Journal of Computational and Applied Mathematics 113 (2000) 123–139

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.nl/locate/cam

Bifurcation and multiplicity results for periodic solutions of a damped wave equation in a thin domain ☆

Russell Johnson[a], Mikhail Kamenskii[b], Paolo Nistri[a,*]

[a] Dipartimento di Sistemi e Informatica, Facoltà di Ingegneria, Università di Firenze, Via di S. Marta 3, 50139 Firenze, Italy

[b] Department of Mathematics, Voronezh State University, Universitetskaja p.1, Voronezh, Russia

Received 22 December 1998

Abstract

We study the bifurcation problem for periodic solutions of a nonautonomous damped wave equation defined in a thin domain. Here the bifurcation parameter is represented by the thinness $\varepsilon > 0$ of the considered domain. This study has as starting point the existence result of periodic solutions already stated by the authors for this equation and it makes use of the condensivity properties of the associated Poincaré map and its linearization around these solutions. We establish sufficient conditions to guarantee that $\varepsilon = 0$ is or not a bifurcation point and a related multiplicity result. These results are in the spirit of those given by Krasnosel'skii and they are obtained by using the topological degree theory for k-condensing operators. © 2000 Elsevier Science B.V. All rights reserved.

MSC: 35B10; 35B32; 47H11

Keywords: Bifurcation; Periodic solutions; Wave equation; Thin domains

1. Introduction

This paper continues the previous work of the authors (see [4–6]) concerning the study of the existence and of the stability properties of periodic solutions of a damped wave equation defined in a thin domain. Here the aim is to investigate a related bifurcation problem for the periodic solutions of the same equation. A multiplicity result is also given. Our starting point is an existence result

☆ Research partially supported by the M.U.R.S.T., the C.N.R. and the R.F.F.I. grant 96-01-00360.

* Corresponding author. Tel.: +39-55-4796356.

E-mail addresses: johnson@ingfi1.ing.unifi.it (R. Johnson), mikhail@kam.vsu.ru (M. Kamenskii), pnistri@ingfi1.ing.unifi.it (P. Nistri)

PII: S0377-0427(99)00248-4

stated in ([4, Theorem 1]) which guarantees the existence, for $\varepsilon > 0$ sufficiently small, of a periodic solution $w^\varepsilon = \binom{u^\varepsilon}{v^\varepsilon}$ with respect to the time t of the following nonautonomous damped wave system:

$$\begin{aligned} \frac{\partial u}{\partial t} &= v, \\ \frac{\partial v}{\partial t} &= \Delta_X u + \frac{\partial^2 u}{\partial Y^2} - \beta v - \alpha u + g(t, X, Y, u) \end{aligned} \tag{1}$$

with Neumann boundary condition

$$\frac{\partial u}{\partial \nu_\varepsilon} = 0 \quad \text{on } \partial Q_\varepsilon, \tag{2}$$

where α and β are positive constants, g is an appropriate smooth function T-periodic with respect to time t and (X, Y) is a generic point of the "thin domain" $Q_\varepsilon = \Omega \times (0, \varepsilon) \subset \mathbb{R}^{N+1}$.

The method employed in [4] consists in assuming that the "reduced" problem at $\varepsilon = 0$ in the domain Ω admits an isolated T-periodic solution $w^0 = \binom{u^0}{v^0}$, and then in giving conditions under which this solution extends to one for problem (1)–(2) in Q_ε. Clearly, to any such solution corresponds a fixed point of the Poincaré map V^ε associated to (1)–(2), which represents the initial condition of the T-periodic solution and viceversa any fixed point of V^ε is the initial condition of a T-periodic solution of (1)–(2).

The Poincaré map V^ε as well as its first approximation $\mathscr{L}^\varepsilon$ are condensing operators with respect to the Kuratowskii measure of noncompactness, generated by a suitable norm in the space Y_ε^1, with the same constant $k < 1$ (see [10,1]). In [7] this condensivity property was proved for a special measure of noncompactness defined by means of the Hausdorff measure of noncompactness. Moreover, it turns out that if $\lambda \in \sigma(\mathscr{L}^\varepsilon)$ satisfies $|\lambda| > k + d$, whenever $d > 0$, then it is an eigenvalue of finite multiplicity (see [1]).

By a suitable change of variable we can reduce the set of fixed points of V^ε, $\varepsilon > 0$ sufficiently small, to the set of zero fixed points of the resulting map F^ε. Our purpose here is to give conditions which guarantee that $\varepsilon = 0$ is or not a bifurcation point of the map F^ε and also to study the multiplicity of the nonzero fixed points of this map. Observe that these results provide an estimation of the number of periodic solutions of the original problem (1)–(2).

These conditions are similar to those given by Krasnosel'skii ([8, Theorem 11.2, p. 225]) and the corresponding results are also of Krasnosel'skii type. Specifically, we refer to the conditions concerning the eigenvalue $\mu(\varepsilon)$ of the linearization L^ε of F^ε for small $\varepsilon > 0$ around the zero fixed point and the successive term of order $h \geqslant 2$. We would like to point out that in the present case, as we will see in the sequel, any nonzero fixed point q_ε of F^ε belongs to a different space depending on the parameter $\varepsilon > 0$. As a consequence we do not have the continuity of $F^\varepsilon(q)$ with respect to both the variables, but we have continuity only on the sequences of fixed points in the sense that: $q_\varepsilon = F^\varepsilon(q_\varepsilon) \to q_0 = F^0(q_0)$ as $\varepsilon \to 0$. Moreover, for studying the bifurcation at $\varepsilon = 0$ we consider here a suitable projector defined by means of the normalized eigenvector corresponding to the simple eigenvalue $\mu(\varepsilon)$, $\varepsilon \geqslant 0$ small, of the linear operator L^ε. This projector turns out to be the Riesz projector for $\varepsilon = 0$ but not for $\varepsilon > 0$, this choice permits to avoid the consideration of the eigenvector of the adjoint operator of L^ε corresponding to $\mu(\varepsilon)$, which does not depend continuously on $\varepsilon > 0$. Observe that here L^ε is not a self-adjoint operator.

The main tool for proving our results is the topological degree for condensing operators and its properties, see [9]. In particular, the reduction property will play a crucial rôle in the proof of the main result. Furthermore, our approach simplifies very much the proof with respect to that given by Krasnosel'skii in [8].

The paper is organized as follows. In Section 2 we provide assumptions, definitions and preliminary results to be used in the sequel. In Section 3 we collect our main results in Theorem 7 and we prove several Lemmas which provide the necessary a priori bounds on the fixed points of the Poincaré map F^{ε}.

2. Assumptions, definitions and preliminary results

For the reader's convenience we report here the necessary background in order to formulate the basic existence result proved in ([4, Theorem 1]).

We assume the following conditions on $g : [0,T] \times \Omega \times [0,\varepsilon_0) \times \mathbb{R} \to \mathbb{R}$:

— g is of class C^1 jointly in the variables t, X, Y and u and it is T-periodic in $t : g(t+T,X,Y,u) \equiv g(t,X,Y,u)$. Moreover, g satisfies the following estimates:

$$|g_X(t,X,Y,u)| \leqslant a(1+|u|^{\theta+1}),$$
$$|g_Y(t,X,Y,u)| \leqslant a(1+|u|^{\theta+1}),$$
$$|g_u(t,X,Y,u)| \leqslant a(1+|u|^{\theta})$$

for all values of its arguments t, X, Y, u. Here $a > 0$ is a suitable constant and $\theta \in [0,\infty)$ if $N = 1$, $\theta \in [0, 2/(N-1))$ if $N > 1$.

Following [3], for fixed $\varepsilon > 0$ we introduce new variables $X = x$, $Y = \varepsilon y$. System (1) becomes

$$\begin{aligned} \frac{\partial u}{\partial t} &= v, \\ \frac{\partial v}{\partial t} &= \Delta_x u + \frac{1}{\varepsilon^2}\frac{\partial^2 u}{\partial y^2} - \beta v - \alpha u + g(t,x,\varepsilon y,u) \end{aligned} \tag{3}$$

and boundary condition (2) takes the form

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial Q,$$

where $Q = \Omega \times (0,1)$ and ν denotes the outward unit normal vector to Q. We suppose that Ω is a C^2-smooth domain.

We now give the most relevant definitions which permit to rewrite (1)–(2) as a fixed point problem in a suitable space. More details can be found in [4].

Following [3] we introduce the following Banach spaces when $\varepsilon > 0$. Let X^1_ε be the space $H^1(Q)$ with the norm

$$\left(\|u\|^2_{1Q} + \frac{1}{\varepsilon^2}\left\|\frac{\partial u}{\partial y}\right\|^2_{0Q} \right)^{1/2}.$$

Here and below, $\|\cdot\|_{0Q}$ denotes the norm in $L^2(Q)$ and $\|\cdot\|_{1Q}$ that in $H^1(Q)$. Let $U_\varepsilon(t)$ be the semigroup generated by the system of linear equations

$$\frac{\partial u}{\partial t} = v,$$
$$\frac{\partial v}{\partial t} = \Delta_x u + \frac{1}{\varepsilon^2}\frac{\partial^2 u}{\partial y^2} - \beta v - \alpha u,$$

with boundary condition (2). It is known (see [3]) that $U_\varepsilon(t)$ is a C_0-semigroup in the space

$$Y_\varepsilon^1 = X_\varepsilon^1 \times L^2(Q) \ni (u,v) = w.$$

One has the exponential estimate

$$\|U_\varepsilon(t)\|_{Y_\varepsilon^1 \to Y_\varepsilon^1} \leqslant c\mathrm{e}^{-\gamma t}, \quad (t \geqslant 0),$$

where $c, \gamma > 0$.

In the sequel by a solution of any differential equation we mean a solution of the corresponding integral equation obtained by the variation-of-constants formula.

Now let $C_T(Y_\varepsilon^1)$ be the space of all continuous, T-periodic functions $w = \binom{u}{v}$ from $\mathbb{R}$ into Y_ε^1 with the usual norm

$$\|w\| = \sup_{t \in [0,T]} \|w(t)\|_{Y_\varepsilon^1}.$$

Define the following maps on $C_T(Y_\varepsilon^1)$:

$$f_\varepsilon(w)(t)(x,y) = \begin{pmatrix} 0 \\ g(t,x,\varepsilon y, u(t,x,y)) \end{pmatrix},$$

and

$$J_\varepsilon w(t) = U_\varepsilon(t)(I - U_\varepsilon(T))^{-1} \int_0^T U_\varepsilon(T-s) w(s)\,\mathrm{d}s + \int_0^t U_\varepsilon(t-s) w(s)\,\mathrm{d}s.$$

Then define

$$\Gamma_\varepsilon(w) = J_\varepsilon f_\varepsilon(w).$$

Using the Sobolev embedding theorem together with the theory of nonlinear Nemytskii operators, it is easy to show that Γ_ε maps $C_T(Y_\varepsilon^1)$ into itself and is completely continuous, i.e. it is continuous and it maps bounded sets into relatively compact sets. We give now the following:

Definition 1. A fixed point of the completely continuous operator

$$\Gamma_\varepsilon : C_T(Y_\varepsilon^1) \to C_T(Y_\varepsilon^1)$$

is a T-periodic solution of (1)–(2).

It is known that a fixed point of Γ_ε is always a T-periodic distributional solution of (1)–(2).

Next, we pose the limit problem at $\varepsilon = 0$. Let $U_0(t)$ $(t \geqslant 0)$ be the semigroup generated by the linear system

$$\frac{\partial u}{\partial t} = v,$$
$$\frac{\partial v}{\partial t} = \Delta_x u - \beta v - \alpha u$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

Let $w_0 = \binom{u_0}{v_0}$ be an element of $H^1(\Omega) \times L^2(\Omega)$. Then $U_0(t)\binom{u_0}{v_0}$ is in $H^1(\Omega) \times L^2(\Omega)$ and one has the estimate

$$\|U_0(t)\|_{H^1(\Omega)\times L^2(\Omega)\to H^1(\Omega)\times L^2(\Omega)} \leqslant c\mathrm{e}^{-\gamma t}$$

where $c, \gamma > 0$.

If $u \in L^2(Q)$ define its projection

$$(\mathscr{M}u)(x) = \int_0^1 u(x, y)\,\mathrm{d}y,$$

so that $\mathscr{M}$ maps $L^2(Q)$ to $L^2(\Omega)$. Correspondingly, we define the projection matrix

$$M = \begin{pmatrix} \mathscr{M} & 0 \\ 0 & \mathscr{M} \end{pmatrix},$$

which maps Y_ε^1 to $H^1(\Omega) \times L^2(\Omega)$.

Defining $i : \Omega \to Q$ by $i(x) = (x, 0)$, we obtain an inclusion $\mathscr{J} : H^1(\Omega) \times L^2(\Omega) \to Y_\varepsilon^1$ with $\mathscr{J}(u, v)(x, y) = (u(x), v(x))$. The map $\mathscr{J}$ is an isometry for all $0 < \varepsilon < \varepsilon_0$, and we identify $U_0(t)\binom{u_0}{v_0}$ with the element $\mathscr{J}U_0(t)\binom{u_0}{v_0}$ of Y_ε^1.

Define an operator Γ_0 on $C_T(H^1(\Omega) \times L^2(\Omega))$ as follows:

$$\Gamma_0(w) = J_0 f_0(w),$$

where J_0 has the same form as J_ε with $U_\varepsilon(t)$ replaced by $U_0(t)$ and

$$f_0(w)(t)(x) = \begin{pmatrix} 0 \\ g(t, x, 0, u(t, x)) \end{pmatrix}.$$

Then $\Gamma_0 : C_T(H^1(\Omega) \times L^2(\Omega)) \to C_T(H^1(\Omega) \times L^2(\Omega))$ and it is completely continuous.

We identify the T-periodic solutions of the system

$$\begin{aligned} \frac{\partial u}{\partial t} &= v, \\ \frac{\partial v}{\partial t} &= \Delta_x u - \beta v - \alpha u + g(t, x, 0, u), \end{aligned} \tag{4}$$

together with the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \tag{5}$$

with the fixed points of the operator Γ_0. The main result proved in [4] is the following existence result.

Theorem 2. *If problem* (4)–(5) *admits an isolated T-periodic solution* $w^0 = \binom{u^0}{v^0} \in C_T(H^1(\Omega) \times L^2(\Omega))$ *with* $\operatorname{ind}(\Gamma_0, w^0) \neq 0$, *then for sufficiently small* $\varepsilon > 0$, *problem* (4)–(5) *admits a T-periodic solution* $w^\varepsilon = \binom{u^\varepsilon}{v^\varepsilon} \in C_T(Y_\varepsilon^1)$ *and*

$$\|w^\varepsilon - \mathscr{J} w^0\|_{C_T(Y_\varepsilon^1)} \to 0 \quad \text{as } \varepsilon \to 0.$$

Here $\operatorname{ind}(\Gamma_0, w^0)$ denotes the topological index of the fixed point w^0 of the map Γ_0. The proof of Theorem 2 is mainly based on the following result, which we repeat here for the reader's convenience since it will be used in the sequel.

Lemma 3. *Suppose that there exist* $r > 0$, $\varepsilon_n \to 0$ *and*

$$\begin{pmatrix} u^* \\ v^* \end{pmatrix} \in C_T(H^1(\Omega) \times L^2(\Omega))$$

such that the problem (1), (2) *admits T-periodic solutions*

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} \in C_T(Y_{\varepsilon_n}^1)$$

with

$$\left\| \begin{pmatrix} u_n \\ v_n \end{pmatrix} - \mathscr{J} \begin{pmatrix} u^* \\ v^* \end{pmatrix} \right\|_{C_T(Y_{\varepsilon_n}^1)} = r.$$

Then there exist a T-periodic solution $\binom{\bar{u}}{\bar{v}}$ *of* (4)–(5) *and a subsequence*

$$\left\{ \begin{pmatrix} u_{k_n} \\ v_{k_n} \end{pmatrix} \right\} \text{ of } \left\{ \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right\}$$

such that

$$\left\| \begin{pmatrix} u_{k_n} \\ v_{k_n} \end{pmatrix} - \mathscr{J} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \right\|_{C_T(Y_{\varepsilon_n}^1)} \to 0 \quad \text{as } n \to \infty$$

with

$$\left\| \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} - \begin{pmatrix} u^* \\ v^* \end{pmatrix} \right\|_{C_T(H^1(\Omega) \times L^2(\Omega))} = r.$$

3. Main result

Throughout this section we assume the conditions of Theorem A. For $\varepsilon > 0$ define the Poincaré map $V^\varepsilon : Y_\varepsilon^1 \to Y_\varepsilon^1$ associated to (1)–(2) as follows:

$$V^\varepsilon(v) = U_\varepsilon(T)v + \int_0^T U_\varepsilon(T-s) f_\varepsilon(w)(s)\, \mathrm{d}s,$$

where $w(t) \in Y_\varepsilon^1$, $t \in [0, T]$ is a solution of (1)–(2) with $w(0) = v$ and T is the period of the nonlinearity g.

As a direct consequence of Theorem A we have that for $\varepsilon > 0$ sufficiently small the Poincaré map has a fixed point $v_\varepsilon \in Y_\varepsilon^1$ which represents the initial condition of the T-periodic solution w^ε of (1) –(2) and vice versa.

Consider the linearization $\mathscr{L}^\varepsilon : Y_\varepsilon^1 \to Y_\varepsilon^1$ of V^ε around w^ε

$$\mathscr{L}^\varepsilon z = U_\varepsilon(T)z + \int_0^T U_\varepsilon(T-s) f_\varepsilon'(w^\varepsilon(s))\psi(s)\,\mathrm{d}s,$$

where $\psi(t) \in Y_\varepsilon^1$, $t \in [0,T]$, is the solution of the linearization of (1) around w^ε such that $\psi(0)=z$.

For $\varepsilon = 0$ we also define $V^0 : H^1(\Omega)\times L^2(\Omega) \to H^1(\Omega)\times L^2(\Omega)$ and the corresponding linearization $\mathscr{L}^0$ around w^0. The linear map $\mathscr{L}^\varepsilon : Y_\varepsilon^1 \to Y_\varepsilon^1$ is k-condensing with respect to the measure of noncompactness of Kuratowskii generated by a suitable equivalent norm in the space Y_ε^1 (see [10]). We note that in [7] this condensivity property was also proved for some special measures of noncompactness defined by means of the Hausdorff measure of noncompactness.

It follows that (see [1]) for any $d > 0$ the points $\lambda \in \sigma(\mathscr{L}^\varepsilon)$ for which $|\lambda| > k + d$ are eigenvalues of finite multiplicity.

For any $\varepsilon > 0$ sufficiently small, let $v = q + v_\varepsilon$, where $q \in Y_\varepsilon^1$ and define

$$F^\varepsilon(q) = V^\varepsilon(q + v_\varepsilon) - v_\varepsilon.$$

Then $V^\varepsilon(v_\varepsilon) = v_\varepsilon$ is equivalent to $F^\varepsilon(0) = 0$. Assume in addition to the previous assumptions that g is h-continuously differentiable ($h \geqslant 2$) with respect to u, then F^ε can be written in the form

$$F^\varepsilon(q) = L^\varepsilon q + C^\varepsilon(q) + D^\varepsilon(q), \tag{6}$$

where $L^\varepsilon q = \mathscr{L}^\varepsilon q$, $C^\varepsilon(\cdot)$ is an homogeneous operator of order $h \geqslant 2$ with respect to q and $D^\varepsilon(\cdot)$ consists of infinitesimals as $q \to 0$ of order higher than h. Clearly F^ε and L^ε have the same condensivity properties as V^ε and $\mathscr{L}^\varepsilon$ respectively. Concerning the spectrum of L^0 we make the following important remark.

Remark 4. Suppose that 1 is a simple eigenvalue of L^0, there is no loss of generality in assuming that $\mu \in \sigma(L^0)$ and $\mu \neq 1$ imply that $|\mu| \leqslant \rho_0 < 1$. In fact, we can readapt to the present situation the arguments in ([8, pp. 226–228]) to show that one can always reduce to this case by redefining the Poincaré map by means of a suitable operator of finite dimension (and thus compact). Moreover, if we denote by p_0 the normalized eigenvector associated to the simple eigenvalue 1 of L^0 and by p_0^* the normalized eigenvector of the adjoint operator $(L^0)^*$ corresponding to the same eigenvalue 1 such that $\langle p_0, p_0^* \rangle = 1$ then we can define the Riesz projector $P_0 : H^1(\Omega)\times L^2(\Omega) \to H^1(\Omega)\times L^2(\Omega)$ as follows:

$$P_0 p = p - \langle p, p_0^* \rangle\, p_0.$$

This projector will play a crucial rôle in what follows. Here $\langle \cdot,\cdot \rangle$ is the inner product in $H^1(\Omega)\times L^2(\Omega)$ associated to the usual norm of this space.

In what follows we will often omit explicit reference to the Banach/Hilbert space in question when indicating norms and inner products.

In the next lemma we give preliminary results which will be useful in the formulation of our main result.

Lemma 5. *Assume that* 1 *is a simple eigenvalue of* L^0 *and that* $\mu \in \sigma(L^0)$ *and* $\mu \neq 1$ *imply* $|\mu| \leqslant \rho_0 < 1$. *Then the following results hold*:
(a) *for any projector* $Q : H^1(\Omega) \times L^2(\Omega) \to H^1(\Omega) \times L^2(\Omega)$ *parallel to* p_0 *we have that*

$$\sigma(QL^0Q) \subset \overline{B(0,\rho_0)} = \{\alpha \in \mathbf{C} : |\alpha| \leqslant \rho_0 < 1\};$$

(b) *for* $\varepsilon > 0$ *sufficiently small there is a continuous function* $\varepsilon \to \mu(\varepsilon) \in \sigma(L^\varepsilon)$ *such that* $\mu(\varepsilon) \to 1$ *as* $\varepsilon \to 0$. *Moreover* $\mu(\varepsilon)$ *is simple.*

Proof. (a) First of all we show that the linear operators L^0 and QL^0Q are condensing with the same constant $\rho_0+\delta$, whenever $\delta > 0$. For this, denote for simplicity by E the Hilbert space $H^1(\Omega)\times L^2(\Omega)$, fix $\delta > 0$, let $P_0 : E \to E$, be the Riesz projector corresponding to $1 \in \sigma(L^0)$, defined before, and let $E_0 = (I - P_0)E$. Therefore

$$\sigma(L^0(I - P_0)) = \sigma(L^0)\setminus\{1\}$$

and we can define an equivalent norm $\|\cdot\|_0$ on E_0 such that $L^0(I - P_0)$ is condensing with the constant $\rho_0 + \delta$ with respect to this norm. Correspondingly, we have defined in E the following equivalent norm:

$$\|p\|_0 = \|(I - P_0)p\|_0 + \|P_0 p\|.$$

Hence $L^0 = L^0(I - P_0) + L^0P_0$ is a condensing operator with constant $\rho_0 + \delta$ since it is the sum of $L^0(I - P_0)$ and of the compact operator L^0P_0. On the other hand, the projector Q does not change the measure of noncompactness of L^0, since Q is the sum of the identity and of a compact operator and so L^0 and QL^0Q are condensing with the same constant $\rho_0 + \delta$. Since $\delta > 0$ is arbitrary we have that $\sigma(L^0)\setminus\overline{B(0,\rho_0)}$ contains only eigenvalues of finite multiplicity of L^0. Let us prove that $1 \notin \sigma(QL^0Q)$. Assume the contrary, then there exists $p \neq 0$ such that $QL^0Qp = p$. Applying Q we get

$$Q(I - L^0)Qp = 0$$

and so

$$(I - L^0)Qp = r\,p_0 \quad \text{for some } r \in \mathbb{R}.$$

If we apply now $(I - L^0)$ we obtain

$$(I - L^0)^2Qp = 0.$$

But $Qp \neq 0$ since $p \neq 0$ and it is linearly independent of p_0, thus $\dim \operatorname{Ker}(I - L^0)^2 \geqslant 2$ which contradicts the simplicity of $1 \in \sigma(L^0)$.

Analogously, if $|\mu| > \rho_0$, $\mu \neq 1$ we can prove that $\mu \notin \sigma(QL^0Q)$. In fact, assume that $\mu \in \sigma(QL^0Q)$, then there exists $p \neq 0$ such that $QL^0Qp = \mu p$. Applying Q we get

$$Q(\mu I - L^0)Qp = 0, \quad Qp \neq 0$$

and so

$$(\mu I - L^0)Qp = r p_0 \quad \text{for some } r \in \mathbb{R}.$$

If we apply now $(\mu I - L^0)$ we obtain

$$(\mu I - L^0)^2Qp = (\mu - 1)\, r p_0.$$

From the two previous relations we obtain

$$-(\mu-1)(\mu I-L^0)Qp+(\mu I-L^0)^2Qp=0.$$

But $\mu\notin\sigma(L^0)$ thus $(\mu I-L^0)^{-1}$ exists and from the last equation we get

$$-\mu Qp+Qp+(\mu I-L^0)Qp=0,$$

or equivalently $Qp=L^0Qp$ and so $Qp=QL^0Qp$, i.e. $1\in\sigma(QL^0Q)$ which is a contradiction.

(b) Since 1 is an isolated eigenvalue of L^0 there exists a closed disc D centered at 1 with radius $0<r_0<1-k$ such that D does not contain points of $\sigma(L^0)$ different from 1. We want to prove that for any $r\in(0,r_0]$ there exists $\varepsilon_r>0$ such that for any $\varepsilon\in(0,\varepsilon_r)$ there is $\mu(\varepsilon)\in\sigma(L^\varepsilon)$ with $|1-\mu(\varepsilon)|<r$. Assume the contrary, thus there exists $\hat r\in(0,r_0]$ and a sequence $\varepsilon_n\to 0$ such that for any $\mu\in\sigma(L^{\varepsilon_n})$ we have $|1-\mu|\geqslant\hat r$. Assume first that there is a sequence $\mu_n\in\sigma(L^{\varepsilon_n})$ such that $|1-\mu_n|=\hat r$ with $\varepsilon_n\to 0$. Then

$$\mu_n p_n=L^{\varepsilon_n}p_n \quad \text{for some } p_n\in Y^1_{\varepsilon_n}, \quad \|p_n\|=1$$

and passing to the limit as $n\to\infty$ by Theorem A we get

$$\mu_0 p_0=L^0 p_0$$

with $|1-\mu_0|=\hat r$, $p_0\in H^1(\Omega)\times L^2(\Omega)$, which is a contradiction. Therefore for $\varepsilon>0$ sufficiently small the Riesz projector

$$\mathscr{P}_\varepsilon p=-\frac{1}{2\pi\mathrm{i}}\int_C(\mu I-L^\varepsilon)^{-1}p\,\mathrm{d}\mu$$

is well defined, where $C=\partial B(1,\hat r)$ and $B(1,\hat r)$ is the ball centered at 1 with radius $\hat r$.

Consider now the normalized eigenvector $p_0\in H^1(\Omega)\times L^2(\Omega)$ corresponding to the eigenvalue $1\in\sigma(L^0)$. By Lemma 3 we get

$$\mathscr{P}_\varepsilon p_0\to P_0p_0\neq 0 \quad \text{as } \varepsilon\to 0.$$

But this contradicts the existence of the sequence $\varepsilon_n\to 0$ such that $|1-\mu|>\hat r$ for any $\mu\in\sigma(L^{\varepsilon_n})$, due to the fact that in this case $\mathscr{P}_{\varepsilon_n}\equiv 0$ for any $n\in\mathbb{N}$.

We prove now that there exist $r_1\in(0,r_0]$ and $\varepsilon_{r_1}>0$ such that for any $\varepsilon\in(0,\varepsilon_{r_1})$ there is at most one $\mu(\varepsilon)\in\sigma(L^\varepsilon)\cap B(1,r_1)$. Assume that this is not the case, thus there exist sequences $\varepsilon_n\to 0,\mu_n,\mu_n'\in\sigma(L^{\varepsilon_n})$ with $\mu_n,\mu_n'\to 1$ and $\mu_n\neq\mu_n'$. Let p_n,p_n' be linearly independent eigenvectors corresponding to μ_n,μ_n' resp. Consider the projector $P_n:Y^1_{\varepsilon_n}\to Y^1_{\varepsilon_n}$ defined by

$$P_np=p-\frac{\langle Mp,p_0^*\rangle}{\langle Mp_n,p_0^*\rangle}p_n,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $H^1(\Omega)\times L^2(\Omega)$ associated to the usual norm of this space. We have

$$P_np_n'=p_n'-\alpha_np_n,$$

where $\alpha_n=\langle Mp_n',p_0^*\rangle/\langle Mp_n,p_0^*\rangle$. Thus

$$p_n'=\alpha_np_n+P_np_n',$$

or equivalently

$$\mu_n'p_n'=(\mu_n+\delta_n)\alpha_np_n+\mu_n'P_np_n',$$

where $\delta_n = \mu_n' - \mu_n \to 0$ as $n \to \infty$. Hence

$$L^{\varepsilon_n} p_n' = \alpha_n L^{\varepsilon_n} p_n + \mu_n' P_n p_n' + \delta_n \alpha_n p_n.$$

From this

$$L^{\varepsilon_n} P_n p_n' = \mu_n' P_n p_n' + \delta_n \alpha_n p_n.$$

Now $P_n p_n' \neq 0$, since otherwise p_n, p_n' are linearly dependent, and so

$$L^{\varepsilon_n} \frac{P_n p_n'}{\|P_n p_n'\|} = \mu_n' \frac{P_n p_n'}{\|P_n p_n'\|} + \delta_n \frac{(I - P_n) p_n'}{\|P_n p_n'\|}.$$

If we pass to the limit as $n \to \infty$ by Lemma 3 and Theorem 2 we obtain

$$L^0 \hat{p}_0 = \hat{p}_0 + \gamma_0 p_0,$$

where

$$\gamma_0 = \lim_{n\to\infty} \frac{\delta_n}{\|P_n p_n'\|}, \qquad \hat{p}_0 = \frac{P_0 p_0}{\|P_0 p_0\|},$$

$\hat{p}_0 \in H^1(\Omega) \times L^2(\Omega)$ and $\|\hat{p}_0\| = 1$. Observe that

$$P_0 p = p - \langle p, p_0^* \rangle p_0$$

is the projection in $H^1(\Omega) \times L^2(\Omega)$ defined in Remark 4.

But $P_0 L^0 P_0 \hat{p}_0 = P_0 \hat{p}_0$ since $\gamma_0 P_0 p_0 = 0$. In conclusion $1 \in \sigma(P_0 L^0 P_0)$ with corresponding eigenvalue $\hat{p}_0$ contradicting Lemma 5(a).

Finally, by using the previous arguments we can show that the eigenvalue $\mu(\varepsilon)$ is simple, and $\varepsilon \to \mu(\varepsilon)$ is continuous. This concludes the proof. □

Observe that part (a) of Lemma 5 is trivially verified for $Q = P_0$ since P_0 is a Riesz operator. However, we have decided to report this result since it seems to us of some interest for its generality.

Let $p_\varepsilon \in Y_\varepsilon^1$, for $\varepsilon > 0$ small, be the normalized eigenvector corresponding to the simple eigenvalue $\mu(\varepsilon) \in \sigma(L^\varepsilon)$, and consider the projector in Y_ε^1 given by

$$P_\varepsilon p = p - \frac{\langle Mp, p_0^* \rangle}{\langle Mp_\varepsilon, p_0^* \rangle} p_\varepsilon.$$

Observe that by Theorem 2 we have that $\langle Mq_\varepsilon, p_0^* \rangle \to \langle q_0, p_0^* \rangle$ as $\varepsilon \to 0$ if $q_\varepsilon = F^\varepsilon(q_\varepsilon)$.

Definition 6. For arbitrary $m > 0$ and $\varepsilon > 0$ sufficiently small we define the cones K_m^ε and K_{-m}^ε in Y_ε^1 as follows:

$$K_m^\varepsilon = \{p \in Y_\varepsilon^1 \colon \langle Mp, p_0^* \rangle \geqslant 0 \text{ and } \|P_0^{\mathscr{J}} p\| \leqslant m \langle Mp, p_0^* \rangle\}$$

and

$$K_{-m}^\varepsilon = \{p \in Y_\varepsilon^1 : -p \in K_m^\varepsilon\},$$

where $P_0^{\mathscr{J}} p = p - \langle Mp, p_0^* \rangle \mathscr{J} p_0$.

Observe that $p_\varepsilon \in K_m^\varepsilon$ for $\varepsilon > 0$ sufficiently small, in fact $p_\varepsilon \to p_0$ as $\varepsilon \to 0$.

We are now in the position to formulate the main result.

Theorem 7. *Let $F^\varepsilon : Y^1_\varepsilon \to Y^1_\varepsilon$ be the map given in* (6). *Assume that for $\varepsilon > 0$ sufficiently small, say $\varepsilon \in (0, \hat{\varepsilon}_0)$, F^ε satisfies the following conditions*:
(H_1) 1 *is a simple eigenvalue of the linear operator L^0 with corresponding normalized eigenvector $p_0 \in H^1(\Omega) \times L^2(\Omega)$. Moreover, if $\mu \in \sigma(L^0)$ and $\mu \neq 1$, then $|\mu| \leqslant \rho_0 < 1$.*
(H_2) $1 \notin \sigma(L^\varepsilon)$ *for any $\varepsilon \in (0, \hat{\varepsilon}_0)$.*
(H_3) $\zeta_0 = \langle C^0(p_0), p_0^* \rangle$ *is different from zero.*
Then there exist $\varepsilon_0 > 0$ and $r_0 > 0$ such that

1. *the equation $F^0(q) = q$ has only the zero solution in the ball $B(0, r_0)$.*
2. *If h is even then for any $\varepsilon \in (0, \varepsilon_0)$ the equation $F^\varepsilon(q) = q$ has at least one nonzero solution $q_\varepsilon \in B(0, r_0)$ such that*

$$\operatorname{sgn}\langle Mq_\varepsilon, p_0^* \rangle = \operatorname{sgn}\zeta_0 \operatorname{sgn}(1 - \mu(\varepsilon)).$$

3. *If h is odd we have the following cases*:
   (a) *if $\operatorname{sgn}\zeta_0 = -\operatorname{sgn}(1 - \mu(\varepsilon))$ for any $\varepsilon \in (0, \varepsilon_0)$ then $\varepsilon = 0$ is not a bifurcation point for F^ε.*
   (b) *if $\operatorname{sgn}\zeta_0 = \operatorname{sgn}(1 - \mu(\varepsilon))$ for any $\varepsilon \in (0, \varepsilon_0)$ then the equation $F^\varepsilon(q) = q$ has at least two nonzero solutions for any $\varepsilon \in (0, \varepsilon_0)$.*

The proof of Theorem 7 relies on the following Lemmas.

Lemma 8. *There exists $r_0 > 0$ such that for any $\tau \in [0, 1]$ we have that $\|q\| \leqslant r_0$ and $q = (I - (1 - \tau)P_0)F^0(q)$ imply that $q = 0$.*

Lemma 9. *There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ there exists a constant $c(\varepsilon) > 0$ for which*

$$\|(I - (I - (1 - \tau)P_\varepsilon)L^\varepsilon)p\| \geqslant c(\varepsilon)\|p\|$$

for any $\tau \in [0, 1]$ and any $p \in Y^1_\varepsilon$.

Lemma 10. *For any $m > 0$ there exist $r_0 > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any $\tau \in [0, 1]$ we have that $\|q\| \leqslant r_0$ and $q = (I - (1 - \tau)P_\varepsilon)F^\varepsilon(q)$ imply that $q \in K^\varepsilon_m \cup K^\varepsilon_{-m}$.*

Proof of Lemma 8. We argue by contradiction, thus assume that there exist sequences $\tau_n \to \tau_0$, $q_n \to 0$, $q_n \neq 0$ and $q_n \in H^1(\Omega) \times L^2(\Omega)$ such that

$$q_n = (I - (1 - \tau_n)P_0)F^0(q_n),$$

or equivalently

$$q_n = (I - (1 - \tau_n)P_0)L^0 q_n + (I - (1 - \tau_n)P_0)(C^0(q_n) + D^0(q_n)).$$

Let $y_n = q_n/\|q_n\|$, thus

$$y_n = (I - (1 - \tau_n)P_0)L^0 y_n + (I - (1 - \tau_n)P_0)\frac{1}{\|q_n\|}(C^0(q_n) + D^0(q_n)). \tag{7}$$

Passing to the limit as $n \to \infty$ we claim that $y_n \to \pm p_0$. In fact, let $y_0 = \lim_{n\to\infty} y_n$ and consider

$$y_0 = (I - (1 - \tau_0)P_0)L^0 y_0 = \tau_0 P_0 L^0 y_0 + (I - P_0)L^0 y_0.$$

Applying P_0 we obtain

$$P_0 y_0 = \tau_0 P_0 L^0 y_0 = \tau_0 P_0 L^0 (P_0 y_0 + (I - P_0) y_0) = \tau_0 P_0 L^0 P_0 y_0.$$

Observe that if $\tau_0 = 0$ then we have $y_0 = \pm p_0$, since $1 \in \sigma(L^0)$ is simple. If, on the other hand, $P_0 y_0 \neq 0$ and $\tau_0 \in (0,1]$, then Lemma 5(a) is contradicted since $P_0 y_0$ is an eigenvector of $P_0 L^0 P_0$ corresponding to the eigenvalue $1/\tau_0 > 1$. Thus either $\tau_0 = 0$ or $P_0 y_0 = 0$. But $P_0 y_0 = 0$ implies $y_0 = \pm p_0$.

Rewrite now (7) in the following form:

$$\|q_n\|^{1-h}(y_n - L^0 y_n) = C^0(y_n) + \frac{D^0(q_n)}{\|q_n\|^h} - \frac{(1-\tau_n)}{\|q_n\|^h} P_0 F^0(q_n).$$

Consider the scalar product of both members of the previous equation with p_0^*. Observe that the right-hand side vanishes and pass to the limit as $n \to \infty$ to obtain

$$\langle C^0(p_0), p_0^* \rangle = 0$$

which contradicts (H_3). This concludes the proof. □

Proof of Lemma 9. Using (H_1) it is possible to show that the set of values $\tau \in [0,1]$ for which 1 is a simple eigenvalue of $(I - (1-\tau)P_0)L^0$ is both open and closed in $[0,1]$ and so this set coincides with $[0,1]$. We now prove that for $\varepsilon > 0$ sufficiently small and any $\tau \in [0,1]$ we have that

$$1 \notin \sigma((I - (1-\tau)P_\varepsilon)L^\varepsilon). \tag{8}$$

Assume this is not the case, then there exist sequences $\tau_n \to \tau_0$ and $\varepsilon_n \to 0$ such that $1 \in \sigma((I - (1-\tau_n)P_{\varepsilon_n})L^{\varepsilon_n})$. By Lemma 5(b) and ($H_2$), for any $n \in \mathbb{N}$ there exists $\mu_n(\varepsilon_n) \neq 1, \mu(\varepsilon_n) \in \sigma(L^{\varepsilon_n})$, with corresponding eigenvector $p_{\varepsilon_n} \in Y^1_{\varepsilon_n}$. Therefore

$$(I - (1-\tau_n)P_{\varepsilon_n})L^{\varepsilon_n} p_{\varepsilon_n} = (I - (1-\tau_n)P_{\varepsilon_n})\mu(\varepsilon_n) p_{\varepsilon_n} = \mu(\varepsilon_n) p_{\varepsilon_n}.$$

Thus, by the same arguments employed in the proof of Lemma 5(b), we can show that $1 \in \sigma((I - (1-\tau_0)P_0)L^0)$ is not simple. Therefore, for $\varepsilon > 0$ small enough and any $\tau \in [0,1]$ there exists

$$(I - (I - (1-\tau)P_\varepsilon)L^\varepsilon)^{-1} : Y^1_\varepsilon \to Y^1_\varepsilon.$$

We show now the existence of a constant $c(\varepsilon) > 0$, independent of $\tau \in [0,1]$, such that

$$\|(I - (I - (1-\tau)P_\varepsilon)L^\varepsilon)^{-1}\| \leqslant c(\varepsilon).$$

We argue again by contradiction, therefore for fixed $\varepsilon > 0$ small enough, we assume the existence of two sequences $\tau_n \to \tau_0$ and $y_n \in Y^1_\varepsilon$, $\|y_n\| = 1$, such that

$$q_n = (I - (I - (1-\tau_n)P_\varepsilon)L^\varepsilon)^{-1} y_n$$

and $\|q_n\| \to \infty$ as $n \to \infty$. Then we have

$$\frac{q_n}{\|q_n\|} = (I - (1-\tau_n)P_\varepsilon)L^\varepsilon \frac{q_n}{\|q_n\|} + \frac{y_n}{\|q_n\|}.$$

Passing to the limit as $n \to \infty$ we get

$$1 \in \sigma((I - (1-\tau_0)P_\varepsilon)L^\varepsilon),$$

which is a contradiction. □

Proof of Lemma 10. Again by contradiction we assume the existence of $m>0$ and sequences $\tau_n \to \tau_0, \varepsilon_n \to 0$ and $q_n \in Y^1_{\varepsilon_n}$ with $q_n \to 0$ such that

$$q_n = (I-(1-\tau_n)P_{\varepsilon_n})F^{\varepsilon_n}(q_n) \tag{9}$$

and

$$\|P_0^{\mathscr{I}} q_n\| > m\|(I-P_0^{\mathscr{I}})q_n\|, \tag{10}$$

where $\|(I-P_0^{\mathscr{I}})q_n\| = \langle Mq_n, p_0^*\rangle$.

Inequality (10) means that $q_n \notin K^\varepsilon_m \cup K^\varepsilon_{-m}$. Using (10) we obtain

$$\|q_n\| < \left(1+\frac{1}{m}\right)\|P_0^{\mathscr{I}} q_n\|. \tag{11}$$

On the other hand, since

$$\|P_\varepsilon - P_0^{\mathscr{I}}\| \to 0 \quad \text{as } \varepsilon \to 0, \tag{12}$$

for sufficiently large n we have

$$\|P_{\varepsilon_n} - P_0^{\mathscr{I}}\| \leqslant \frac{m}{2(1+m)}.$$

Using (11) we have

$$\|P_{\varepsilon_n} q_n\| \geqslant \|P_0^{\mathscr{I}} q_n\| - \|P_{\varepsilon_n} - P_0^{\mathscr{I}}\|\,\|q_n\| > \frac{m}{1+m}\|q_n\| - \frac{m}{2(1+m)}\|q_n\| = \frac{m}{2(1+m)}\|q_n\|.$$

Furthermore,

$$\|P_{\varepsilon_n} q_n\| \leqslant \|P_0^{\mathscr{I}} q_n\| + \|P_{\varepsilon_n} - P_0^{\mathscr{I}}\|\,\|q_n\| \leqslant \|q_n\| + \frac{m}{2(1+m)}\|q_n\| = \frac{2+3m}{2(1+m)}\|q_n\|.$$

In conclusion

$$\frac{m}{2(1+m)}\|q_n\| \leqslant \|P_{\varepsilon_n} q_n\| \leqslant \frac{2+3m}{2(1+m)}\|q_n\|. \tag{13}$$

Returning to (9) we write

$$q_n = (I-P_{\varepsilon_n})L^{\varepsilon_n} q_n + \tau_n P_{\varepsilon_n} L^{\varepsilon_n} q_n + (I-(1-\tau_n))P_{\varepsilon_n}[C^{\varepsilon_n}(q_n) + D^{\varepsilon_n}(q_n)]. \tag{14}$$

Applying P_{ε_n} we get

$$P_{\varepsilon_n} q_n = \tau_n P_{\varepsilon_n} L^{\varepsilon_n} q_n + \tau_n P_{\varepsilon_n}[C^{\varepsilon_n}(q_n) + D^{\varepsilon_n}(q_n)].$$

But

$$P_\varepsilon L^\varepsilon q = P_\varepsilon L^\varepsilon P_\varepsilon q + P_\varepsilon L^\varepsilon (I-P_\varepsilon) q = P_\varepsilon L^\varepsilon P_\varepsilon q,$$

for any $\varepsilon \geqslant 0$. Therefore

$$P_{\varepsilon_n} q_n = \tau_n P_{\varepsilon_n} L^{\varepsilon_n} P_{\varepsilon_n} q_n + \tau_n P_{\varepsilon_n}[C^{\varepsilon_n}(q_n) + D^{\varepsilon_n}(q_n)]. \tag{15}$$

Now from (13) we have, for n sufficiently large, $\|P_{\varepsilon_n} q_n\| \neq 0$. Let us define now

$$z_n = \frac{P_{\varepsilon_n} q_n}{\|P_{\varepsilon_n} q_n\|},$$

thus (15) can be rewritten in the form

$$z_n = \tau_n P_{\varepsilon_n} L^{\varepsilon_n} z_n + \tau_n P_{\varepsilon_n} \frac{[C^{\varepsilon_n}(q_n) + D^{\varepsilon_n}(q_n)]}{\|P_{\varepsilon_n} q_n\|}. \tag{16}$$

From (13) and the assumptions on the operators C, D we get that

$$\frac{\|C^{\varepsilon_n}(q_n) + D^{\varepsilon_n}(q_n)\|}{\|P_{\varepsilon_n} q_n\|} \to 0 \quad \text{as } n \to \infty.$$

In conclusion taking the limit as $n \to \infty$ in (16) we obtain

$$z_0 = \tau_0 P_0 L^0 z_0, \quad \|z_0\| = 1 \quad \text{and} \quad \tau_0 \neq 0.$$

But $P_0 z_0 = z_0$ and so $\tau_0 P_0 L^0 P_0 z_0 = P_0 z_0$, namely $P_0 z_0$ is an eigenvector of $P_0 L^0 P_0$ corresponding to the eigenvalue $1/\tau_0 > 1$ contradicting Lemma 5(a). This concludes the proof. □

The following result is a direct consequence of Lemma 8 and the upper-semicontinuity property of the multivalued map $\varepsilon \to \mathrm{Fix}((I - (1-\tau)P_\varepsilon)F^\varepsilon)$ at $\varepsilon = 0$ for any $\tau \in [0, 1]$, where $\mathrm{Fix}(G)$ denotes the set of fixed points of the map G. This property can be easily verified by using Lemma 3 and (12).

Lemma 11. *Let $r_0 > 0$ be given by Lemma 8. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in [0, \varepsilon_0)$ and any $\tau \in [0, 1]$ we have that $\|q\| \neq r_0$, $q \in Y_\varepsilon^1$, implies that $q \neq (I - (1-\tau)P_\varepsilon)F^\varepsilon(q)$.*

Furthermore, as a consequence of Lemma 9 we have the following result.

Lemma 12. *Let $\varepsilon_0 > 0$ be given by Lemma 9. For any $\varepsilon \in (0, \varepsilon_0)$ there exists $\rho(\varepsilon) > 0$ such that, for any $\tau \in [0, 1]$, if*

$$\|q\| < \rho(\varepsilon) \quad \text{and} \quad q = (I - (1-\tau)P_\varepsilon)F^\varepsilon(q),$$

then $q = 0$.

Proof of Lemma 12. By the properties of the operators C, D and the estimation methods by Hale and Raugel [3] there exists a constant $\gamma > 0$ such that

$$\|(I - (1-\tau)P_\varepsilon)[C^\varepsilon(q) + D^\varepsilon(q)]\| \leqslant \gamma \|q\|^h \tag{17}$$

for any $\varepsilon \in (0, \varepsilon_0)$. Therefore, taking $c(\varepsilon) > 0$ as in Lemma 9 and choosing $\rho(\varepsilon) > 0$ such that

$$c(\varepsilon) > \gamma \rho^{h-1}(\varepsilon)$$

we get the conclusion. □

Finally, we are in a position to prove our main result.

Proof of Theorem 7. Assertion 1 is proved in Lemma 8. Furthermore, for fixed $m > 0$, the previous results guarantee the existence of an $\varepsilon_0 > 0$, $r_0 > 0$ and $\rho(\varepsilon) > 0$ such that the operators

$$I - (I - P_\varepsilon)F^\varepsilon \quad \text{and} \quad I - F^\varepsilon,$$

are homotopic, via the homotopy $I-(I-(1-\tau)P_\varepsilon)F^\varepsilon, \tau\in[0,1]$, in each of the two disjoint open sets:

$$V^{\pm}_{m,\varepsilon}=(B(0,r_0)\cap \operatorname{int} K^{\varepsilon}_{\pm m})\backslash B(0,\rho(\varepsilon))$$

for any $\varepsilon\in(0,\varepsilon_0)$. Here "int" denotes the interior of a set. Therefore

$$\deg(I-(I-P_\varepsilon)F^\varepsilon, V^{\pm}_{m,\varepsilon},0)=\deg(I-F^\varepsilon, V^{\pm}_{m,\varepsilon},0)$$

for any $\varepsilon\in(0,\varepsilon_0)$.

By the reduction property of the topological degree we have

$$\deg(I-(I-P_\varepsilon)F^\varepsilon, V^{\pm}_{m,\varepsilon},0)=\deg(I-(I-P_\varepsilon)F^\varepsilon, V^{\pm}_{m,\varepsilon}\cap \operatorname{span}\{p_\varepsilon\},0).$$

Note that $p_\varepsilon\in \operatorname{int} K^\varepsilon_m$ for sufficiently small $\varepsilon>0$, since $p_0\in \operatorname{int} K^\varepsilon_m$ for any $m>0$ and $p_\varepsilon\to p_0$ as $\varepsilon\to 0$. Let us calculate now the restriction of the operator $(I-(I-P_\varepsilon)F^\varepsilon)$ to the linear space $\operatorname{span}\{p_\varepsilon\}$:

$$\begin{aligned}
&\rho p_\varepsilon-(I-P_\varepsilon)F^\varepsilon(\rho p_\varepsilon)\\
&\quad=\rho p_\varepsilon-(I-P_\varepsilon)[L^\varepsilon(\rho p_\varepsilon)+C^\varepsilon(\rho p_\varepsilon)+D^\varepsilon(\rho p_\varepsilon)]\\
&\quad=\rho(1-\mu(\varepsilon))p_\varepsilon-\rho^h\zeta_\varepsilon p_\varepsilon-\langle D^\varepsilon(\rho p_\varepsilon),p_\varepsilon\rangle p_\varepsilon.
\end{aligned}$$

with $\rho\in\mathbb{R}$. For $\varepsilon=0$ we have

$$\rho p_0-(I-P_0)F^0(\rho p_0)=-\rho^h\zeta_0 p_0-\langle D^0(\rho p_0),p_0\rangle p_0,$$

where $\zeta_\varepsilon=\langle MC^\varepsilon(p_\varepsilon),p_0^*\rangle,\ \varepsilon\geqslant 0$.

We prove now assertions 2 and 3 of Theorem 7. We start with assertion 1, let h be even and assume that $\zeta_0<0$ to be definite. Let $\tilde r_0\in(0,r_0]$ be such that

$$\operatorname{sgn}[-\tilde r_0^h\zeta_0-\langle D^0(\tilde r_0 p_0),p_0\rangle]=-\operatorname{sgn}\zeta_0.$$

Since

$$\lim_{\varepsilon\to 0}[-\tilde r_0^h\zeta_\varepsilon-\langle D^\varepsilon(\tilde r_0 p_\varepsilon),p_\varepsilon\rangle]=-\tilde r_0^h\zeta_0-\langle D^0(\tilde r_0 p_0),p_0\rangle,$$

there is $\hat\varepsilon_0\in(0,\varepsilon_0]$ such that

$$\operatorname{sgn}[-\tilde r_0^h\zeta_0-\langle D^0(\tilde r_0 p_0),p_0\rangle]=\operatorname{sgn}[-\tilde r_0^h\zeta_\varepsilon-\langle D^\varepsilon(\tilde r_0 p_\varepsilon),p_\varepsilon\rangle]$$

for any $\varepsilon\in(0,\hat\varepsilon_0)$, and

$$\sup_{\varepsilon\in(0,\hat\varepsilon_0)}[\tilde r_0^h\zeta_\varepsilon+\langle D^\varepsilon(\tilde r_0 p_\varepsilon),p_\varepsilon\rangle]=\alpha<0.$$

Furthermore, since $\mu(\varepsilon)\to 1$ as $\varepsilon\to 0$ we have that there exists $\tilde\varepsilon_0\in(0,\hat\varepsilon_0]$ for which the inequality

$$\tilde r_0^h\zeta_\varepsilon+\langle D^\varepsilon(\tilde r_0 p_\varepsilon),p_\varepsilon\rangle<\tilde r_0(1-\mu(\varepsilon))$$

is satisfied for any $\varepsilon\in(0,\tilde\varepsilon_0)$.

Finally, since $h\geqslant 2$, for any $\varepsilon\in(0,\tilde\varepsilon_0)$ there exists $\tilde\rho(\varepsilon)$ such that

$$\operatorname{sgn}[\rho(1-\mu(\varepsilon))-\rho^h\zeta_\varepsilon-\langle D^\varepsilon(\rho p_\varepsilon),p_\varepsilon\rangle]=\operatorname{sgn}(1-\mu(\varepsilon)),$$

for any $\rho\in(0,\tilde\rho(\varepsilon)]$.

For simplicity, let us rename $\tilde{\varepsilon}_0, \tilde{r}_0$ and $\tilde{\rho}(\varepsilon)$ as ε_0, r_0 and $\rho(\varepsilon)$ resp., and redefine the open set $V_{m,\varepsilon}^+$ by means of these values. Now, if $\operatorname{sgn}(1-\mu(\varepsilon)) = \operatorname{sgn}\zeta_0$ for $\varepsilon \in (0, \varepsilon_0)$ then

$$\deg(I-(I-P_\varepsilon)F^\varepsilon, V_{m,\varepsilon}^+ \cap \operatorname{span}\{p_\varepsilon\}, 0) = 1$$

and so

$$\deg(I-F^\varepsilon, V_{m,\varepsilon}^+, 0) = 1$$

This implies the existence of a fixed point q_ε of F^ε such that $\langle Mq_\varepsilon, p_0^*\rangle > 0$. The other cases for h even are handled in the same way.

Suppose now that h is odd and $\operatorname{sgn}(1-\mu(\varepsilon)) = -\operatorname{sgn}\zeta_0$ in a right neighborhood of $\varepsilon = 0$ and assume for contradiction the existence of a sequence $q_n, q_n \in K_m^+$ for definiteness, such that

$$q_n = L^{\varepsilon_n} q_n + C^{\varepsilon_n}(q_n) + D^{\varepsilon_n}(q_n)$$

with $\varepsilon_n \to 0$ and $q_n \to 0$. Let $y_n = q_n/\|q_n\|$, then

$$y_n = L^{\varepsilon_n} y_n + \|q_n\|^{h-1} C^{\varepsilon_n}(y_n) + \frac{D^{\varepsilon_n}(q_n)}{\|q_n\|}.$$

By Lemma 3 we have that $y_n \to p_0$ as $n \to \infty$. By using the fact that $L^{\varepsilon_n}(I-P_{\varepsilon_n})y_n = \mu(\varepsilon_n)(I-P_{\varepsilon_n})y_n$ we can rewrite the last equation in the form

$$(1-\mu(\varepsilon_n))(I-P_{\varepsilon_n})y_n + (I-L^{\varepsilon_n})P_{\varepsilon_n} y_n = \|q_n\|^{h-1} C^{\varepsilon_n}(y_n) + \frac{D^{\varepsilon_n}(q_n)}{\|q_n\|}.$$

Applying the projector P_{ε_n} we obtain

$$P_{\varepsilon_n}(I-L^{\varepsilon_n})P_{\varepsilon_n} y_n = \|q_n\|^{h-1} P_{\varepsilon_n} C^{\varepsilon_n}(y_n) + \frac{P_{\varepsilon_n} D^{\varepsilon_n}(q_n)}{\|q_n\|}. \tag{18}$$

By the upper semicontinuity of the spectrum $\sigma(P_\varepsilon L^\varepsilon P_\varepsilon)$ with respect to $\varepsilon > 0$ (see [6]) we have that $1 \notin \sigma(P_{\varepsilon_n} L^{\varepsilon_n} P_{\varepsilon_n})$. Furthermore, again using results of [6], for $n \in \mathbb{N}$ sufficiently large, we obtain for some positive constant Λ

$$\|(I - P_{\varepsilon_n} L^{\varepsilon_n} P_{\varepsilon_n})^{-1}\| \leqslant \Lambda.$$

Thus from (17) we get

$$\|P_{\varepsilon_n} y_n\| \leqslant \gamma \Lambda \|q_n\|^{h-1}.$$

Dividing (18) by $\|q_n\|^{h-1}$ and passing to the limit as $n \to \infty$ by Lemma 3 we obtain

$$\| \, \|q_n\|^{1-h} P_{\varepsilon_n} y_n - \mathscr{J}(I-L^0)^{-1} P_0 C^0(p_0)\| \to 0.$$

Thus

$$\frac{1-\mu(\varepsilon_n)}{\|q_n\|^{h-1}} \to \zeta_0.$$

This is impossible, since $\operatorname{sgn}(1-\mu(\varepsilon_n)) = -\operatorname{sgn}\zeta_0$ for sufficiently large $n \in \mathbb{N}$. The case when $q_n \in K_m^-$, $n \in N$, is treated in the same way.

Finally, we pass to the case when h is odd and $\operatorname{sgn}(1-\mu(\varepsilon)) = \operatorname{sgn}\zeta_0$ in a right neighborhood of $\varepsilon = 0$. Arguing as in the proof of the case when h is even and using the symmetry with respect to the origin of the principal part $\rho(1-\mu(\varepsilon)) - \rho^h \zeta_\varepsilon$, for $\varepsilon > 0$ fixed small, we can prove both

$\deg(I - F^{\varepsilon}, V^{+}_{m,\varepsilon}, 0)$ and $\deg(I - F^{\varepsilon}, V^{-}_{m,\varepsilon}, 0)$ are different from zero and there must exist at least two nonzero fixed points of F^{ε} for $\varepsilon > 0$ sufficiently small. We leave the details to the reader. □

We end the paper with the following.

Remark 13. Note that the conditions on the $\mathrm{sgn}(1 - \mu(\varepsilon))$ in Theorem 7 not only guarantee the existence but also the stability or instability of the T-periodic solutions of (1)–(2) corresponding to the fixed points of the Poincaré operator F^{ε}. Specifically, in the case when $\mathrm{sgn}(1 - \mu(\varepsilon)) = 1$ we have stability, while if $\mathrm{sgn}(1 - \mu(\varepsilon)) = -1$ we have instability. In fact, we have $|\mu| < 1$ for all $\mu \in \sigma(L^{\varepsilon})$ in the first case and $|\mu(\varepsilon)| > 1$ in the second one, therefore we can apply the results of [6] to draw the conclusion, (see also [2]).

References

[1] R.R. Akhmerov, M.I. Kamenskii et al., Measures of Noncompactness and Condensing Operators, Birkhäuser, Basal, 1992.

[2] Ju.L. Daleckii, M.G. Krein, Stability of solutions of differential equations in Banach spaces, Amer. Math. Soc. Trans. Math. Monographs, Vol. 43, Providence, RI, 1974.

[3] J. Hale, G. Raugel, A damped hyperbolic equation on thin domains, Trans. Am. Math. Soc. 329 (1992) 185–219.

[4] R. Johnson, M.I. Kamenskii, P. Nistri, On periodic solutions of a damped wave equation in a thin domain using degree theoretic methods, J. Differential Equations 40 (1997) 186–208.

[5] R. Johnson, M.I. Kamenskii, P. Nistri, Existence of periodic solutions of an autonomous damped wave equation in thin domains, J. Dyn. Differential Equations 10 (1998) 409–424.

[6] R. Johnson, M.I. Kamenskii, P. Nistri, Stability and instability of periodic solutions of a damped wave equation in a thin domain, in: J. Schauder (Ed.), Lecture Notes in Nonlinear Analysis, Vol. 2, Center for Nonlinear Studies, N. Copernicus Univ., Toruń, Poland, 1998, pp. 199–213.

[7] M. Kamenskii, On the study of stability of periodic solutions for a new class of quasi-linear equations in Banach spaces, Doklady Accad. Nauk. (1997) 13–16 (in Russian).

[8] M.A. Krasnosel'skii, The operator of translation along the trajectories of differential equations, Trans. Math. Monographs 19 (1968).

[9] N.G. Lloyd, Degree Theory, Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 1978.

[10] O. Lopes, S. Ceron, α-contractions and attractors for dissipative semilinear hyperbolic equations and systems, Ann. Mat. Pura Appl. 160 (1991) 193–206.