ and to their behaviour when ϵ tends to zero. It is still convenient for our purposes to consider here the uniform topology for the x-variable and the weak topology for the y-variable. In fact, with this choice we will be able to show the upper semicontinuity at $\epsilon = 0$ of a suitably defined condensing operator, whose fixed points represent the T-periodic solutions of our problem. Roughly speaking, we will show that if the reduced problem at $\epsilon = 0$ admits isolated T-periodic solutions with topological degree different from zero then we provide sufficient conditions to guarantee the existence of T-periodic solutions (x_ϵ, y_ϵ) for small $\epsilon > 0$ and also that for every sequence $\epsilon_n \to 0$ the sequence $(x_{\epsilon_n}, y_{\epsilon_n})$ converge, in the above topology, to a T-periodic solution of the reduced problem. Observe that here the topological degree is that for condensing operators in locally convex spaces.

The methods presented in this chapter are similar to those of [2], but here we use different measures of noncompactness which turn out to be more suitable for the present problem. Furthermore, the topological degree theory is used in a different way.

The chapter is organized as follows. In Section 1 we state the problem and we formulate the assumptions which permit to solve it. Furthermore, we introduce convenient operators in order to rewrite our problem in terms of a multivalued fixed point problem. In Section 2 we prove in Theorem 1 the relevant properties of the resulting fixed point operator, in particular the condensivity with respect to a suitably introduced measure of noncompactness and the upper semicontinuity in the considered topology. These properties will permit to use the topological degree theory for condensing operators in locally convex spaces in order to prove the existence of fixed points for $\epsilon > 0$ sufficiently small and their behaviour when ϵ tends to zero.

1. STATEMENT OF THE PROBLEM, DEFINITIONS AND ASSUMPTIONS

Through this chapter we consider a system of differential inclusions of the form

$$\begin{aligned} x'(t) &\in A\,x(t) + f_1(t, x(t), y(t)), \quad t \geq 0 \\ \epsilon y'(t) &\in B\,y(t) + f_2(t, x(t), y(t)), \end{aligned} \tag{1}$$

where A and B are infinitesimal generators of C^0-semigroups of linear operators e^{At} and e^{Bt}, $t \geq 0$, respectively, acting in the Banach spaces E_1 and E_2 with E_2 satisfying the Radon-Nikodym condition (see [7]), $\epsilon > 0$ is a small parameter and f_i: $\mathbf{R}_+ \times E_1 \times E_2 \to Kv(E_i), i = 1, 2$, are multivalued operators. Here $Kv(E)$ denotes the set of all the nonempty, convex, compact subsets of the Banach space E.

Statement of the Problem: we want to provide conditions under which system (1) can be rewritten as a fixed point problem in a suitable chosen functional space $\mathcal{F}$ for a multivalued condensing operator F_ϵ. Moreover, these conditions must guarantee that F_ϵ is upper semicontinuous at any $\epsilon \geq 0$ in a prescribed topology of the underlying functional space $\mathcal{F}$.

As a result we can apply the related topological degree theory for multivalued condensing operator defined in $\mathcal{F}$ (see e.g., [1,4]) to derive the following relevant property:

If there exists an open set $U \subset \mathcal{F}$ such that

$$\deg(I - F_0, \bar{U}) \neq 0$$

then for $\epsilon > 0$ sufficiently small the set Σ_ϵ of periodic solutions of (1) is nonempty and the map $\epsilon \to \Sigma_\epsilon$ is upper semicontinuous in the considered topology of $\mathcal{F}$.

To make precise the setting in which we will solve the above problem we first choose for the functions $t \to x(t)$ and $t \to y(t)$ the functional spaces $C_T(E_1)$ and $L^1_T(E_2)$ respectively, and so $\mathcal{F} = C_T(E_1) \times L^1_T(E_2)$. We recall that $C_T(E)$ denotes the space of T-periodic continuous functions x: $[0, T] \to E$ equipped with the uniform norm: $\max_t \| x(t) \|_E$ and $L^1_T(E)$ is the space of T-periodic strongly measurable functions x: $[0, T] \to E$ having finite norm $\| x \|_{L^1_T} := \int_0^T \| x(t) \|_E \, dt$. In the sequel by ${}^wE$ we will denote the space E equipped with the weak topology of E, while $Kv - w(E)$ will denote the set of all the nonempty, convex, weakly compact subsets of E.
We assume that

(S_0) there exist positive constants $\gamma_1, \gamma_2 > 0$ such that

$$\| e^{At} \|_{E_1} \leq e^{-\gamma_1 t} \quad \text{and}$$
$$\| e^{Bt} \|_{E_2} \leq e^{-\gamma_2 t}$$

for any $t \geq 0$. Moreover $D(B^*)$, the domain of the adjoint operator B^*, is dense in E_2 (see [14]).

(A_0) f_i: $\mathbf{R}_+ \times E_1 \times E_2 \to Kv(E_i), i = 1, 2$, are T-periodic with respect to time, that is

$$f_i(t + T, x, y) = f_i(t, x, y)$$

for any $t \geq 0$ and $(x, y) \in E_1 \times E_2$. Furthermore, the Nemytskii operators Φ_i: $C_T(E_1) \times L^1_T(E_2) \to Kv{-}w(L^1_T(E_i))$ generated by f_i, $i = 1, 2$, as follows

$$\Phi_i(x, y) = \big\{ g \in L^1_T(E_i) : g(t) \in f_i(t, x(t), y(t)) \\ \text{for almost all (a.a) } t \in [0, T] \big\}$$

are well defined.

The following assumptions are formulated in terms of the Nemytskii operators $\Phi_i, i = 1, 2$.

(A_1) For any pair of bounded sets $\Omega_1 \subset C_T(E_1), \Omega_2 \subset L^1_T(E_2)$ there exists a function $\varphi \in L^1_T(\mathbf{R})$ such that

$$\| g_i(t) \|_{E_i} \leq \varphi(t)$$

for a.a. $t \in \mathbf{R}$ and any $g_i \in \Phi_i(x, y), i = 1, 2$, whenever $(x, y) \in \Omega_1 \times \Omega_2$.

(A_2) Φ_i are upper semicontinuous multivalued operators from $C_T(E_1) \times {}^wL^1_T(E_2)$ to ${}^w L^1_T(E_i)$.

Remark 1: Explicit conditions on $f_i, i = 1, 2$, which ensure that the related Nemytskii operators are well defined will be given in Section 3. For the finite dimensional case (see [3]).

We also need suitable compactness conditions on $\Phi_i, i = 1, 2$, expressed in terms of the Hausdorff measure of noncompactness. To this aim we give the following definitions.

Definition 1: Let E be a Banach space. Let $\Omega \subset E$ be a bounded set. The Hausdorff measure of noncompactness $\chi_E(\Omega)$ of the set Ω is the infimum of the numbers $\alpha > 0$ such that Ω has a finite α-net in E. For the relevant properties of χ_E we refer to [1].

Definition 2: Let E be a Banach space. Let $\Omega \subset E$ be a bounded set of E. The measure of weak noncompactness $\chi_w(\Omega)$ of the set Ω is the infimum of the number $\alpha > 0$ such that Ω has a weakly compact α-net in E. This measure of weak compactness and its properties have been studied by De Blasi in [6].

Definition 3: Let Ω be a bounded set of $L^1_T(E)$. A function $b \in L^1_T(\mathbf{R})$ is called a weak bound for the Hausdorff measure of noncompactness $\chi_{L^1_T(E)}$ of the set Ω if for every $\delta > 0$ there exist a measurable set $e_\delta \subset [0, T]$ and a compact set $K_\delta \subset E$ such that meas $e_\delta < \delta$ and for every $f \in \Omega$ there exists $g \in L^1_T(E)$ satisfying $g(t) \in K_\delta$ for a.a. $t \in [0, T]$ and

$$\| f(t) - g(t) \|_E \leq b(t) + \delta$$

for a.a. $t \in [0, T] \backslash e_\delta$.

In the sequel the set of all the functions $b \in L^1_T(\mathbf{R})$ with the previous properties will be denoted by $WB(\Omega)$. Observe that we can always assume that $g(t) = 0$ for $t \in e_\delta$.

We introduce now the operators $F_\epsilon, \epsilon \in [0, 1]$, whose fixed points will represent the T-periodic solutions of (1). For this, we need first to define the linear operators Λ_ϵ: $L^1_T(E_1) \times L^1_T(E_2) \to C_T(E_1) \times L^1_T(E_2)$ defined as follows

$$\Lambda_\epsilon \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}(t) = \begin{pmatrix} \Lambda_1 g_1 \\ \Lambda_2(\epsilon) g_2 \end{pmatrix}(t) = \begin{pmatrix} \int_{-\infty}^{t} e^{A(t-s)} g_1(s)\, ds \\ \frac{1}{\epsilon} \int_{-\infty}^{t} e^{(1/\epsilon)B(t-s)} g_2(s)\, ds \end{pmatrix}, \quad \epsilon > 0.$$

While, for $\epsilon = 0$ we set $\Lambda_2(0) = -B^{-1}$. Finally we pose

$$F_\epsilon(x, y) = \left\{ \Lambda_\epsilon \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} : g_1 \in \Phi_1(x, y),\ g_2 \in \Phi_2(x, y) \right\}$$

We formulate now the assumptions under which $F_\epsilon, \epsilon \in [0,1]$, is a well defined condensing operator.

(A_3) There exists closed, convex set $Q \subset {}^wL^1_T(E_2)$ such that Q is bounded in $L^1(E_2)$ and

$$\Lambda_2(\epsilon)\ \Phi_2 : C_T(E_1) \times Q - \circ Q$$

(A_4) There exist positive constants k_{11} and k_{12} such that for any pair of bounded sets $\Omega_1 \subset C_T(E_1)$ and $\Omega_2 \subset Q$ the constant function

$$K = k_{11} \sup_t \chi_{E_1}(\Omega_1(t)) + k_{12}\chi_w(\Omega_2)$$

belongs to $WB(\Phi_1(\Omega_1 \times \Omega_2))$.

(A_5) There exist positive constants k_{21} and k_{22} such that for any pair of bounded sets $\Omega_1 \subset C_T(E_1)$ and $\Omega_2 \subset Q$ one has

$$\chi_w(\Phi_2(\Omega_1 \times \Omega_2)) \leq k_{21} \sup_t \chi_{E_1}(\Omega_1(t)) + k_{22}\chi_w(\Omega_2).$$

Finally, we now formulate the last assumption.

(A_6) The eigenvalues λ of the matrix

$$H = \begin{pmatrix} k_{11}/\gamma_1 & k_{12}/\gamma_1 \\ k_{21}/\gamma_2 & k_{22}/\gamma_2 \end{pmatrix}$$

satisfy $|\lambda| < 1$.

Remark 2: The assumption (A_3) is verified if, for instance, there exist positive constants M and l such that

$$\| f_2(t,x,y) \|_{E_2} \leq M + l \| y \|$$

with $l/\gamma_2 < 1$. In this case, we have $Q = Q_R$, where

$$Q_R := \Big\{ g \in L^1_T(E_2) : \| g(t) \|_{E_2} \leq R, \quad \text{for a.a. } t \in \mathbf{R} \Big\}.$$

and $R > 0$ is sufficiently large.

Definition 4: By a T-periodic solution of system (1) where $\epsilon \geq 0$, we mean a fixed point $(x_0, y_0) \in C_T(E_1) \times L^1_T(E_2)$ of the operator F_ϵ.

2. RESULTS

In order to formulate our results we introduce first two measures of noncompactness. The first is defined as follows: given a bounded set $\Omega \subset C_T(E_1) \times L^1_T(E_2)$ we put

$$\overline{\chi}(\Omega) = \begin{pmatrix} \chi_{C_T(E_1)}(P_1(\Omega)) \\ \chi_w(P_2(\Omega)) \end{pmatrix},$$

where P_1 is the projector on the first coordinate of the Cartesian product $C_T(E_1)\times L^1_T(E_2)$, while P_2 is the projector on the second coordinate of the same space. The second measure of noncompactness is defined as

$$\overline{\nu}(\Omega) = \begin{pmatrix} \sup_t \, \chi_{E_1}(P_1(\Omega(t))) \\ \chi_w(P_2(\Omega)) \end{pmatrix}.$$

We can now prove the following.

Theorem 1: *Assume that the conditions* $(S_0), (A_0) \div (A_5)$ *are satisfied, then the operator* F_ϵ: $C_T(E_1) \times Q \to C_T(E_1) \times Q$ *is upper semicontinuous at any* $\epsilon \in [0,1]$. *Furthermore* F_ϵ *has nonempty, compact, convex values and it is* $(H, \overline{\chi}, \overline{\nu})$*-bounded with respect to all its variables, namely*

$$\overline{\chi}\left(\bigcup_{\epsilon\in[0,1]} F_\epsilon(\Omega)\right) \le H\overline{\nu}(\Omega) \tag{2}$$

for any bounded set $\Omega \subset C_T(E_1) \times L^1_T(E_2)$, *where the inequality is understood in the sense of the semiorder induced by the cone* $\mathbf{R}^2_+$.

Proof: Let Ω be a bounded set in $C_T(E_1) \times Q$, then $\Omega \subset P_1\Omega \times P_2\Omega$. By condition (A_4) we have

$$k_{11} \sup_t \chi_{E_1}(P_1\Omega(t)) + k_{12}\chi_w(P_2\Omega) \in WB(\Phi_1(P_1\Omega \times P_2\Omega_2)|_{[0,T]}).$$

Therefore for a given $\delta > 0$ there exist $e_\delta \subset [0,T]$, a compact $K_\delta \subset E_1$ and a set $G \subseteq L^1_T([0,T], E_1)$ such that meas $e_\delta < \delta$ and for all $g \in G$ one has $g(t) \in K_\delta$ for $t \in [0,T]\backslash e_\delta, g(t) = 0$ for $t \in e_\delta$. We can extend any $g \in G$ to $\mathbf{R}$ by T-periodicity. For simplicity, we still denote by g such an extension and by G the corresponding set. Observe that $\Lambda_1 G$ is relatively compact in $C_T(E_1)$. Let $f \in \Phi_1(P_1\Omega \times P_2\Omega)$ and evaluate $\| \Lambda_1 f(t) - \Lambda_1 g(t) \|_{E_1}$ as follows

$$\begin{aligned} \| \Lambda_1 f(t) - \Lambda_1 g(t) \|_{E_1} &\le \int_{-\infty}^t \mathrm{e}^{-\gamma_1(t-s)} \| f(s) - g(s) \|_{E_1} \, ds \\ \le \int_{-\infty}^t & e^{-\gamma_1(t-s)}(k_{11} \sup_t \chi(P_1\Omega(t)) + k_{12}\chi_w(P_2(\Omega)) \, ds \\ + \int_{-\infty}^t & e^{-\gamma_1(t-s)}\psi_\delta(s) \| f(s) \|_{E_1} \, ds, \end{aligned}$$

where ψ_δ is the characteristic function of the set $e_\delta + jT, j \in \mathbf{Z}$. Since the function $z(t) = \int_{-\infty}^t e^{-\gamma_1(t-s)}\psi_\delta(s)\| f(s) \|_{E_1} \, ds$ is T-periodic then it is sufficient to consider z only for $t \in [0,T]$. Let $t \in [0,T]$ and consider

$$\begin{aligned} z(t) = e^{-\gamma_1 t} \sum_{j=0}^{\infty} \int_{-(j+1)T}^{-jT} e^{\gamma_1 s}\psi_\delta(s) \| f(s) \|_{E_1} \, ds \\ + \int_0^t e^{-\gamma_1(t-s)}\psi_\delta(s) \| f(s) \|_{E_1} \, ds \end{aligned}$$

$$
\begin{aligned}
&= e^{-\gamma_1 t}\sum_{j=0}^{\infty} e^{-j\gamma_1 T}\int_0^T e^{-\gamma_1(T-s)}\psi_\delta(s)\| f(s)\|_{E_1}\,ds \\
&\quad + \int_0^t e^{-\gamma_1(t-s)}\psi_\delta(s)\| f(s)\|_{E_1}\,ds \\
&= e^{-\gamma_1 t}(1-e^{-\gamma_1 T})^{-1}\int_0^T e^{-\gamma_1(T-s)}\psi_\delta(s)\| f(s)\|_{E_1}\,ds \\
&\quad + \int_0^t e^{-\gamma_1(t-s)}\psi_\delta(s)\| f(s)\|_{E_1}\,ds \\
&\le \frac{2}{1-e^{-\gamma_1 T}}\int_0^T \psi_\delta(s)\| f(s)\|_{E_1}\,ds.
\end{aligned}
$$

Since $\delta > 0$ is arbitrary the last term is zero by (A_1). From (2) we obtain

$$
\begin{aligned}
&\chi_{C_T(E_1)}(\Lambda_1\Phi_1(P_1\Omega\times P_2\Omega)) \\
&\le \frac{k_{11}}{\gamma_1}\sup_t \chi_{E_1}(P_1\Omega(t)) + \frac{k_{12}}{\gamma_2}\chi_w(P_2\Omega).
\end{aligned}
$$

By the monotonicity of the Hausdorff measure of noncompactness we get

$$
\chi_{C_T(E_1)}(\Lambda_1\Phi_1(\Omega)) \le \chi_{C_T(E_1)}(\Lambda_1\Phi_1(P_1\Omega\times P_2\Omega))
$$

and so

$$
\chi_{C_T(E_1)}(\Lambda_1\Phi_1(\Omega)) \le \frac{k_{11}}{\gamma_1}\sup_t \chi_{E_1}(P_1\Omega(t)) + \frac{k_{12}}{\gamma_2}\chi_w(P_2\Omega). \tag{3}
$$

Let us now show that the operator F_ϵ: $C_T(E_1)\times Q \to C_T(E_1)\times Q$ is upper semicontinuous at $\epsilon = 0$. Observe that taking into account that the theory of the topological degree for condensing operators is constructed by means of the restriction of the involved operators to a fundamental compact set and the Theorem of Šmulian, we can verify the upper semicontinuity of F_ϵ on the sequences. For this, by assumption (A_2), it is enough to show that the linear operator $\Lambda(\epsilon)$: ${}^wL^1_T(E_2)\to {}^wL^1_T(E_2)$ is sequentially continuous with respect to ϵ. It is clear that it is continuous for $\epsilon > 0$, let us show that it is continuous at $\epsilon = 0$. To this aim, consider $g_n \to g_0$ weakly in $L^1_T(E_2)$ and $\epsilon_n \to 0$. Let v^* be the functional generated by the function

$$
y^*(t) = \sum_{i=0}^{m-1} y_i^*\psi_{[t_i,t_{i+1})}(t), \tag{4}
$$

where $y_i^* \in E_2^*$ and $0 = t_0 < t_1 < \cdots < t_m = T$, then

$$
\begin{aligned}
&\int_0^T \left\langle y^*(t), \frac{1}{\epsilon_n}\int_{-\infty}^t e^{(1/\epsilon_n)B(t-s)}g_n(s)\,ds\right\rangle dt \\
&= \sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} \left\langle y_i^*, \frac{1}{\epsilon_n}\int_{-\infty}^t e^{(1/\epsilon_n)B(t-s)}g_n(s)\,ds\right\rangle dt
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \left\langle y_i^*, \frac{1}{\epsilon_n} \int_{-\infty}^{t_i} e^{(1/\epsilon_n)B(t-s)} g_n(s)\, ds \right\rangle dt \\
&\quad + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \left\langle y_i^*, \frac{1}{\epsilon_n} \int_{t_i}^{t} e^{(1/\epsilon_n)B(t-s)} g_n(s)\, ds \right\rangle dt \\
&= \sum_{i=0}^{m-1} \int_{-\infty}^{t_i} \left\langle \frac{1}{\epsilon_n} \int_{t_i}^{t_{i+1}} e^{(1/\epsilon_n)B^*(t-s)} y_i^*\, dt, g_n(s) \right\rangle ds \\
&\quad + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \left\langle \frac{1}{\epsilon_n} \int_{s}^{t_{i+1}} e^{(1/\epsilon_n)B^*(t-s)} y_i^*\, dt, g_n(s) \right\rangle ds \\
&= \sum_{i=0}^{m-1} \int_{-\infty}^{t_i} \left\langle (B^*)^{-1} \left(e^{(1/\epsilon_n)B^*(t_{i+1}-s)} - e^{(1/\epsilon_n)B^*(t_i-s)} \right) y_i^*, g_n(s) \right\rangle ds \\
&\quad + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \left\langle (B^*)^{-1} \left(e^{(1/\epsilon_n)B^*(t_{i+1}-s)} - I \right) y_i^*, g_n(s) \right\rangle ds \\
&= \sum_{i=0}^{m-1} \int_{-\infty}^{t_{i+1}} \left\langle (B^*)^{-1} e^{(1/\epsilon_n)B^*(t_{i+1}-s)} y_i^*), g_n(s) \right\rangle ds \\
&\quad - \sum_{i=0}^{m-1} \int_{-\infty}^{t_i} \left\langle (B^*)^{-1} e^{(1/\epsilon_n)B^*(t_i-s)} y_i^*, g_n(s) \right\rangle ds \\
&\quad - \int_0^T \left\langle (B^*)^{-1} y^*(s), g_n(s) \right\rangle ds.
\end{aligned}
$$

We now prove that the first two terms tend to zero as $n \to \infty$. For this, note that, since $g_n \to g_0$ weakly in $L_T^1(E_2)$, we have that for any $\delta > 0$ there exists $\mu > 0$ such that for any set $e \subset [0, T]$ with meas$(e) < \mu$ it follows $\int_e \| g_n(s) \|_{E_2} ds < \delta$ for any $n \in \mathbf{N}$, (see [7]). Let $t \in [0, T]$, consider

$$
\begin{aligned}
&\int_{-\infty}^{t} \left\langle (B^*)^{-1} e^{(1/\epsilon_n)B^*(t-s)} y_i^*, g_n(s) \right\rangle ds \\
&= \int_{-\infty}^{0} \left\langle (B^*)^{-1} e^{(1/\epsilon_n)B^*(t-s)} y_i^*, g_n(s) \right\rangle ds \\
&\quad + \int_{0}^{t} \left\langle (B^*)^{-1} e^{(1/\epsilon_n)B^*(t-s)} y_i^*, g_n(s) \right\rangle ds \\
&= \int_{0}^{T} \left\langle (B^*)^{-1} e^{(1/\epsilon_n)B^*t} (I - e^{(1/\epsilon_n)B^*T})^{-1} e^{(1/\epsilon_n)B^*(T-s)} y_i^*, g_n(s) \right\rangle ds \\
&\quad + \int_{0}^{t} \left\langle (B^*)^{-1} e^{(1/\epsilon_n)B^*(t-s)} y_i^*, g_n(s) \right\rangle ds.
\end{aligned}
$$

We first write, $\int_0^T = \int_0^{T-\mu} + \int_{T-\mu}^T$ and $\int_0^t = \int_0^{t-\mu} + \int_{t-\mu}^t$. Then we can estimate by means of (S_0) these integrals and taking into account the observation above we can conclude that

$$
\int_{-\infty}^{t} \left\langle (B^*)^{-1} e^{(1/\epsilon_n)B^*(t-s)} y_i^*, g_n(s) \right\rangle ds \to 0
$$

as $n \to \infty$. We leave the details to the reader.

Finally, the last term tends to

$$-\int_0^T \langle y^*(s), B^{-1}g_0(s)\rangle \, ds.$$

In conclusion, we have weak convergence of $\frac{1}{\epsilon_n}\int_{-\infty}^t e^{(1/\epsilon_n)B(t-s)}g_n(s)\,ds$ to $-B^{-1}g_0$ with respect to the functionals v^*. On the other hand E_2 has the Radon-Nikodym property, then by (see Theorem 1 [7:p. 98]) we have that $(L_T^1(E_2))^* = L_T^\infty(E_2^*)$. Now we can approximate in the "almost every" convergence any function $y \in L_T^\infty(E_2^*)$ by a function of the form (4). For this it is possible to consider the continuous function

$$z_h(s) = \frac{1}{h}\int_s^h y(t)\,dt, \quad s \in [0, T]$$

which tends to $y(s)$ as $h \to 0$ for a.a. $s \in [0, T]$ (see Theorem 9 [7:p. 49]) and then approximate $z_h(\cdot)$ by step functions. Applying now Egorov's Theorem we finally obtain that for any $y \in L_T^\infty(E_2^*)$

$$\int_0^T \left\langle y(t), \frac{1}{\epsilon_n}\int_{-\infty}^t e^{(1/\epsilon_n)B(t-s)}g_n(s)\,ds\right\rangle dt$$

tends to

$$-\int_0^T \langle y(s), B^{-1}g_0(s)\rangle \, ds$$

as $n \to \infty$. Therefore, the operator $\Lambda(\epsilon)\Phi_2(x, y)$ is upper semicontinuous from $[0, 1] \times C_T(E_1) \times Q$ to Kv-$w(Q)$. Observe that if $C \subset L_T^1(E_2)$ is any weakly compact set then $\bigcup_{\epsilon\in[0,1]} \Lambda_2(\epsilon)C$ is weakly compact. To conclude the proof we show that $\Lambda_2(\epsilon)$, $\epsilon \in [0, 1]$, is a $\frac{1}{\gamma_2}$-contraction from $L_T^1(E_2)$ to $L_T^1(E_2)$. In fact, let $\epsilon > 0$, and consider

$$\begin{aligned}
&\int_0^T \frac{1}{\epsilon}\left\| \int_{-\infty}^t e^{B(t-s)}g(s)\,ds \right\|_{E_2} dt \le \int_0^T \frac{1}{\epsilon}\int_{-\infty}^t e^{-(1/\epsilon)\gamma_2(t-s)}\| g(s) \|_{E_2}\,ds\,dt \\
&= \int_0^T \frac{1}{\epsilon}\int_{-\infty}^0 e^{-(1/\epsilon)\gamma_2(t-s)}\| g(s) \|_{E_2}\,ds\,dt + \int_0^T \frac{1}{\epsilon}\int_0^t e^{-(1/\epsilon)\gamma_2(t-s)}\| g(s) \|_{E_2}\,ds\,dt \\
&= \int_{-\infty}^0 \frac{1}{\epsilon}\int_0^T e^{-(1/\epsilon)\gamma_2(t-s)}\| g(s) \|_{E_2}\,ds\,dt + \int_0^T \frac{1}{\epsilon}\int_s^T e^{-(1/\epsilon)\gamma_2(t-s)}\| g(s) \|_{E_2}\,ds\,dt \\
&= \frac{1}{\gamma_2}\left(\int_{-\infty}^0 e^{(1/\epsilon)\delta s}\| g(s) \|_{E_2}\,ds - \int_{-\infty}^0 e^{-(1/\epsilon)(T-s)}\| g(s) \|_{E_2}\,ds\,dt\right. \\
&\quad \left. + \int_0^T (1 - e^{-(1/\epsilon)\gamma_2(T-s)})\| g(s) \|_{E_2}\,ds\right) = \frac{1}{\gamma_2}\int_0^T \| g(s) \|_{E_2}\,ds.
\end{aligned}$$

Finally, for $\epsilon = 0$, since $B^{-1} = \int_0^{+\infty} e^{Bt}\,dt$, we have $\| B^{-1} \| \le \frac{1}{\gamma_2}$. In conclusion

$$\begin{aligned}
&\chi_w\left(\bigcup_{\epsilon\in[0,1]} \Lambda_2(\epsilon)\Phi_2(\Omega)\right) \le \frac{1}{\gamma_2}\chi_w(\Phi_2(\Omega)) \\
&\le \frac{k_{21}}{\gamma_2}\sup_t \chi_{E_1}(P_1\Omega(t)) + \frac{k_{22}}{\gamma_2}\chi_w(P_2\Omega).
\end{aligned} \tag{5}$$

In fact, if C is a weakly compact set which constitutes the α-net of $\Phi_2(\Omega)$, then $\bigcup_{\epsilon\in[0,1]}\Lambda_2(\epsilon)C$ is a weakly compact set which represents the $\frac{\alpha}{\gamma_2}$-net of $\bigcup_{\epsilon\in[0,1]}\Lambda_2(\epsilon)\Phi_2(\Omega)$.

From (3) and (5) we have (2). Inequality (2) implies that the values of F_ϵ are compact and as it is easy to see they are also convex and nonempty. This concludes the proof. □

As a consequence of the previous result we have the following.

Corollary 1: *Assume that the assumptions $(S_0), (A_0) \div (A_6)$ are satisfied, then the operator $(\epsilon, x, y) \to F_\epsilon(x, y)$ is $\overline{\chi}$-condensing with respect to all the variables.*

Proof: As it is easy to see $\overline{\nu}(\Omega) \le \overline{\chi}(\Omega)$. Therefore, if for a bounded set $\Omega \subset C_T(E_1) \times Q$ we have

$$\overline{\chi}(\Omega) \le \overline{\chi}\left(\bigcup_{\epsilon\in[0,1]} F_\epsilon(\Omega)\right) \tag{6}$$

then by Theorem 1 we obtain

$$\overline{\nu}(\Omega) \le H\overline{\nu}(\Omega). \tag{7}$$

Applying inequality (7) m times we get

$$\overline{\nu}(\Omega) \le H^m\overline{\nu}(\Omega). \tag{8}$$

By condition (A_6) we have that $\| H^m \| \to 0$ as $m \to \infty$, hence from (8) we derive $\overline{\nu}(\Omega) = 0$. If we apply Theorem 1 by means of (6) we have $\overline{\chi}(\Omega) \le H\overline{\nu}(\Omega) = 0$. Therefore $\overline{\chi}(\Omega) = 0$ and so Ω is relatively compact in $C_T(E_1) \times Q$. □

Now, using standard methods of the topological degree theory for multivalued condensing operators in locally convex spaces (see [4]) we can derive the following existence result for system (1).

Theorem 2: *Assume that condition (S_0) and $(A_0) \div (A_6)$ are satisfied and assume that there exists a relative, open, bounded set $U \subset C_T(E_1) \times Q$ for which*

$$\deg(I - F_0, \bar{U}) \ne 0.$$

Then there exists $\epsilon_0 > 0$ such that for all $\epsilon \in [0, \epsilon_0]$ the set Σ_ϵ of the solutions of system (1) belonging to the set U is nonempty and upper semicontinuous with respect to ϵ in the $C_T(E_1) \times {}^w L^1_T(E_2)$ topology.

3. EXAMPLE

In what follows we provide an example illustrating how the assumptions on the Nemytskii operators $\Phi_i, i = 1, 2$, presented in the previous Section can be verified. This will be done by specifying a possible choice and the properties of the nonlinear operators f_i, which generate $\Phi_i, i = 1, 2$. This example has been formulated having in mind a concrete application of our abstract results to a control problem in infinite dimensional spaces of the type of those considered in [?], as already pointed out in the Introduction. Specifically, we consider the following form for $f_i, i = 1, 2$.

$$f_1(t,x,y) = \psi_1(t,x) + b_{11}(x)y \tag{9}$$

$$f_2(t,x,y) = \psi_2(t,x) + b_{21}(x)y + b_{22}y. \tag{10}$$

We assume the following conditions.

(a_0) The multivalued operators ψ_i: $\mathbf{R} \times E_1 \to Kv(E_i)$ are T-periodic in t, and for any $x \in E_1$ there exists a selection $g(t) \in \psi_i(t,x)$, for a.a. $t \in \mathbf{R}$, belonging to $L^1_T(\mathbf{R})$, $i = 1, 2$.

(a_1) For a.a. $t \in \mathbf{R}$ the operators $\psi_i(t,\cdot), i = 1, 2$, are upper semicontinuous.

(a_2) There exist positive constants l_{i1} such that

$$\chi_{E_i}(\psi_i(t,\Omega)) \leq l_{i1}\chi_{E_i}(\Omega), \quad i = 1, 2.$$

(a_3) There exist positive constants M_i such that

$$\| \psi_i(t,x) \|_{E_i} \leq M_i, \quad i = 1, 2.$$

We now formulate the assumptions on the operators

$$b_{i1} : E_1 \to LK(E_2, E_i), \quad i = 1, 2, \tag{11}$$

where $LK(E_2, E_i)$ denotes the space of linear compact operators acting from E_2 to E_i.

(a_4) There exist positive constants $m_{i1}, i = 1, 2$, such that

$$\| b_{i1}(x) \|_i \leq m_{i1}$$

for any $x \in E_1$. Here $\| \cdot \|_i$ denotes the operator norm in $LK(E_2, E_i)$.

(a_5) The maps $x \to b_{i1}(x), i = 1, 2$, are continuous.

(a_6) There exists a positive constant l_{22} such that the bounded linear operator b_{22}: $E_2 \to E_2$ satisfies

$$\| b_{22} \| \leq l_{22}. \tag{12}$$

(a_7) Finally, we assume the following

$$\frac{l_{11}}{\gamma_1} < 1 \quad \text{and} \quad \frac{m_{22} + l_{22}}{\gamma_2} < 1 \tag{13}$$

Let $R > 0$ sufficiently large and let

$$Q_R := \{ y \in L^1_T(E_2) : \| y(t) \|_{E_2} \leq R, \quad \text{for a.a.} \quad t \in \mathbf{R} \}.$$

Now we prove that by (13) we get: $\Lambda_2(\epsilon)\Phi_2(C_T(E_1) \times Q_R)_R$. This was already noticed in Remark 1 (with $Q = Q_R$). For this, let $\epsilon > 0$ and for a.a. $t \in \mathbf{R}$ we have

$$\| \Lambda_2(\epsilon)\Phi_2(x,y)(t) \|_{E_2} \leq \frac{1}{\epsilon} \int_{-\infty}^{t} e^{-(1/\epsilon)\gamma_2(t-s)} [M_2 + (m_{22} + l_{22})R] \, ds$$
$$\leq \frac{M_2 + (m_{22} + l_{22})R}{\gamma_2}.$$

For $\epsilon = 0$, for a.a., $t \in \mathbf{R}$ we have

$$\| \Lambda_2(0)\Phi_2(x,y)(t) \|_{E_2} \leq \frac{M_2 + (m_{22} + l_{22})R}{\gamma_2}.$$

By (13) if $R > \frac{M_2}{\gamma_2 - m_{22} - l_{22}}$ then we get the conclusion.

Let us prove now that the multivalued operators Φ_i are upper semicontinuous from $C_T(E_1) \times {}^wL^1_T(E_2)$ to $Kv\text{-}w(L^1_T(E_i)), i = 1, 2$. First observe that under our assumptions on ψ_i we have that the associated Nemytskii operator are upper semicontinuous (see [13]). Therefore, it is sufficient to verify that the operators

$$(x, y) \to b_{i1}(x(\cdot))y(\cdot)$$

are continuous in the topologies which we have introduced in the previous section. From (a_5) we have that if $x_n \to x_0$ in $C_T(E_1)$ and $y_n \to y_0$ in ${}^wL^1_T(E_i)$ then

$$\langle y^*, b_{i1}(x_n)y_n\rangle \to \langle y^*, b_{i1}(x_0)y_0\rangle. \tag{14}$$

Let us now verify conditions (A_4) and (A_5). From (a_3) we have that for any $\Omega \subset C_T(E_1)$ we have

$$\chi_{E_i}(\psi_i(t, \Omega(t))) \leq l_{i1} \sup_t \chi_{E_1}(\Omega(t)), \quad i = 1, 2.$$

Therefore if

$$\Gamma_i(\Omega) := \{g : g \in L^1_T(E_i), g(t) \in \psi_i(t, x(t)), \text{ a.a. } t \in \mathbf{R} \textit{ and } x \in \Omega\},$$

then

$$\chi_{E_i}(\Gamma_i(\Omega)(t)) \leq l_{i1} \sup_t \chi(\Omega(t)).$$

Observe that $\chi_{E_i}(b_{i1}(\Omega_1(t))\Omega_2(t)) = 0$ and so by [5], (11) and (a_5) we have that for $\Omega_1 \subset C_T(E_1)$ and $\Omega_2 R$ we obtain

$$l_{i1} \sup_t \chi_{E_1}(\Omega_1(t)) \in WB(\Gamma_i(\Omega_1) + b_{i1}(\Omega_1)\Omega_2).$$

Consequently,

$$l_{i1} \sup_t \chi_{E_1}(\Omega_1(t)) \in \chi_{WB}(\Phi_1(\Omega_1 \times \Omega_2)).$$

Furthermore, observe that if $\| \Omega(t) \| \leq p(t), p \in L^1_T(\mathbf{R})$ and $\gamma \in \chi_{WB}$ then

$$\chi_w(\Omega) \leq \int_0^T \gamma(t)\, dt$$

and

$$\chi_w(b_{22}\Omega) \leq l_{22}\chi_w(\Omega)$$

we obtain (A_4) and (A_5) with $k_{11} = l_{11}, k_{12} = 0, k_{21} = T\, l_{21}$ and $k_{22} = l_{22}$.

Finally, it is immediate to see that all the possible solutions (x, y) of the reduced system at $\epsilon = 0$ are bounded as follows:

$$\| x \|_{C_T(E_1)} \leq M_1 + m_{11} R \quad \text{and} \quad \| y \|_{L_T(E_2)} \leq T\,R.$$

Hence

$$\deg(I - F_0, \bar{U}) \neq 0,$$

where $U = B_{C_T(E_1)}(0, \rho) \times Q_R$ with $\rho > M_1 + m_{11}$. This concludes the example.

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