

FEEDBACK STABILITY OF CLOSED SETS FOR NONLINEAR CONTROL SYSTEMS*

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Dedicated to Alfonso Vignoli on the occasion of his 60th anniversary

Summary: Using methods from the multivalued analysis we show the existence of feedback controls which “stabilizes” a given closed set K of \mathbf{R}^n , which satisfies a suitable regularity property, with respect to the dynamics of a nonlinear control system. For this, we study the structure and the properties of the external contingent Bouligand cone to K and we use a suitable selection theorem of the regulation map associated to the system.

Keywords: φ -convex set, proximate retract, sleek set, external contingent Bouligand cone, regulation map, feedback control.

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0 INTRODUCTION

In this paper we consider an autonomous nonlinear control system described by the differential equations

$$\dot{x} = f(x, u), \tag{1}$$

where f is a continuous function. The state variable x belongs to \mathbf{R}^n and the control variable u belongs to a set $U(x) \subset \mathbf{R}^m$ depending on x . The assumptions on the multivalued map $x \rightarrow U(x)$ will be precised later. The control problem for system (1) that we consider in this paper is in the class of control problems involving sliding manifolds. Specifically, the problem that we consider here can be formulated as follows.

- * Given a “sufficiently” smooth, nonempty, closed set $K \subset \mathbf{R}^n$. Given a suitable neighborhood I of K and an initial state $x_0 \in I \setminus K$. We want, by means of a control u which takes value in $U(x)$, $x \in I \setminus K$, to steer in finite time and then hold the state x of system (1) in a prescribed ϵ -neighborhood of K .

The set K represents the required behaviour of the controlled dynamics. In the case when K can be defined as the zeros of a continuously differentiable map $s : \mathbf{R}^n \rightarrow \mathbf{R}^m$, $m \leq n$, $s = (s_j)_{j=1}^m$ and one uses feedback control laws $u = u(x)$ which are discontinuous along the surfaces $s_j(x) = 0$, usually $u_j(x) = -\text{sgn } s_j(x)$, $j = 1, 2, \dots, m$, then this is the classical nonlinear variable structure control problem for which there is a very broad literature, see e.g. the monographs [16] and [17].

Another approach for solving the proposed problem is based on the theory of singularly perturbed ordinary differential equations and it was proposed in [7]. It consists in defining a dynamical feedback controller as the solution of a differential equation containing a small parameter $\epsilon > 0$. This equation is directly derived from the dynamics f and the function s . In fact, the states corresponding to such controls enter any prescribed neighborhood of K and remain in it for all the future times. One of the advantage of this approach is that these controls are absolutely continuous and so the corresponding dynamics (1) does not require any regularization, for instance in the sense of Filippov or Krasowski, as in the case of discontinuous feedback controls. Furthermore, the so-called chattering phenomenon, which is one of the main drawbacks of discontinuous feedback controls, is eliminated.

In this paper we aim at solving the above problem in a different way, that is we will use tools and methods of the multivalued analysis, in particular we will consider the external contingent Bouligand cone $\tilde{T}_K(x)$ to the set K , $x \in I$, and the selection theory for multivalued maps. Indeed, multivalued analysis is of great relevance for the study of control problems: viability, invariance and stability problems for system (1), among many others, are widely investigated by methods of the multivalued analysis, see the monographs [1], [2], [3], [4], [14] and the paper [12]. One of the main difficulties here is to establish the lower semicontinuity of the external contingent Bouligand cone $\tilde{T}_K(x)$ at any $x \in I$, in order to use a selection result from [5] for a suitably defined regulation map associated to (1). This permits to derive the existence of a trajectory of (1) starting from x_0 and having the required

property. We would like also to point out that we do not require any particular conditions on (1), like f to be affine in u , in order to ensure the lower semicontinuity of the considered regulation map. In fact, the employed selection result does not require such a condition. Therefore the advantages of our approach are related to the generality of the control system (1) and of the control set $U(x)$.

The paper is organized as follows. In Section 1, for a large class of sets K , Proposition 1 provides the lower semicontinuity of the cone-valued map $x \rightarrow \tilde{T}_K(x)$, $x \in I$. For this, we prove a suitable representation for the cone $\tilde{T}_K(x)$, which is known in the case when K is a convex set. In Proposition 2 and 3 under different assumptions we show the lower semicontinuity of suitable cone-valued selections $\tilde{T}_{K,\gamma}(x)$ of $\tilde{T}_K(x)$, $x \in I$, which permit to define the suitable regulation map for (1) as the intersection of $\tilde{T}_{K,\gamma}(x)$ with the set of velocities $F(x) := f(x, U(x))$, $x \in I$. Finally, in Section 2 we show how to solve the proposed problem. Indeed, by using both a selection result of [5] for the regulation map and a result for the approximation in graph of multivalued maps of [11], we prove in Theorem 2 the existence of a trajectory $x = x(t)$, $t \geq 0$, $x(0) = x_0 \in I$, of (1) corresponding to a control $u(t) \in U(x(t))$ which reaches in finite time a prescribed neighborhood of K and remains in it for all future times. Moreover, we also outline other possible controllability results which can be obtained by means of the proposed procedure.

1 EXTERNAL CONTINGENT BOULIGAND CONE

In this section we establish some relevant properties of the external contingent Bouligand cone for a class of nonempty, closed sets of \mathbf{R}^n . These properties will be employed in the next section for solving the control problem for (1) stated above. We start with some definitions.

Definition 1 Let $K \subset \mathbf{R}^n$ be nonempty. Let $x \in \overline{K}$, the closure of K . The contingent Bouligand cone $T_K(x)$ is defined by

$$T_K(x) := \{v \in \mathbf{R}^n : \liminf_{\tau \rightarrow 0^+} \frac{d_K(x + \tau v)}{\tau} = 0\}.$$

Here $d_K(\cdot)$ is the distance function defined as $d_K(x) := \inf_{y \in K} d(x, y)$ where $d(x, y) = |x - y|$ and $|\cdot|$ denotes the Euclidean norm in \mathbf{R}^n . It turns out

that $T_K(x)$ is a closed cone. If the cone-valued map $x \rightarrow T_K(x)$ is lower semicontinuous at any $x \in K$, the set K is said to be sleek. Finally, the polar cone of $T_K(x)$ is the normal cone to K , denoted by $N_K(x)$.

Definition 2 Let $K \subset \mathbf{R}^n$ be a nonempty, closed set. Denote by $\pi_K(\cdot)$ the metric projection on K defined by

$$\pi_K(x) := \{y \in K : |x - y| = d_K(x)\}.$$

We assume the following conditions on K .

- (C) There exists an open neighborhood I of K in \mathbf{R}^n such that for any $x \in I \setminus K$ the metric projection $\pi_K(x)$ is single valued.

Remark 1 In ([10], Theorem 4.8) for a nonempty, closed set $K \subset \mathbf{R}^n$ satisfying (C) it is shown that $x \rightarrow \pi_K(x)$ is continuous in I and locally Lipschitz in $I \setminus K$ together with $\text{grad } d_K(x)$. Furthermore, in ([10], Theorem 4.18) it is proved, in an implicit way, that condition (C) is equivalent to the following property.

- (Γ) There exists a continuous function $\varphi : K \times K \rightarrow \mathbf{R}_+$ such that for all $x, y \in K, v \in N_K(x)$ we have that $\langle v, y - x \rangle \leq \varphi(x, y)|v| |y - x|^2$.

Here $\langle \cdot, \cdot \rangle$ denotes the inner product in \mathbf{R}^n . A nonempty, closed set satisfying this property is called φ -convex set (see [6]). In [8] it is shown that a φ -convex set of an infinite dimensional Hilbert space satisfies (C) with $x \rightarrow \pi_K(x)$ continuous in I . Furthermore, a nonempty, closed set K of a Banach space which satisfies (C) with $\pi_K(x) : I \rightarrow K$ continuous (metric retraction) is called proximate retract (see e.g. [5], [13] and [15]). Therefore, in a finite dimensional space, the class of φ -convex sets coincides with that of proximate retracts. It is still an open question if a proximate retract in an infinite dimensional Hilbert space is φ -convex.

Remark 2 Note that any φ -convex set K is sleek (see [9]). Moreover, convex sets as well as sets with $C^{1,1}$ -boundary are φ -convex.

For any $x \in I$, consider the set $K(x)$ defined as follows:

$$K(x) := K + d_K(x)B_1,$$

where $B_1 \subset \mathbf{R}^n$ denotes the unit closed ball centered at the origin. We pose the following problem.

(P) To show that the cone-valued map defined in I by

$$x \rightarrow T_{K(x)}(x)$$

is lower semicontinuous at any $x \in I$, i.e. the set $K(x)$ is sleek at any $x \in I$.

To solve problem (P) is one of the aims of this section.

Remark 3 Note that by ([3], Theorem 4.1) we have that if $x \rightarrow T_{K(x)}(x)$ is lower semicontinuous at x then the cone $T_{K(x)}(x)$ is convex.

The first step to solve (P) is to prove the following result which is know in the case when K is convex ([3], p. 141).

Lemma 1 *Assume that K is a nonempty, closed set satisfying condition (C). Then*

$$T_{K(x)}(x) = T_K(\pi_K(x)) + T_{d_K(x)B_1}(x - \pi_K(x)), \quad x \in I.$$

Proof: Obviously if $x \in K$ there is nothing to prove. Therefore let $x \in I \setminus K$ and assume that $v \in T_{K(x)}(x)$, this means that there exist sequences $\tau_n \rightarrow 0+$, $v_n \rightarrow v$ such that $x_n := x + \tau_n v_n \in K(x)$, i.e. $d_{K(x)}(x_n) = 0$. By property (C) and Remark 1, there exists $l = l(x) > 0$ such that $|y - y_n| \leq l|x_n - x|$ for n sufficiently large with $y = \pi_K(x)$ and $y_n = \pi_K(x_n)$. On the other hand for $n \in \mathbf{N}$ there exists $b_n \in B_1$ such that $x_n = y_n + |x - y|b_n$, since $|x_n - y_n| = d_K(x_n) \leq d_K(x)$. In fact, from $x_n \in K(x)$ it follows that there exist $\hat{y}_n \in K$ and $\hat{b}_n \in B_1$ such that $x_n = \hat{y}_n + d_K(x)\hat{b}_n$, thus $d_K(x_n) = d_K(\hat{y}_n + d_K(x)\hat{b}_n) \leq d_K(\hat{y}_n) + d_K(x)|\hat{b}_n| \leq d_K(x)$. Furthermore, by passing to a subsequence if necessary, we have that $b_n \rightarrow b$, where $b = \frac{(x - y)}{|x - y|}$. Rewrite $x_n = x + \tau_n v_n$ as follows

$$x + \tau_n v_n = y + |y_n - y| \frac{y_n - y}{|y_n - y|} + (x - y) + |x - y| |b_n - b| \frac{b_n - b}{|b_n - b|}, \quad (2)$$

where $b_n := \frac{x_n - y_n}{|x - y|}$. Observe that $y_n = y + |y_n - y| \frac{y_n - y}{|y_n - y|} \in K$, with $|y_n - y| \rightarrow 0+$ and $|x - y|b_n = (x - y) + |x - y| |b_n - b| \frac{b_n - b}{|b_n - b|} \in d_K(x)B_1$

with $|x - y| |b_n - b| \rightarrow 0+$. Then $w_0 := \lim_{n \rightarrow \infty} \frac{y_n - y}{|y_n - y|} \in T_K(y)$ and $z_0 := \lim_{n \rightarrow \infty} \frac{b_n - b}{|b_n - b|} \in T_{d_K(x)B_1}(x - y)$.

On the other hand from (2) we have

$$v_n = \frac{|y_n - y|}{\tau_n} \frac{y_n - y}{|y_n - y|} + \frac{|x - y| |b_n - b|}{\tau_n} \frac{b_n - b}{|b_n - b|}, \quad (3)$$

with $\frac{|y_n - y|}{\tau_n} \leq l|v_n| \leq M$ for n sufficiently large. Thus by passing to a subsequence if necessary, we have

$$\frac{|y_n - y|}{\tau_n} \rightarrow \alpha_0 \geq 0 \quad \text{and} \quad \frac{|x - y| |b_n - b|}{\tau_n} \rightarrow \beta_0 \geq 0$$

obtaining from (3)

$$v = \alpha_0 w_0 + \beta_0 z_0 \in T_K(y) + T_{d_K(x)B_1}(x - y).$$

Viceversa, let us prove now that from $w \in T_K(y)$ and $z \in T_{d_K(x)B_1}(x - y)$, where $y = \pi_K(x)$ and $x \in I \setminus K$, it follows that $w + z \in T_{K(x)}(x)$. Observe that, without loss of generality, we can assume that $|z| = |x - y|$. Assume $w \in T_K(y)$ then there exist sequences $\hat{\tau}_n \rightarrow 0+$ and $w_n \rightarrow w$ such that $y + \hat{\tau}_n w_n \in K$. We show now the existence of a sequence $z_n \rightarrow z$, such that $|z_n| = |x - y|$ and

$$(x - y) + \hat{\tau}_n z_n \in d_K(x)B_1.$$

In fact, for any $z \in T_{d_K(x)B_1}(x - y)$, with $|z| = |x - y|$, and any sequence $\tau_n \rightarrow 0+$ there exists a sequence $z_n \rightarrow z$ such that $|z_n| = |x - y|$ for any $n \in \mathbf{N}$, and

$$(x - y) + \tau_n z_n \in d_K(x)B_1,$$

for n sufficiently large. In particular,

$$(x - y) + \hat{\tau}_n z_n \in d_K(x)B_1.$$

In conclusion,

$$y + \hat{\tau}_n w_n + (x - y) + \hat{\tau}_n z_n = x + \hat{\tau}_n (w_n + z_n) \in K(x)$$

with $\hat{\tau}_n \rightarrow 0+$, thus $w + z \in T_{K(x)}(x)$. This completes the proof. \square

Remark 4 As a consequence of Lemma 1 and the closedness of $T_{K(x)}(x)$ we have that, in our case, the set

$$T_K(\pi_K(x)) + T_{d_K(x)B_1}(x - \pi_K(x)), \quad x \in I,$$

is closed.

Definition 3 Let $\gamma > 0$ and K as in Lemma 1, for any $x \in I \setminus K$, we define the closed affine cone $\tilde{T}_{K,\gamma}(x)$ as follows

$$\tilde{T}_{K,\gamma}(x) := \{v \in \mathbf{R}^n : \liminf_{\tau \rightarrow 0^+} \frac{d_K(x + \tau v) - d_K(x)}{\tau} \leq -\gamma\}.$$

For simplicity, we denote by $\tilde{T}_K(x)$ the external contingent Bouligand cone $\tilde{T}_{K,0}(x)$. Observe that if $x \in K$ then $\tilde{T}_{K,0}(x)$ reduces to $T_K(x)$.

We can now prove the following result.

Lemma 2 $\tilde{T}_K(x) = T_{K(x)}(x)$ for any $x \in I$.

Proof: Let $v \in T_{K(x)}(x)$, i.e. $\liminf_{\tau \rightarrow 0^+} \frac{d_{K(x)}(x + \tau v)}{\tau} = 0$. On the other hand $d_{K(x)}(x + \tau v) = \inf_{z \in K(x)} d(x + \tau v, z)$, where $z = \zeta + d_K(x)b$, $\zeta \in K$, $b \in B_1$. Thus

$$d_{K(x)}(x + \tau v) \geq d_K(x + \tau v) - d_K(x),$$

since $d(x + \tau v, \zeta + d_K(x)b) \geq d(x + \tau v, \zeta) - d_K(x)$, for any $\zeta \in K$ and any $b \in B_1$. In conclusion

$$0 \geq \liminf_{\tau \rightarrow 0^+} \frac{d_K(x + \tau v) - d_K(x)}{\tau},$$

which implies $v \in \tilde{T}_K(x)$. Viceversa, assume that $v \in \tilde{T}_K(x)$ hence there exist sequences $\tau_n \rightarrow 0^+$, $v_n \rightarrow v$ such that

$$d_K(x + \tau_n v_n) \leq d_K(x). \quad (4)$$

Let $y_n = \pi_K(x + \tau_n v_n)$, thus $|x - y_n + \tau_n v_n| = d_K(x + \tau_n v_n)$ and by (4) we get $|x - y_n + \tau_n v_n| \leq |x - y_n|$ for any $n \in \mathbf{N}$. This implies that $x + \tau_n v_n \in K(x)$, since $x + \tau_n v_n \in y_n + d_K(x)B_1$. Consequently, we obtain $d_{K(x)}(x + \tau_n v_n) = 0$ for any $n \in \mathbf{N}$, hence $v \in T_{K(x)}(x)$. This completes the proof. \square

We have the following result.

Proposition 1 *Assume that $K \subset \mathbf{R}^n$ is a nonempty, closed set which satisfies condition (C). Then the cone-valued map $x \rightarrow \bar{T}_K(x)$ is lower semicontinuous at any $x \in I$ with nonempty, closed, convex values.*

Proof: The proof easily follows from Lemmas 1 and 2, taking into account Remarks 1 and 3. \square

Definition 4 Let $\psi : \text{Dom}(\psi) \rightarrow \mathbf{R}$ be a function and $x \in \text{Dom}(\psi)$. We define the contingent epiderivative $D_{\uparrow}\psi(x)(v)$ of ψ at x in the direction v as follows

$$D_{\uparrow}\psi(x)(v) := \liminf_{\substack{\tau \rightarrow 0^+ \\ v' \rightarrow v}} \frac{\psi(x + \tau v') - \psi(x)}{\tau}.$$

As pointed out in Remark 1, if K is a nonempty, closed set satisfying condition (C) then $x \rightarrow d_K(x)$ is continuously differentiable in $I \setminus K$. Therefore, in this case we have that

$$D_{\uparrow}d_K(x)(v) = \langle \text{grad } d_K(x), v \rangle$$

for any $x \in I \setminus K$ and $v \in \mathbf{R}^n$. Furthermore, if K is any nonempty, closed set of \mathbf{R}^n and $x_0 \in \mathbf{R}^n \setminus K$ is such that $x \rightarrow d_K(x)$ is differentiable at x_0 then $\pi_K(x_0)$ is a singleton (see [10]). Finally, since the distance function $x \rightarrow d_K(x)$, $x \in \mathbf{R}^n$, is Lipschitz of constant 1, then by ([3], Proposition 6.1.7) it follows that for any $x \in \mathbf{R}^n$

$$(i) \quad D_{\uparrow}d_K(x)(v) = \liminf_{\tau \rightarrow 0^+} \frac{d_K(x + \tau v) - d_K(x)}{\tau}, \text{ for any } v \in \mathbf{R}^n.$$

$$(ii) \quad |D_{\uparrow}d_K(x)(v)| \leq |v|, \text{ for any } v \in \mathbf{R}^n.$$

In particular from (ii) it follows that d_K is contingently epiderivable, i.e. $D_{\uparrow}d_K(x)(0) = 0$ at any $x \in \mathbf{R}^n$, or equivalently $D_{\uparrow}d_K(x)(v) > -\infty$ at any $x \in \mathbf{R}^n$ whenever $v \in \mathbf{R}^n$. Finally, observe that if the epigraph of $d_K : E_p(d_K)$ is sleek with $\text{Dom } d_K = D$, i.e. $x \rightarrow T_{E_p(d_K)}(x, d_K(x))$ is lower semicontinuous at any $x \in D$, then the cone-valued map $x \rightarrow E_p(D_{\uparrow}d_K(x))$ is also lower semicontinuous with closed, convex values at any $x \in D$; in fact $T_{E_p(d_K)}(x, d_K(x)) = E_p(D_{\uparrow}d_K(x))$, see ([3], proposition 6.1.4).

The following two results are in the spirit of those of ([1], Section 9.4.4).

Proposition 2 *Assume that $K \subset \mathbf{R}^n$ is a nonempty, closed set which satisfies condition (C). Moreover, assume that for γ sufficiently small and any $x \in I \setminus K$ there exists $\bar{v} \in \mathbf{R}^n$ such that $D_{\uparrow}d_K(x)(\bar{v}) < -\gamma$. Then the cone-valued map $x \rightarrow \tilde{T}_{K,\gamma}(x)$, $x \in I \setminus K$, is lower semicontinuous with nonempty, closed, convex values.*

Proof: For $x \in I \setminus K$, let us define the map

$$\widehat{T}_{K,\gamma}(x) := \{v \in \mathbf{R}^n : D_{\uparrow}d_K(x)(v) < -\gamma\}.$$

By assumption this set is nonempty. We show that the Graph $\widehat{T}_{K,\gamma}$ is open, where $\text{Dom } \widehat{T}_{K,\gamma} = I \setminus K$. For this, let $(x, v) \in \text{Graph } \widehat{T}_{K,\gamma}$, for any sequence $(x_n, v_n) \rightarrow (x, v)$ we have that

$$\lim_{n \rightarrow \infty} D_{\uparrow}d_K(x_n)(v_n) = D_{\uparrow}d_K(x)(v) < -\gamma.$$

Furthermore, the set $\widehat{T}_{K,\gamma}(x)$ is convex, since $D_{\uparrow}d_K(x)(v) = \langle \text{grad } d_K(x), v \rangle$. Finally, consider the map

$$x \rightarrow \tilde{T}_K(x) \cap \widehat{T}_{K,\gamma}(x) = \widehat{T}_{K,\gamma}(x), \quad x \in I \setminus K,$$

it is lower semicontinuous, since it is the intersection of the lower semicontinuous map $x \rightarrow \tilde{T}_K(x)$ (Proposition 1) with a map having open graph. Furthermore, it is convex valued and so $x \rightarrow \widehat{T}_{K,\gamma}(x) = \widehat{T}_{K,\gamma}(x)$ is also lower semicontinuous. As a consequence $x \rightarrow \tilde{T}_{K,\gamma}(x)$ is lower semicontinuous at any $x \in I \setminus K$. \square

Finally, we can also prove the following result, which provides the same conclusion of Proposition 2 under different assumptions.

Proposition 3 *Let $K \subset \mathbf{R}^n$ be a nonempty, closed set. Let I be a neighborhood of K . Assume that $E_p(d_K)$ is sleek with $\text{Dom } d_K = I$, assume that for $\gamma > 0$ sufficiently small and any $x \in I \setminus K$ there exists $\bar{v} \in \mathbf{R}^n$ such that $D_{\uparrow}d_K(x)(\bar{v}) < -\gamma$. Then the cone-valued map $x \rightarrow \tilde{T}_{K,\gamma}(x)$, $x \in I \setminus K$, is lower semicontinuous with nonempty, closed, convex values.*

Proof: Let $x_n \rightarrow x$ and $v \in \tilde{T}_{K,\gamma}(x)$, hence $(v, -\gamma) \in E_p(D_{\uparrow}d_K(x))$. By the lower semicontinuity of the map $x \rightarrow E_p(D_{\uparrow}d_K(x))$ there exist sequences $v_n \rightarrow v$ and $\epsilon_n \geq 0$, $\epsilon_n \rightarrow 0$ such that

$$(v_n, -\gamma - \epsilon_n) \in E_p(D_{\uparrow}d_K(x_n)).$$

Let $\bar{v} \in \tilde{T}_{K,\gamma}(x)$, hence $\bar{u} = -\gamma - D_{\uparrow}d_K(x)(\bar{v}) > 0$ and so $(\bar{v}, -\gamma - \bar{u}) \in E_p(D_{\uparrow}d_K(x))$. Therefore there exist sequences $\bar{v}_n \rightarrow \bar{v}$ and $\epsilon'_n > 0$, $\epsilon'_n \rightarrow 0$ such that

$$(\bar{v}_n, -\gamma - \bar{u} + \epsilon'_n) \in E_p(D_{\uparrow}d_K(x_n)).$$

Define now $w_n = (1 - \theta_n)v_n + \theta_n\bar{v}_n$, where $\theta_n = \frac{\epsilon_n}{2(\epsilon_n + \epsilon'_n)} \in [0, 1]$. We have that

$$(w_n, -\gamma - \epsilon_n/2) = (1 - \theta_n)(v_n, -\gamma - \epsilon_n) + \theta_n(\bar{v}_n, -\gamma - \bar{u} + \epsilon_n).$$

Since $E_p(D_{\uparrow}d_K(x_n))$ is convex it follows that $(w_n, -\gamma - \epsilon_n/2) \in E_p(D_{\uparrow}d_K(x_n))$, that is $D_{\uparrow}d_K(x_n)(w_n) \leq -\gamma - \epsilon_n/2$ and so $w_n \in \tilde{T}_{K,\gamma}(x_n)$ with $w_n \rightarrow v$. Moreover, $\tilde{T}_{K,\gamma}(x)$ is convex. Indeed, the fact that $E_p(d_K)$ is sleek implies the upper semicontinuous of the map $(x, v) \rightarrow D_{\uparrow}d_K(x)(v)$, $x \in I \setminus K$ and $v \in \mathbf{R}^n$. Now, for $v_1, v_2 \in \tilde{T}_{K,\gamma}(x)$, define $E = \{\theta \in [0, 1] : D_{\uparrow}d_K(x)(\theta v_1 + (1 - \theta)v_2) < -\gamma\}$. It is nonempty by assumption, moreover it is easy to verify that E is both closed and open relatively to $[0, 1]$ and so $E = [0, 1]$. \square

Remark 5 To the best knowledge of the authors the characterization of the nonempty, closed sets K for which $E_p(d_K)$ is sleek is an open question.

2 CONTROL PROBLEMS

In this section we consider the autonomous nonlinear control system (1). We assume the following conditions

- (A₀) $f : \bar{I} \times \mathbf{R}^m \rightarrow \mathbf{R}^n$ is a continuous map and I is an open set of \mathbf{R}^n .
The control parameter u belongs to the set $U(x)$ depending on $x \in I$.
The set-valued map $x \rightarrow U(x) \subset \mathbf{R}^m$ is upper semicontinuous with nonempty, compact, convex values.

We aim at solving the following control problem for system (1).

- (CP) Given a nonempty, closed set $K \subset \mathbf{R}^n$ satisfying condition (C). Given $x_0 \in I \setminus K$, with I neighborhood of K assigned by condition (C). We want to show the existence of a trajectory $x = x(t)$, $t \geq 0$, $x(0) = x_0 \in I \setminus K$, of the control system (1) corresponding to a control $u(t) \in U(x(t))$, $t \geq 0$, which reaches a prescribed neighborhood of K in finite time and remains in it for all future times.

Observe that here problem (CP) is formulated having in mind Proposition 1 and 2 as tools to solve it. Clearly, if one wants to use Proposition 3 then K and I are the sets satisfying the conditions of Proposition 3. Moreover, in this case we also assume that K is sleek.

Consider the set of velocities

$$F(x) := f(x, U(x)), \quad x \in I.$$

It results that the map $x \rightarrow F(x)$, $x \in I$, is upper semicontinuous with nonempty, compact values. Furthermore we assume the following condition.

(A₁) $F(x)$ is a convex set for any $x \in I$.

Definition 5 For $x \in I \setminus K$, we define the regulation map associated to (1) as follows

$$R_{K,\gamma}(x) := F(x) \cap \tilde{T}_{K,\gamma}(x).$$

We assume that

(A₂) $R_{K,\gamma}(x)$ is nonempty for any $x \in I \setminus K$ and for $\gamma > 0$ sufficiently small.

Proposition 4 Under assumptions (A₀)-(A₂) and the assumptions of Proposition 2, for any $\delta > 0$ there exists a continuous map $g : I \rightarrow \mathbf{R}^n$ such that

- (i) $g(x) \in \tilde{T}_{K,\gamma}(x)$ for any $x \in I \setminus K$;
- (ii) g is a δ -approximation in graph of the map F . That is $\text{Graph}(g + \delta B_1) \subseteq \text{Graph}(F) + \delta B_1$.

Proof: Under our assumption the regulation map $x \rightarrow R_{K,\gamma}(x)$, $x \in I \setminus K$, has nonempty, compact, convex values. Thus the proof is a direct consequence of Lemma 5.1 and Remark 5.2 of [5]. \square

The following result provides the relevant behaviour of the trajectories of the dynamical system $\dot{x} = g(x)$.

Theorem 1 Under assumptions of Proposition 4 any solution of the initial value problem

$$\begin{cases} \dot{x} = g(x) \\ x(0) = x_0, \quad x_0 \in I \setminus K, \end{cases} \quad (5)$$

reaches the set K in finite time $t_0 > 0$ and remains in it for all $t \geq t_0$. Moreover $t_0 \leq d_K(x_0)/\gamma$.

Proof: Let $x = x(t)$ be any solution of the initial value problem (5). Let $t \geq 0$ and $d(t) := d_K(x(t))$. Observe that $t \rightarrow d_K(x(t))$ is an absolutely continuous function. For $\tau > 0$ we can write $x(t + \tau) = x(t) + \tau\dot{x}(t) + \tau\alpha(\tau)$, where $\alpha(\tau) \rightarrow 0$ as $\tau \rightarrow 0$. We have that

$$\begin{aligned} \dot{d}(t) &= \lim_{\tau \rightarrow 0^+} \frac{d_K(x(t + \tau)) - d_K(x(t))}{\tau} = \\ &= \lim_{\tau \rightarrow 0^+} \frac{d_K(x(t) + \tau\dot{x}(t) + \tau\alpha(\tau)) - d_K(x(t))}{\tau} \leq -\gamma. \end{aligned}$$

Since $\dot{x}(t) = g(x(t)) \in \tilde{T}_{K,\gamma}(x(t))$. Thus for $t \geq 0$ such that $x(t) \in I \setminus K$ we have proved that $\dot{d}(t) \leq -\gamma$, namely

$$d(t) \leq d(0) - \gamma t = d_K(x_0) - \gamma t.$$

Let $t_0 > 0$ such that $d(t_0) = 0$ and $d(t) > 0$ for all $0 \leq t < t_0$, then $t_0 \leq \frac{d_K(x_0)}{\gamma}$. \square

As a straightforward consequence of ([11], Theorem 1, p. 87) and ([3], Theorem 8.2.10), we can derive the following.

Theorem 2 *For any $\epsilon > 0$ there exists $\delta > 0$ such that if $g : I \setminus K \rightarrow \mathbf{R}^n$ is a continuous δ -approximation in graph of the multivalued map F then, for any solution $y = y(t)$, $t \geq 0$, of the initial-value problem (5), there exists a solution $x = x(t)$, $t \geq 0$, of the initial-value problem*

$$\begin{cases} \dot{x} \in F(x) \\ x(0) = x_0 \end{cases}$$

such that for any time interval $[0, a]$ we have

$$\max_{t \in [0, a]} |x(t) - y(t)| \leq \epsilon.$$

Moreover, there exists a measurable control $u(t) \in U(x(t))$, such that $\dot{x}(t) = f(x(t), u(t))$ for a.a. $t \geq 0$.

It is now evident that, as a consequence of Theorems 1 and 2, the trajectory $x = x(t)$, $t \geq 0$, reaches in finite time the ϵ -neighborhood of K and remains in it for all the future times. Hence, problem (CP) is solved.

Remark 6 If $x \rightarrow F(x)$ is lower semicontinuous, then the celebrated Michael selection theorem ensures the existence of a continuous selection $g(x)$ of the regulation map $x \rightarrow R_{K,\gamma}(x)$, $x \in I \setminus K$, and so we can take $x(t) = y(t)$, $t \geq 0$. Note that, under our assumptions, $x \rightarrow F(x)$ is lower semicontinuous if $x \rightarrow U(x)$ is lower semicontinuous.

Remark 7 Observe that if, in Theorem 1, we assume the conditions of Proposition 1 then by means of the above arguments we can show the existence of a trajectory $x = x(t)$, $t \geq 0$, of the control system which does not leave a prescribed ϵ -neighborhood of I . Finally, observe that the condition $F(x) \subseteq \tilde{T}_{K,\gamma}(x)$, or $F(x) \subseteq \tilde{T}_K(x)$, allows to prove the same behaviour for all the trajectories of the control system.

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