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A Second Order Sliding Control Approach for Vibration Reduction

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Abstract

In this paper we discuss the problem of reducing vibrations of flexible systems. Flexible systems are characterized by an infinite number of under-damped resonances (modes), which can magnify the effect of narrow-band disturbancies with frequencies close to system modes. A possible way to overcome this drawback is to increase the damping at selected frequency locations. We implement this solution via a control design based on a second order sliding manifold approach. In particular, the stable sliding surface consists of a time-invarying part which increases the damping of selected modes without affecting the remaining dynamics, and a time-varying part which bounds the control amplitude in the

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transient response. Moreover, robustness issues are addressed, showing that the closed-loop system exhibits strong robustness properties typical of high-gain control systems.

1 Introduction

Vibration alleviation in flexible systems is one of the most challenging problems in Automatic Control field. In many applications (consider for instance aerospace, automotive and robotic systems) the need to reduce the weight of the vehicles or manipulators while assuring economically viable payloads calls for the use of light structures. However, in this case the elastic behaviour of the structures can no longer be neglected, and vibrations must be reduced to prevent excessive stress on the structures.

Passive control methods have been widely used to reduce unwanted vibrations. Unfortunately, the efficacy of passive strategies decreases with vibration frequency, so that the reduction of low frequency vibrations requires the use of active control techniques. Some particular characteristics of flexible systems must be addressed and possibly exploited in designing the control laws. The drawbacks are that flexible structures are infinite dimensional systems, e.g. they are described by means of linear and nonlinear partial differential operators. Thus they present a theoretically infinite number of resonance peaks and the associated damping factors are very small. Moreover, as in most physical systems, some significant parameters, like damping factors, moment of inertia, etc. are uncertain. The positive aspects are that flexible systems are passive, hence open-loop stable. This is one of the key point that will be exploited in this paper.

In order to define in more detail the specifics of the control, we must observe that the presence of system natural resonances amplifies the effects of (tonal) disturbances with frequencies close to the system natural frequencies. Hence, a possible control objective is to reduce the vibration corresponding to the first system natural frequencies, where, generally, the main part of the elastic energy is stored.

A great amount of work has been carried out by many authors [1]-[6], and different control strategies have been proposed taking into account particular aspects of the structures. Since the distributed structure of these systems is hard to treat both from a practical and mathematical point of view, the usual approach is to replace the partial differential model by an equivalent infinite set of ordinary differential equations, and then to consider only a finite set of system modes, thus neglecting the high frequency dynamics to obtain a finite-dimension mathematical description of the structures. Unfortunately, since sensors and actuators act both on modelled and neglected dynamics, the latter ones may be driven to instability (spillover phenomenon [1]). More specifically, the action exerted by actuators on the unmodelled dynamics is called control spillover, whereas the component of the unmodelled dynamics measured by the sensors is called observation spillover [5]. The presence of either kind of spillover degrades system performance, while the simultaneous action of both control and observation spillover may cause instability.

A possible strategy to avoid control spillover is to impose smooth control signals, with a limited rate of variation [6], so that the unmodelled dynamics are not affected by the control action.

In this paper, we will focus our attention on two of the topics discussed above. First, the control law will be designed to increase the closed-loop system robustness with respect to uncertain parameters. Second, in the case of narrow-band disturbances, since the worst situation occur when disturbance frequencies are close to system natural frequencies, our objective will be also to increase the system damping in particular frequency ranges, while leaving unaltered the remaining frequency regions so as to avoid spillover. The use of the smooth control law mentioned above is possible following the arguments given in [7] and [8].

More specifically, we propose a controller which is able to increase the damping factor of r selected modes of the structure (where r is the number of available inputs) leaving the remaining dynamics unaffected.

In [8] it has been shown that, by using a first-order sliding manifold approach and singular perturbation theory, the robustness properties of high gain control may be retained avoiding peaks in the control signal. In [7] it has been noted that, during the "reaching phase" (i.e. the transient in which the control signal approaches the equivalent control), the rate of variation can be limited so as to obtain a smooth control action. In [9], again in the singular perturbation framework, using the above results it has been shown that it is possible to synthesize a control signal which does not generate spillover. Although the first-order sliding strategy has proved effective, it employs a full-state feedback, hence deflection and velocity state variables must be available.

In this paper a second order sliding manifold approach is proposed and it is shown that by using piezoelectric actuators in self-sensing configuration [10], the deflection measures are sufficient to synthesize a control signal which allows the closed loop system to meet the given specifications (see Remark 5).

The use of a second order sliding control technique, in fact, in the general case is useful since it requires half the number of sensors as compared with the first order case. A preliminary version of this result can be found in [11].

2 System Modelling

In this paper we deal with the transversal vibration of a pinned-pinned beam in the configuration shown in Fig. 1. The system dynamics are described by the equations of linear elasticity as described by the following partial differential equation (1), whose derivation may be found by the interested reader in the standard texts on the subject, e.g. see [12].

$$\frac{\partial^2}{\partial z^2}\left[EI(z)\frac{\partial^2 v(z,t)}{\partial z^2}\right] + m(z)\frac{\partial^2 v(z,t)}{\partial t^2} = \sum_{j=1}^{r+r_d} f_j(z,t) \tag{1}$$

with boundary conditions

$$v(0,t) = 0, v(L,t) = 0, \left.\frac{\partial^2 v}{\partial z^2}\right|_{z=0} = 0, \left.\frac{\partial^2 v}{\partial z^2}\right|_{z=L} = 0 \tag{2}$$

where $v(z,t)$ is the deflection at abscissa z and time t, $f_j(z,t)$, $j = 1,\ldots,r+r_d$ external inputs: r the number of control inputs and r_d the number of disturbances, L the beam length, $m(z)$ the mass per unit of length, $I(z)$ the moment of inertia and E the Youngs modulus.

A technique usually used to deal with this class of problems, that we briefly recall for the sake of completeness, is to start by solving the associated homogeneous equation

$$\frac{\partial^2}{\partial z^2}\left[EI(z)\frac{\partial^2 v(z,t)}{\partial z^2}\right] + m(z)\frac{\partial^2 v(z,t)}{\partial t^2} = 0 \tag{3}$$

assuming that the solution $v(z,t)$ to be separable in time and space, and of the form

$$v(z,t) = V(z)\eta(t) \tag{4}$$

Substituting (4) into (3) we have:

$$\frac{1}{m(z)V(z)}\frac{d^2}{dz^2}\left[EI(z)\frac{d^2V(z)}{dz^2}\right] = -\frac{1}{\eta(t)}\frac{d^2\eta(t)}{dt^2} \tag{5}$$

Since the left hand side of (5) depends only on z and the right hand side only on t, equation (5) has a solution only if both sides are equal to a given positive constant, say ω^2 (for physical reasons we choose a *positive* constant).

Thus, solving equation (5) requires the solution of the following *eigenvalue problem*:

$$\frac{d^2}{dz^2}\left[EI(z)\frac{d^2V(z)}{dz^2}\right] - \omega^2 m(z)V(z) = 0 \tag{6}$$

$$\frac{d^2\eta(t)}{dt^2} + \omega^2\eta(t) = 0 \tag{7}$$

where, for each eigenvalue ω_i^2, $i = 1, 2, \ldots, +\infty$, we can find an associated eigenfunction $V_i(z)$ such that equation (6) has a nontrivial solution and the boundary conditions are satisfied.

Since the eigenfunctions are orthogonal, we can normalize them to obtain an orthonormal set satisfying the relations

$$\int_0^L m(z)V_i(z)V_s(z)dz = \delta_{is} \tag{8}$$

$$\int_0^L V_i(z)\frac{d^2}{dz^2}\left[EI(z)\frac{d^2V_s(z)}{dz^2}\right]dz = \omega_i^2\delta_{is}. \tag{9}$$

Now we can turn to the solution of the original problem (1). By using the expansion theorem, we assume the solution of (1) to be of the form

$$v(z,t) = \sum_{i=1}^{\infty} V_i(z)\eta_i(t), \tag{10}$$

Moreover, we assume that this solution satisfies the boundary conditions. If we replace this solution in (1) multiply by V_i and integrate over the beam length, keeping in mind (8) and (9) we have the infinite set of ODE's

$$\ddot{\eta}_i(t) + \omega_i^2 \eta_i(t) = \sum_{j=1}^{r+r_d} N_{ij}(t);\ i = 1, 2, \ldots, \infty \tag{11}$$

where $\eta_i(t)$ are the so-called *modal coordinates*, $\omega_i > 0$ the ith modal frequency and $N_{ij}(t)$ is the projection of $f_j(z,t)$ on the ith mode, i.e.

$$N_{ij}(t) = \int_0^L V_i(z) f_j(z,t) dz. \tag{12}$$

So far, we have derived the equations of an *undamped* model of the beam. From a physical point of view however, a (possibly small) amount of damping is always present. The above strategy can be used to solve equation where a *viscous damping* is also present. However, the usual practice is simply to add a damping term in the set of ODE's (11), thus obtaining

$$\ddot{\eta}_i(t) + 2\xi_i \omega_i \dot{\eta}_i(t) + \omega_i^2 \eta_i(t) = \sum_{j=1}^{r+r_d} N_{ij}(t);\ i = 1, 2, \ldots, \infty, \tag{13}$$

where ξ_i is the ith modal damping ratio, that is generally estimated experimentally.

The above discussion is also the starting point for the approximate solution of PDE's. In most practical cases the need to satisfy the boundary conditions makes the eigenvalue problem hard to solve. However, a number of approximate techniques (assumed modes, Rayleigh-Ritz method, Galerkin's method, finite element method) have been developed and are presented in many textbooks (see for instance [12]), all leading to a system of ODE's of the form (13).

In this paper, we assume point force disturbances located at abscissae z_j, $j = 1, \ldots, r_d$ and piezoelectric plate devices as actuators at abscissae z_j, $j = r_d + 1, \ldots, r_d + r$.

For the force disturbance then we have

$$f_j(z,t) = d_j(t)\delta(z - z_j), \quad j = 1, \ldots, r_d \tag{14}$$

hence

$$N_{ij}(t) = V_i(z_j) d_j(t), \qquad j = 1, \ldots, r_d, \tag{15}$$

where $V_i(z)$ is the normalized shape of the i-th mode and $d_j(t)$ is the time behavior of the disturbance.

For the control action, the piezoelectric plate devices exert on the structure on which they are bonded a couple of moments applied in the correspondence of the piezoelectric plates' end points (say the abscissae z_j^- and z_j^+). The effect of these devices on the structure is modelled by considering the equivalent force (see [10] for a complete derivation)

$$N_{ij}(t) = \left(\frac{d\,V_i(z_j^+)}{d\,z} - \frac{d\,V_i(z_j^-)}{d\,z}\right) u_j(t); \quad j = r_d + 1, \ldots, r_d + r \tag{16}$$

where $u_j(t)$ is the time behavior of the j-th control input.

From a practical point of view, infinite order models are very hard to treat, hence a *reduced order model* of the structure is obtained by projecting the distributed parameter system onto some n-order subspace H_n and neglecting the projection of the system onto the complement of H_n (*residual modes*, see [2]).

This can be done considering only n modes in the system of equations (13), say the first n after a possible arrangement of the variables. Thus the finite order system of ODE

$$\ddot{\eta}_i(t) + 2\xi_i\omega_i\dot{\eta}_i(t) + \omega_i^2\eta_i(t) = \sum_{j=1}^{r+r_d} N_{ij}(t); \; i = 1, 2, \ldots, n \tag{17}$$

is obtained.

By defining the state vector $x \in \mathbf{R}^{2n}$ as
$x = (x_1^T, x_2^T)^T = (\eta_1, \ldots, \eta_n, \dot{\eta}_1, \ldots, \dot{\eta}_n)^T$, the equations of the modeled dynamics may be summarized in the state space model

$$\dot{x} = Ax + B_u u + B_d d, \qquad x(0) = x_0 \tag{18}$$

where u is the r-vector, $0 < r \leq n$, of the control inputs and d the r_d-vector of disturbances. Moreover,

$$A = \begin{pmatrix} 0_{n\times n} & I_n \\ K & Q \end{pmatrix}, B_u = \begin{pmatrix} 0_{n\times r} \\ B_2 \end{pmatrix}, B_d = \begin{pmatrix} 0_{n\times r_d} \\ B_{d2} \end{pmatrix} \tag{19}$$

where $K \in \mathbf{R}^{n\times n}$ and $Q \in \mathbf{R}^{n\times n}$ are diagonal matrices, $B_2 \in \mathbf{R}^{n\times r}$ is a full rank matrix and $B_{d2} \in \mathbf{R}^{n\times r_d}$, $B_2 \in \mathbf{R}^{n\times r}$ and $B_{d2} \in \mathbf{R}^{n\times r_d}$ are the matrices obtained from (16) and (15) considering, respectively, the control inputs and the disturbances. In the sequel $Re\lambda_{\min}(X)$ will denote the minimum of the real part of the eigenvalues of the matrix X and $\mathcal{R}(X)$ its range. Moreover, for the sake of simplicity, we will omit the notation tr for the transpose for all the state vectors.

After a rearrangement of the indexes, if necessary, B_2 can be written in the form

$$B_2 = \begin{pmatrix} VT \\ T \end{pmatrix}, \tag{20}$$

where $T \in \mathbf{R}^{r\times r}$ is an invertible matrix and $V \in \mathbf{R}^{(n-r)\times r}$. The matrix B_2 is injective, hence there exists a left pseudoinverse $B_2^+ \in \mathbf{R}^{r\times n}$ of B_2, of the form

$$B_2^+ = \begin{pmatrix} 0_{r\times(n-r)} & T^{-1} \end{pmatrix}. \tag{21}$$

Clearly, $B_2B_2^+B_2B_2^+ = B_2B_2^+$ and so

$$B_2B_2^+ = \begin{pmatrix} 0_{(n-r)\times(n-r)} & V \\ 0_{r\times(n-r)} & I_r \end{pmatrix} \tag{22}$$

is a projection. It turns out that $P = (I - B_2B_2^+)$ is a projection onto $\mathcal{R}(B_2)^\perp$. Let

$$\mathcal{P} = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}. \tag{23}$$

Observe that if we write $x_1 = (x_1^1, x_1^2) \in \mathbf{R}^{n-r} \times \mathbf{R}^r$ and $x_2 = (x_2^1, x_2^2) \in \mathbf{R}^{n-r} \times \mathbf{R}^r$ then $(I-\mathcal{P})(x_1, x_2) = ((Vx_1^2, x_1^2), (Vx_2^2, x_2^2))$, where the vector (x_1^2, x_2^2) corresponds in (19) to the r selected modes x_1^2 to be controlled.

Finally, we consider the matrix A_1 defined as follows

$$A_1 = \begin{pmatrix} 0_{(n-r)\times(n-r)} & I_{n-r} \\ K_1 & Q_1 \end{pmatrix} \tag{24}$$

where $K_1, Q_1 \in \mathbf{R}^{(n-r)\times(n-r)}$ are the diagonal matrices having as diagonal entries the first $(n-r)$ diagonal entries of the matrices K and Q

respectively. In other words A_1 corresponds to the dynamics of the state (x_1^1, x_2^1) in the open loop system (18).

3 Control Strategy

In this section we propose a second order sliding strategy that we will use to control the vibrating system (18). For this, first we define a suitable sliding manifold as follows

$$\mathcal{S} = \{(x_1, t) \in \mathbf{R}^n \times \mathbf{R}_+ \ : \ s(x_1, t) = 0\} \tag{25}$$

where $s : \mathbf{R}^n \times \mathbf{R}_+ \to \mathbf{R}^r$ is given by

$$s(x_1, t) = H\left(-x_1 + (I\ 0)e^{Wt}x_0\right) \tag{26}$$

$x_0 = x(0)$ and $H \in \mathbf{R}^{r \times n}$, $W \in \mathbf{R}^{2n \times 2n}$, with

$$W = \begin{pmatrix} 0 & I \\ W_1 & W_2 \end{pmatrix} \tag{27}$$

are real matrices to be chosen in the following Theorem in a convenient way. Here all the submatrices of W are $n \times n$. In the sequel the function $s(x_1, t)$ defined by (26) will be simply denoted by s.

Then we introduce the control law $u(t) \in \mathbf{R}^r$ as the solution of the second order differential equation

$$\epsilon \ddot{u} = \frac{\partial s}{\partial x_1}\dddot{x}_1 + \frac{\partial^2 s}{\partial t^2} + \Gamma\left(\frac{\partial s}{\partial t} + \frac{\partial s}{\partial x_1}\dot{x}_1\right) + Ms \tag{28}$$

where ϵ is a positive "small" scalar parameter and $\Gamma, M \in \mathbf{R}^{r \times r}$ are real matrices such that equation (28) at $\epsilon = 0$ be stable. The meaning of this assumption will be clarified in Remark 5. Observe that this equation can be rewritten as

$$\ddot{s} + \Gamma\dot{s} + Ms = 0 \tag{29}$$

where the dot denotes the total derivative with respect to the time.

The following theorem shows that the control strategy (28) is stabilizing for the system, allows damping of the r selected x_1^2 modes and ensures robustness against parameter uncertainties.

Theorem. Consider the system of differential equations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ K & Q \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ B_2 \end{pmatrix} u + \begin{pmatrix} 0 \\ B_{d2} \end{pmatrix} d \tag{30}$$

$$\epsilon \dot{u} = \frac{\partial s}{\partial x_1}\ddot{x}_1 + \frac{\partial^2 s}{\partial t^2} + \Gamma\left(\frac{\partial s}{\partial t} + \frac{\partial s}{\partial x_1}\dot{x}_1\right) + Ms \tag{31}$$

with $(x(0), u(0)) = (x_0, u_0) \in \mathbf{R}^{2n} \times \mathbf{R}^r$.

Let γ and δ be given positive real numbers. Assume that $H = UB_2^+$ and $U \in \mathbf{R}^{r\times r}$ satisfies

$$(i) \qquad Re\lambda_{\min}(U) > 0.$$

Moreover, assume that

$$(ii) \qquad Re\lambda_{\min}(-W) > \gamma,$$

and the following matching condition

$$(iii) \qquad \mathcal{R}(B_{d2}) \subseteq \mathcal{R}(B_2).$$

Then there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$, $(x_1(t,\epsilon), x_2(t,\epsilon), u(t,\epsilon))$ the solution of (30)–(31) is such that

$$\|x(t,\epsilon)\| \le \delta + ae^{-\sigma t} \quad \textit{for any} \quad t \in [0, +\infty), \tag{32}$$

where $\sigma = \min\{\gamma, Re\lambda_{\min}(-A_1)\} > 0$, a is a positive constant depending on the data and

$$u(t,\epsilon) = \frac{1}{\epsilon}\left\{\frac{d}{dt}s(x(t,\epsilon),t) + \Gamma s(x(t,\epsilon),t) + M\int_0^t s(x(\tau,\epsilon),\tau)d\tau\right\} + u_0. \tag{33}$$

Furthermore, the controlled part $(I - \mathcal{P})x(t,\epsilon)$ of the state $x(t,\epsilon)$ of system (30)–(31) satisfies the inequality

$$\|(I - \mathcal{P})x(t,\epsilon)\| \le \delta + \hat{a}e^{-\gamma t} \quad \textit{for any} \quad t \in [0, +\infty), \tag{34}$$

where $\hat{a}$ is a positive constant depending on the data.

Proof. The proof consists essentially in showing that under our assumptions we can apply Tikhonov's Theorem (see Appendix) for singularly perturbed systems on the infinite time horizon. For this, it is sufficient to prove the stability of the boundary layer and that of the reduced order system corresponding to (30)–(31). That is conditions $(H6)$ and $(H7)$ of the Appendix. In fact, the remaining conditions are easily verified by means of the linearity of the differential equations (30)–(31) and of the invertibility of the matrix U. For this, first we compute the so-called equivalent control u_{eq} by letting $\epsilon = 0$ in (31) and solving the corresponding algebraic equation with respect to u. This equation is uniquely solvable in virtue of the assumption of nonsingularity of the $r \times r$ matrix U. In fact, it turns out that

$$\begin{aligned} u_{eq} = -U^{-1}[(HK + MH)x_1 + (HQ + \Gamma H)x_2 + HB_{d2}d - \\ - (H(W_1\ W_2) + \Gamma H(0\ I) + MH(I\ 0))\, e^{Wt} x_0] \end{aligned} \tag{35}$$

For fixed $(x, t) \in \mathbf{R}^{2n} \times \mathbf{R}_+$, $x = (x_1, x_2)$, consider the boundary layer problem

$$\dot{z} = -Uz - b, \quad z(0) = u_0, \tag{36}$$

where $b = b(x,t)$ is the term in (35) contained in the square brackets. Since, by assumption (i), $-U$ is a Hurwitz matrix it is easy to show that the solution $z(t)$ tends to $u_{eq}(x,t)$ as $t \to +\infty$. Therefore the zero solution of the boundary layer is exponentially stable.

Consider now the *reduced order system* obtained by replacing the equivalent control u_{eq} in (30).

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 \\ \dot{\tilde{x}}_2 &= \left(PK - B_2U^{-1}MUB_2^+\right)\tilde{x}_1 + \left(PQ - B_2U^{-1}\Gamma UB_2^+\right)\tilde{x}_2 + \\ &\quad + PB_{d2}d + B_2U^{-1}Le^{Wt}x_0, \end{aligned} \tag{37}$$

where $P = (I - B_2B_2^+)$ and $PB_{d2}d = 0$ by the matching condition (iii) and

$$L = [H(W_1\ W_2) + \Gamma H(0\ I) + MH(I\ 0)].$$

Observe that the last term in (37) converges exponentially to zero as $t \to +\infty$ with exponential rate $-\gamma$ by assumption (ii). We show that the equilibrium point $(\tilde{x}_1, \tilde{x}_2) = (0,0)$ of the reduced system (37) is exponentially stable. For this, observe that if $(\tilde{x}_1(t), \tilde{x}_2(t))$ is the solution of (37) satisfying $(\tilde{x}_1(0), \tilde{x}_2(0)) = x_0 \in \mathbf{R}^{2n}$ then $s(\tilde{x}_1(t), t) = 0$ for any $t \geq 0$. In fact, at $\epsilon = 0$ we have $s(\tilde{x}_1(0), 0) = \dot{s}(\tilde{x}_1(0), 0) = 0$ and so the unique corresponding solution of $\ddot{s} + \Gamma\dot{s} + Ms = 0$ is the zero solution.

Therefore

$$H\left(-\tilde{x}_1(t) + (I\ 0)e^{Wt}x_0\right) = 0 \tag{38}$$

and

$$H\left(-\dot{\tilde{x}}_1(t) + (0\ I)e^{Wt}x_0\right) = 0 \tag{39}$$

for any $t \geq 0$.

On the other hand, $H = H(I-P)$ and so (38) and (39) are equivalent respectively to

$$H\left[-(I-P)\tilde{x}_1(t) + (I-P)(I\ 0)e^{Wt}x_0\right] = 0 \tag{40}$$

and

$$H\left[-(I-P)\tilde{x}_2(t) + (I-P)(0\ I)e^{Wt}x_0\right] = 0 \tag{41}$$

for any $t \geq 0$. But $H = UB_2^+$, with U nonsingular, therefore Ker $H =$ Ker $B_2^+ =$ Ker $B_2B_2^+$, hence

$$\begin{aligned} (I-P)\tilde{x}_1(t) &= (I-P)(I\ 0)e^{Wt}x_0 \\ (I-P)\tilde{x}_2(t) &= (I-P)(0\ I)e^{Wt}x_0. \end{aligned} \tag{42}$$

Namely, by assumption (ii), we have that

$$\lim_{t\to\infty}(I-P)\tilde{x}_1(t) = \lim_{t\to\infty}(I-P)\tilde{x}_2(t) = 0 \tag{43}$$

exponentially with exponential rate $-\gamma$.

Furthermore, $((I-P)\tilde{x}_1(t), (I-P)\tilde{x}_2(t))$ is the solution of the Cauchy problem

$$\begin{aligned} &(I-P)\dot{\tilde{x}}_1 = (I-P)\tilde{x}_2, \\ &(I-P)\dot{\tilde{x}}_2 = -B_2U^{-1}MH(I-P)\tilde{x}_1 - B_2U^{-1}\Gamma H(I-P)\tilde{x}_2 + \\ &\qquad +B_2U^{-1}Le^{Wt}x_0, \\ &((I-P)\tilde{x}_1(0), (I-P)\tilde{x}_2(0)) = ((I-P)x_{1,0}, (I-P)x_{2,0}) \end{aligned} \tag{44}$$

where $(x_{1,0}, x_{2,0}) = x_0 \in \mathbf{R}^{2n}$. In fact, $(I-P) = B_2 B_2^+$, $(I-P)B_2 = B_2$ and $(I-P)P = 0$. Note that the right hand side of the second equation is equal to $(I-P)(W_1, W_2)e^{Wt}x_0$ as can be proved by using (42).

Moreover, from (37) by applying P we obtain

$$\begin{aligned} P\dot{\tilde{x}}_1 &= P\tilde{x}_2 \\ P\dot{\tilde{x}}_2 &= P(K\tilde{x}_1 + Q\tilde{x}_2) \end{aligned} \tag{45}$$

or equivalenty

$$\begin{aligned} P\dot{\tilde{x}}_1 &= P\tilde{x}_2 \\ P\dot{\tilde{x}}_2 &= PKP\tilde{x}_1 + PQP\tilde{x}_2 + PK(I-P)\tilde{x}_1 + PQ(I-P)\tilde{x}_2. \end{aligned} \tag{46}$$

Here, in virtue of the previous analysis the term $PK(I-P)\tilde{x}_1 + PQ(I-P)\tilde{x}_2$ tends to zero exponentially as $t \to \infty$ with exponential rate $-\gamma$. On the other hand, the system

$$\begin{aligned} P\dot{x}_1 &= Px_2 \\ P\dot{x}_2 &= PKPx_1 + PQPx_2 \end{aligned} \tag{47}$$

is exponentially stable. Indeed, we have that $Px_1 = (x_1^1 - Vx_1^2, 0)$ and $Px_2 = (x_2^1 - Vx_2^2, 0)$, with $x_1^1, x_2^1 \in \mathbf{R}^{n-r}$. It is immediate to verify that $PKP = KP$ and $PQP = QP$, since K and Q are diagonal matrices. Therefore the dynamics of the nonzero components of the vector (Px_1, Px_2) is governed by the matrix

$$A_1 = \begin{pmatrix} 0_{(n-r)\times(n-r)} & I_{n-r} \\ K_1 & Q_1 \end{pmatrix}, \tag{48}$$

where $K_1, Q_1 \in \mathbf{R}^{(n-r)\times(n-r)}$ are the diagonal matrices having as diagonal entries the first $(n-r)$ diagonal entries of the matrices K and Q respectively. On the other hand, the matrix A is an Hurwitz matrix with eigenvalues $\lambda_i = \omega_i(-\zeta_i \pm \sqrt{\zeta_i^2 - 1})$, where $\omega_i, \zeta_i > 0$, for any $i = 1, \ldots, n$. Hence A_1 is also a Hurwitz matrix and the solution $(P\tilde{x}_1(t), P\tilde{x}_2(t))$ of the nonhomogeneous system (46) satisfying the initial condition $(P\tilde{x}_1(0), P\tilde{x}_2(0)) = (Px_{1,0}, Px_{2,0})$ is such that

$$||(P\tilde{x}_1(t), P\tilde{x}_2(t))|| \to 0$$

as $e^{-\sigma t}$, where $\sigma = \min\{\gamma, Re\lambda_{\min}(-A_1)\}$.

Therefore, for $\epsilon > 0$ sufficiently small, say $\epsilon < \epsilon_0$, we have for any $t \geq 0$

$$\|x(t,\epsilon)\| \leq \|x(t,\epsilon) - \tilde{x}(t)\| + \|\tilde{x}(t)\| \leq \delta + ae^{-\sigma t} \tag{49}$$

for some costant $a > 0$ depending on the data. In fact, the inequality $\|x(t,\epsilon) - \tilde{x}(t)\| < \delta$ for any $t \geq 0$ follows from application of the Tikhonov's Theorem in the time interval $[0, +\infty)$, while the inequality $\|\tilde{x}(t)\| \leq ae^{-\sigma t}$, $t \geq 0$ follows from the exponential stability of the reduced order system.

Finally, from (42) by using the same argument it follows that for ϵ sufficiently small we have for any $t \geq 0$

$$\|(I - \mathcal{P})x(t,\epsilon)\| \leq \delta + \hat{a}e^{-\gamma t} \tag{50}$$

for some constant $\hat{a} > 0$ depending on the data. Hence $\|(x_1^2(t,\epsilon), x_2^2(t,\epsilon))\|$ also satisfies the inequality above. This concludes the proof.

Remark 1. As apparent from expression (33) our control strategy turns out to be a robust high-gain PID feedback. Indeed, we find the three actions of a PID controller, namely the derivative, proportional and integral action, while the term $1/\epsilon$ is responsible for the gain increase. However, a classic high-gain control strategy exhibits an initial peak in the control signal, while in our strategy $u(0) = u_0$.

Remark 2. It is easy to show that the unmodeled residual dynamics is excited by the modeled dynamics due to the feedback action. However, since the modeled dynamics are not affected by the residual modes, no destabilizing spillover phenomenon occurs, and the closed-loop system is stable even in the presence of unmodeled dynamics.

Remark 3. Following the guidelines in [7], it is possible to limit the initial rate of variation of the control signal to reduce the high-frequency components of the input and the excitation of high-frequency modes.

Remark 4. In the proof of the theorem, we have shown that the last r modes are controlled, while the first $n - r$ modes are unaffected by the control action. This result is based on the decomposition (20) of the matrix B_2. Obviously, different decompositions are possible: for instance, if we want to dampen the first r modes and leave the remaining $n - r$

modes unaffected, we can write B_2 as

$$B_2 = \begin{pmatrix} T \\ VT \end{pmatrix}, \tag{51}$$

and compute the pseudo-inverse in (21) as

$$B_2^+ = \begin{pmatrix} T^{-1} & 0_{r\times(n-r)} \end{pmatrix} \tag{52}$$

In this case, the proof of the theorem applies in a straightforward way. In general, we can always extract m linearly independent rows from the matrix B_2 and compute the appropriate pseudo-inverse. Thus, we can select the modes to dampen by using the suitable expression for the pseudo-inverse and for the matrix H.

Remark 5. Since in our strategy we have $s(x_1(0),0) = \dot{s}(x_1(0),0) = 0$, equation (29) ensures that $s(x_1(t),t) = \dot{s}(x_1(t),t) = 0$ for all $t \geq 0$, hence the proposed approach is a second-order sliding strategy. Note that even if the initial state $x_0 = (x_{0,1}, x_{0,2})$ is not exactly known, due to the stability assumption on equation (29) we have

$$\lim_{t\to\infty} s(x_1(t),t) = \lim_{t\to\infty} \dot{s}(x_1(t),t) = 0 \tag{53}$$

This situation occours, for istance, in the case when the second component $x_{0,2}$ of the initial state x_0 is not available due to the lack of velocity measurements. Observe that we can choose the matrices M and Γ to impose the rate of the exponential decay in the limits above.

4 Simulation Results

In this section we show the properties of the proposed controllers on a uniform cross section beam with the following numerical values

$$L = 3\ [m];\ m = 19.61\ [kg/m];\ I = 6.28 \cdot 10^{-7}\ [m^4];$$

$$\xi_i = 0.01 \text{ for any } i;\ E = 2.07 \cdot 10^{11}\ [N/m^2]$$

The model used for the control design has been obtained as shown in Section 2 considering the first 4 modes.

nominal (4-modes)	8-modes
$-0.9 \pm 89.3j$	$-0.9 \pm 89.3j$
$-3.6 \pm 357.1j$	$-3.6 \pm 357.1j$
$-8 \pm 803.6j$	$-8 \pm 803.6j$
$-14.3 \pm 1428j$	$-14.3 \pm 1428j$
	$-22.3 \pm 2232j$
	$-32.1 \pm 3214j$
	$-43.8 \pm 4375j$
	$-57.1 \pm 5714j$

Table 1: Eigenvalues of the 4-modes beam model (nominal) and of the 8-modes model.

As shown in Fig. 1, there are three disturbances acting at abscissae $z = \{L/6,\ L/2,\ 6L/7\}$ and two piezoelectric plate actuators of length $\Delta_z = 30$ $[mm]$ located at abscissae $z = \{L/4,\ 3L/5\}$; moreover, for output deflection measurements we assume the presence of two piezoelectric plates in self-sensing configuration.

The disturbances, expressed in Newton, [N], acting on the beam are

$$\begin{aligned} d_1(t) &= 30\,(1 + 0.1 \sin 0.2t) \sin 85t + \sin(320t + \sin 0.1t) && (54)\\ d_2(t) &= 30\,(1 + 0.1 \sin 0.2t) \sin\left(85t + \frac{\pi}{2}\right) \\ &\quad + \sin(320t + \sin 0.1t) && (55)\\ d_3(t) &= 30\,(1 + 0.1 \sin 0.2t) \sin\left(85t + \frac{\pi}{4}\right) \\ &\quad + \sin\left(320t + \frac{\pi}{3} + \sin 0.1t\right) && (56) \end{aligned}$$

The fundamental frequencies is $\omega_{d1} = 85$ $[rad/s]$, $\omega_{d2} = 320$ $[rad/s]$, they are close to the system's first two resonances, respectively (see Table 1).

In implementing our control strategy, we have used the following parameters

$$\epsilon = 10^{-3};\ U = I;\ M = \begin{pmatrix} 90 & 0 \\ 0 & 1200 \end{pmatrix};$$

$$\Gamma = \begin{pmatrix} 9 & 0 \\ 0 & 36 \end{pmatrix}$$

The matrix W has been selected as the solution of an LQ problem [14] for the plant (18) with weighting matrices $Q_{LQ} = I_8$, $R_{LQ} = I_2$.

The controller designed on the 4-modes model has been tested on a more accurate 8-modes model. The output deflections are reported in Figs. 2 and 3.

In Fig. 4 open and close loop maximum singular values are compared. It is evident that the controller increases damping factor of the first two modes leaving unaffected the other dynamics.

5 Conclusions

In this paper the design of an active control strategy to reduce the vibration of flexible systems has been presented. The controller is derived via a second order sliding manifold methodology, and its objective is to increase the damping of lightly damped flexible structures in the frequency ranges where narrow-band disturbancies may significatively excite the structure. Moreover, robustness issues are addressed, resulting in a proposed strategy with strong robustness properties like those of high-gain systems, but without the peaking phenomenon typical of the latter.

The effectiveness of the proposed strategy is tested via computer simulations in the case of a pinned-pinned beam, and the controller is first designed on a reduced order model of the beam, then tested on a higher order model, taking into account neglected high-order dynamics. The results show that the damping is increased in selected frequency ranges, while the remaining modes are unaffected, which is a desirable result in controlling spillover-prone structures.

Work in progress addresses both the limitation of the initial rate of the control and the implementation of the proposed control strategy via an output feedback, controlling the whole infinite-dimensional system.

6 Appendix: Tikhonov's Theorem

In this appendix we recall the statement of Tikhonov's Theorem on the infinite time horizon. For a complete presentation and proof we refer to [13].

Consider the system

$$\begin{aligned} \dot{x} &= f(t,x,y,\epsilon) \quad x(0)=x_0 && (57)\\ \epsilon\dot{y} &= g(t,x,y,\epsilon) \quad y(0)=y_0 && (58) \end{aligned}$$

where $x \in \mathbf{R}^N$, $y \in \mathbf{R}^M$, $t \in [0,+\infty)$ and $\epsilon > 0$ is a "small" parameter. Let $\rho > 0$ and $B_\rho = \{(x,y) \in \mathbf{R}^N \times \mathbf{R}^M : |x| + |y| \le \rho\}$.

We assume the following conditions.

(H1) The system (57) has an unique solution $x = x(t), y = y(t)$ defined on $[0,+\infty)$.

(H2) $f, g, f_x, f_y, g_t, g_x, g_y \in C([0,+\infty) \times B_\rho \times [0,\epsilon_0])$, for any $\rho > 0$. Here f_x denotes the Jacobian matrix $(\partial f_i/\partial x_j)$, $i,j = 1,2,\cdots n$, and similarly for f_y, g_x and g_y.

(H3) There exists a bounded, twice continuously differential function $y = y^*(t,x)$ such that $g(t,x,y^*(t,x),0) = 0$ for all $(t,x) \in [0,+\infty)\times B^x_\rho$, where B^x_ρ is the restriction of B_ρ to $\mathbf{R}^N$.

(H4) There exists $\mu > 0$ such that, if $(t,x) \in [0,+\infty) \times B^x_\rho$, $|v - y^*(t,x)| < \mu$, and $v \neq y^*(t,x)$, then $g(t,x,v,0) \neq 0$.

Observe that there is no loss of generality in assuming that $y^*(t,x) = 0$ for all $(t,x) \in [0,+\infty) \times B_\rho$. Indeed, if this is not the case, the trasformation

$$\begin{aligned} x &= w\\ y &= z + y^*(t,w) \end{aligned}$$

takes the system (57) into

$$\begin{aligned}
\dot{w} &= f(t,w,z+y^*(t,w),\epsilon) = F(t,w,z,\epsilon) \\
\epsilon\dot{z} &= g(t,w,z+y^*(t,w),\epsilon) - \\
&- \epsilon[y_t^*(t,w) + y_x^*(t,w)\, f(t,w,z+y^*(t,w),\epsilon)] = G(t,w,z,\epsilon)
\end{aligned}$$

which is of the same form as (57) with $G(t,w,0,0) = 0$ for all $[0,+\infty) \times B_\rho^x$.

Finally, we assume the following conditions which are now formulated for $y^*(t,x) = 0$ for all $(t,x) \in [0,+\infty) \times B_\rho^x$.

(H5) The function f is continuous at $y = 0$, $\epsilon = 0$ uniformly in $(t,x) \in [0,+\infty) \times B_\rho^x$ and $f(t,x,0,0)$ and $f_x(t,x,0,0)$ are bounded in $[0,+\infty) \times B_\rho^x$. Moreover, the function g is continuous at $\epsilon = 0$ uniformly in $(t,x,y) \in [0,+\infty) \times B_\rho$ and $g(t,x,y,0)$ and its derivatives g_t, g_x, g_y are bounded on $[0,+\infty) \times B_\rho$.

(H6) The zero solution of the boundary layer, i.e. of the equation $\dot{\zeta} = g(t_0,x_0,\zeta,0)$, with $(t_0,x_0) \in [0,+\infty) \times B_\rho^x$, is uniform-asymptotically stable uniformly with respect to the parameters $(t_0,x_0) \in [0,+\infty) \times B_\rho^x$.

(H7) The origin is an uniformly asymptotically stable equilibrium point of $\dot{x} = f(t,x,0,0)$.

For the convenience of the reader we give the following.

Definition. A point $(t_0,x_0,y_0) \in [0,+\infty] \times \mathbf{R}^N \times \mathbf{R}^M$ such that the solution $\zeta(t,t_0,y_0)$ of the Cauchy problem

$$\begin{aligned}
\dot{\zeta} &= g(t_0,x_0,\zeta) \\
\zeta(t_0) &= y_0
\end{aligned}$$

satisfies $\lim_{t\to+\infty}\zeta(t,t_0,y_0) = 0$ is said to belong to the domain of influence of the zero solution of the boundary layer.

Now we can state the Tikhonov's Theorem.

Tikhonov's Theorem. Suppose that assumptions (H1)–(H6) are satisfied. Let $(0,x_0,y_0) \in [0,+\infty) \times \mathbf{R}^N \times \mathbf{R}^M$ be a point in the domain of influence of $\overline{y} = y^*(0, x_0)$. Then the solution $(x(t,\epsilon), y(t,\epsilon))$ of the Cauchy problem

$$\begin{aligned} (\dot{x}, \epsilon\dot{y}) &= (f(t,x,y), g(t,x,y)) \\ (x(0), y(0)) &= (x_0, y_0) \end{aligned}$$

has the following properties

$$\lim_{\epsilon \to 0} x(t,\epsilon) = x_0(t) \quad \text{uniformly in } [0, +\infty], \tag{59}$$

$$\lim_{\epsilon \to 0} y(t,\epsilon) = y_0(t) \quad \text{uniformly in } [t_1, +\infty] \tag{60}$$

whenever $0 < t_1 < +\infty$. Here $(x_0(t), y_0(t))$, $t \in [0, +\infty)$, is the solution of the "reduced" system

$$\begin{aligned} \dot{x} &= f(t,x,y) \\ x(0) &= x_0 \\ 0 &= g(t,x,y) \end{aligned}$$

so that in particular $y_0(t) = y^*(t, x_0(t))$.

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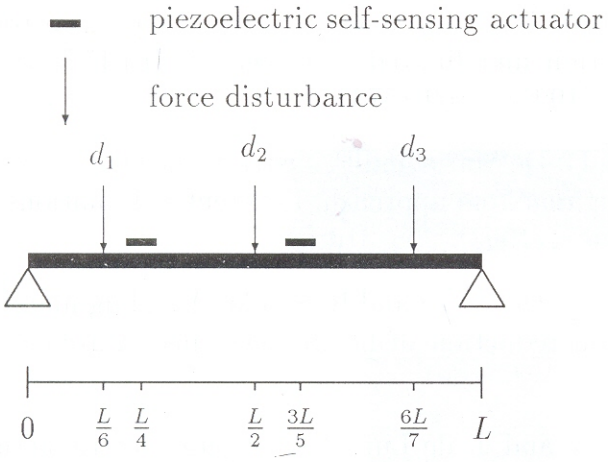

Figure 1: Flexible Structure.

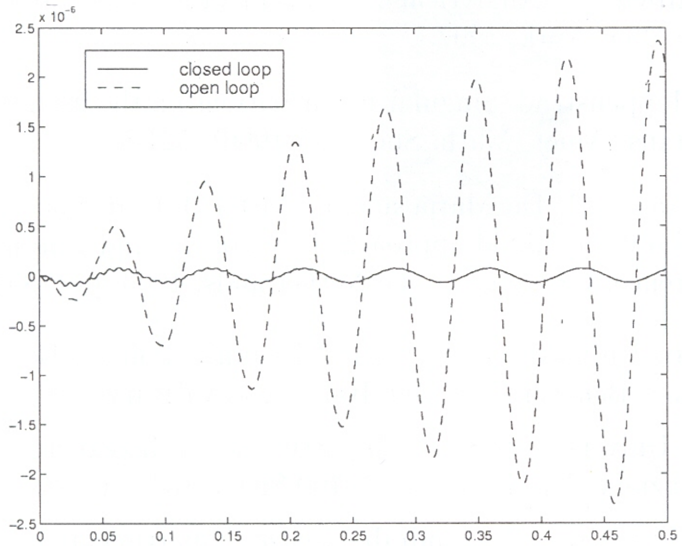

Figure 2: Open-loop and closed-loop deflection at abscissa $z = L/4$.

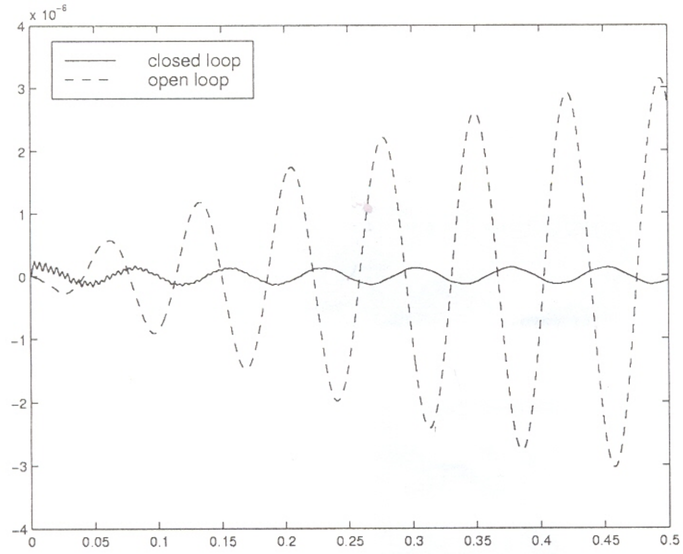

Figure 3: Open-loop and closed-loop deflection at abscissa $z = 3L/5$.

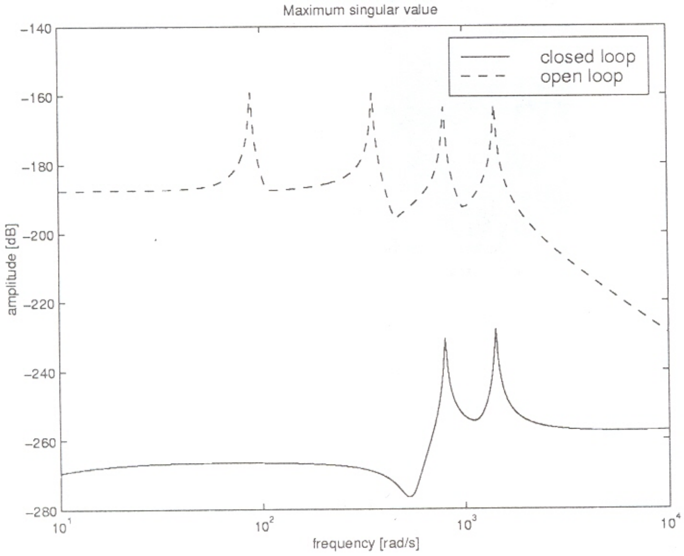

Figure 4: Maximum singular values of 4-modes model open loop (dashed) and closed loop (solid).