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On the Existence of Periodic Solutions of the Navier–Stokes Equations in a Thin Domain Using the Topological Degree

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We use the method of the topological degree, the theory of fractional powers of positive operators, and the Grisvard formula together with results proved by G. Raugel and G. R. Sell to study the periodic solutions of the incompressible Navier–Stokes equations in a thin three-dimensional domain.

KEY WORDS: Periodic solutions; Navier–Stokes equations; fractional powers; thin domains.
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1. INTRODUCTION

The purpose of this paper is to study the periodic solutions of the incompressible Navier–Stokes equations in a thin three-dimensional domain $Q_\varepsilon = \Omega \times (0, \varepsilon)$. Here $\Omega \subset \mathbf{R}^2$ is a rectangle, and ε is a small parameter. The Navier–Stokes equations are supplemented by periodic boundary conditions.

The problem we pose is the following. Recall that Raugel and Sell [23, 24, 22] introduced the "reduced" Navier–Stokes equations in the two-dimensional domain Ω. These consist of the incompressible Navier–Stokes equations in Ω together with a third equation. We will suppose that the

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reduced equations admit a periodic solution with a nonzero topological degree. We prove general results to the effect that such a periodic solution perturbs to a periodic solution of the Navier–Stokes equations in the three-dimensional domain Q_ε. This is nontrivial to prove because of the singular nature of the problem as $\varepsilon \to 0$.

Partial differential equations on thin domains were first studied systematically by Hale and Raugel; see, e.g., [10, 11, 22]. Periodic solutions of hyperbolic partial differential equations in thin domains were then considered in [13, 14] (see also [15]). The Navier–Stokes equations in the domain Q_ε were studied in the above mentioned papers by Raugel and Sell. Our study of the *periodic* solutions of the Navier–Stokes equations complements their work in the sense that they emphasized the problems of long-time existence of solutions and semicontinuity properties of global attractors. We note that the topological index of the set of all the T-periodic solutions of the 2- or 3-dimensional Navier–Stokes equations is equal to 1. Indeed, the proof of this fact given in [12] for Dirichlet boundary conditions carries over to our case. We wish to emphasize that we have found the methods of topological degree theory to be very useful in studying periodic solutions; our general reference is [17].

In addition to the topological degree theory, we will also make essential use of the theory of fractional powers of positive self-adjoint operators. Fractional powers were applied to the Navier–Stokes equations in basic papers by Sobolevskii [27] and Fujita–Kato [2] and by several authors since (see, e.g., [3, 4]). The presentation of this theory in the book [18] is particularly appropriate for our purposes. We will also find an important resolvent formula of Grisvard [8] to be very useful.

Let us be somewhat more specific about the problem we will study and results we will prove. Let $Q_\varepsilon = \Omega \times (0, \varepsilon)$ be as above. The incompressible Navier–Stokes equations are

$$\begin{cases} \dfrac{\partial U}{\partial t} = \nu \, \Delta U - \nabla P - (U \cdot \nabla)\, U + F \\ \nabla \cdot U = 0 \end{cases} \tag{1}$$

for $t > 0$ and $(x_1, x_2, x_3) \in Q_\varepsilon$. As usual $U = (U_1, U_2, U_3)$ is a 3-dimensional vector function of (t, x_1, x_2, x_3) which represents the velocity of a fluid element at time t and at position (x_1, x_2, x_3). The coefficient ν is the kinematic viscosity. The scalar quantity $P = P(t, x_1, x_2, x_3)$ is the pressure, and $F = F(t, x_1, x_2, x_3)$ is an external force.

We impose periodic boundary conditions on (1); thus we are in effect studying the Navier–Stokes equations in a three-dimensional torus which is thin in one direction. It turns out that, with these boundary conditions, problem (1) has a natural limit as $\varepsilon \to 0$; this limit is defined by the reduced Navier–Stokes equations in Ω. The passage to the limit is effectuated using properties of the Green's function of a certain ordinary differential operator. The fact that the Green's function behaves regularly as $\varepsilon \to 0$ is of basic significance and seems to be noted explicitly for the first time in the present paper.

We will assume that the external force F satisfies

$$\int_{Q_\varepsilon} F(t, x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3 = 0 \qquad \text{for all} \quad t \geqslant 0$$

(If F is independent of t, this condition is actually necessary for the existence of nontrivial periodic solutions of (1).) We will also assume that the reduced Navier–Stokes equations admit a T-periodic solution u_0; for simplicity we will assume that $\int_\Omega u_0(0, x_1, x_2)\, dx_1\, dx_2 = 0$. We present two main results. First, if F is T-periodic with respect to t, and if the topological index of u_0 with respect to a nonlinear mapping defined by the reduced equations is not zero, then (1) admits a T-periodic solution u_ε which tends to u_0 in the L_2-sense as $\varepsilon \to 0$. The second result is proved under the assumption that F is autonomous: $F = F(x_1, x_2, x_3)$. One imposes mild conditions of a standard type on the linearization at u_0 of the reduced equations. It is then shown that there exist a function $\varepsilon \to T_\varepsilon$ and a T_ε-periodic solution u_ε of (1) such that $T_\varepsilon \to T$, and $u_\varepsilon \to u$ in the L_2-sense as $\varepsilon \to 0$. The proof of the second result uses the method of functionalization of a parameter. This important technique is discussed in [17] and has been applied successfully to several problems (see, e.g., [6]).

This paper is organized as follows. In Section 2, we discuss certain basic facts concerning the formulation of problem (1). We introduce the reduced Navier–Stokes equations in Ω, and state our main results. In Section 3 we prove these results.

It goes without saying that we have benefited from the vast and rich literature regarding the mathematical formulation of the Navier–Stokes equations and the properties of their solutions. We make no attempt to review this literature. However, we wish to acknowledge once and for all our repeated use of basic facts and concepts contained in the texts [1, 5, 19, 28, 29]. The specific approach we adopt for the study of our periodic solution problem is motivated in substantial part by the papers [25–27, 2]. For the application of other methodologies to the study of the mathematical properties of solutions of the Navier–Stokes equations, we refer to [9, 16, 20]. For an application of the averaging method to a problem in some sense related to the one we consider here, see [21].

2. PRELIMINARIES AND STATEMENT OF RESULTS

Let us write again the problem which we will study:

$$\begin{cases} \dfrac{\partial U}{\partial t} - \nu\, \Delta U + (U \cdot \nabla)\, U + \nabla P = F(t, x_1, x_2, x_3) \\ \nabla \cdot U = 0 \end{cases} \tag{1}$$

where $t > 0$ and $(x_1, x_2, x_3) \in Q_\varepsilon$. Here $Q_\varepsilon = \Omega \times (0, \varepsilon)$, where $\Omega \subset \mathbf{R}^2$ is the rectangle $[0, \ell_1] \times [0, \ell_2]$, and ε is a small positive parameter. The problem (1) is to be viewed as an evolution equation for the velocity $U = U(t, x_1, x_2, x_3)$ and the pressure $P = P(t, x_1, x_2, x_3)$. We will study (1) in the presence of periodic boundary conditions:

$$U(t, 0, x_2, x_3) = U(t, \ell_1, x_2, x_3) \tag{2a}$$

$$U_{x_1}(t, 0, x_2, x_3) = U_{x_1}(t, \ell_1, x_2, x_3) \tag{2b}$$

$$U(t, x_1, 0, x_3) = U(t, x_1, \ell_2, x_3) \tag{2c}$$

$$U_{x_2}(t, x_1, 0, x_3) = U_{x_2}(t, x_1, \ell_2, x_3) \tag{2d}$$

$$U(t, x_1, x_2, 0) = U(t, x_1, x_2, \varepsilon) \tag{2e}$$

$$U_{x_3}(t, x_1, x_2, 0) = U_{x_3}(t, x_1, x_2, \varepsilon) \tag{2f}$$

With our methods we can treat various types of external force field F; we indicate two possibilities. It is useful to think of F as a function depending parametrically on ε: we write $F = F_\varepsilon(t, x_1, x_2, x_3)$.

(i) Suppose that there exists $\varepsilon_0 > 0$ and a continuous function F: $[0, \infty) \times [0, \ell_1] \times [0, \ell_2] \times [0, \varepsilon_0] \to \mathbf{R}^3$; such that $F_\varepsilon(t, x_1, x_2, x_3) = F(t, x_1, x_2, x_3)$. In this case, we require that F be T-periodic in t for a fixed $T > 0$, and that $\int_{Q_\varepsilon} F(t, x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3 = 0$ for all $t \geqslant 0$ and $0 \leqslant \varepsilon \leqslant \varepsilon_0$. This last condition is satisfied if, for instance, $F(t, x_1, x_2, x_3) = \sum_{n=0}^{\infty} F_n(t, x_1, x_2)\, x_3^n$, where F_n is T-periodic with $\int_\Omega F_n(t, x_1, x_2)\, dx_1\, dx_2 = 0$ $(n \geqslant 0,\ t \geqslant 0)$.

(ii) Write $y = x_3/\varepsilon$ so that $0 \leqslant y \leqslant 1$, and let F be a continuous T-periodic mapping from $[0, \infty)$ to $L_2(Q)$ where $Q = [0, \ell_1] \times [0, \ell_2] \times [0, 1]$. Let $F_\varepsilon(t, x_1, x_2, x_3) = F(t, x_1, x_2, x_3/\varepsilon)$ for $0 \leqslant \varepsilon \leqslant \varepsilon_0$. We can write $F(t, x_1, x_2, y) = \sum_{n=-\infty}^{\infty} F_n(t, x_1, x_2)\, e^{2\pi i n y}$; we impose the sole condition that $\int_\Omega F_0(t, x_1, x_2)\, dx_1\, dx_2 = 0$ for all $t \geqslant 0$. We see that, as $\varepsilon \to 0$, F_ε converges weakly in $L_2(Q)$ to F_0 for each $t \geqslant 0$; it will turn out that this weak convergence is sufficient for the validity of our results.

Clearly, the above considerations apply also to the case when F does not depend on t.

We also suppose that the initial value $U_0(x_1, x_2, x_3) = U_0(0, x_1, x_2, x_3)$ satisfies $\int_{Q_\varepsilon} U_0(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3 = 0$. As remarked in [23], this implies that $\int_{Q_\varepsilon} U(t, x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3 = 0$ for all $t \geqslant 0$.

We are going to study the problem of the existence of periodic solutions in time of (1)–(2). As we will see, it will not be necessary to discuss the question of the existence and regularity of the solution corresponding to general initial values U and P. The existence and regularity question for the thin domain Q_ε was studied in [22–24]. Of course, for the 2-dimensional Navier–Stokes equations, one has a complete theory of existence and regularity of solutions of (1)–(2); see, e.g., [1, 19, 28]. For the Navier–Stokes equations in a general three-dimensional domain the existence and regularity problem has been resolved only for short time intervals.

We introduce the change of variables

$$\begin{aligned} x_1 &= x_1 \\ x_2 &= x_2 \\ x_3 &= \varepsilon y \\ u(t, x_1, x_2, y) &= U(t, x_1, x_2, \varepsilon y) \\ p(t, x_1, x_2, y) &= P(t, x_1, x_2, \varepsilon y) \end{aligned}$$

Then Eqs. (1) take the form

$$\frac{\partial u}{\partial t} - \nu\, \Delta_\varepsilon u + (u \cdot \nabla_\varepsilon)\, u + \nabla_\varepsilon p = F(t, x_1, x_2, \varepsilon y) \tag{1_ε}$$

where $\nabla_\varepsilon = (\partial/\partial x_1, \partial/\partial x_2, (1/\varepsilon)(\partial/\partial y))$ and $\Delta_\varepsilon = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + (1/\varepsilon)(\partial^2/\partial y^2)$ are singular differential operators. These differential operators act on functions defined on the fixed domain

$$Q = (0, \ell_1) \times (0, \ell_2) \times (0, 1)$$

The boundary conditions (2) become

$$u(t, 0, x_2, y) = u(t, \ell_1, x_2, y) \tag{$2a_\varepsilon$}$$

$$u_{x_1}(t, 0, x_2, y) = u_{x_1}(t, \ell_1, x_2, y) \tag{$2b_\varepsilon$}$$

$$u(t, x_1, 0, y) = u(t, x_1, \ell_2, y) \tag{$2c_\varepsilon$}$$

$$u_{x_2}(t, x_1, 0, y) = u_{x_2}(t, x_1, \ell_2, y) \tag{$2d_\varepsilon$}$$

$$u(t, x_1, x_2, 0) = u(t, x_1, x_2, 1) \tag{$2e_\varepsilon$}$$

$$u_y(t, x_1, x_2, 0) = u_y(t, x_1, x_2, 1) \tag{$2f_\varepsilon$}$$

Let $L_2(Q)$ be the set of $\mathbf{R}^3$-valued vector functions u on Q with finite norm $\|u\|_2 = (\int_Q |u(x_1, x_2, y)|^2 \, dx_1 \, dx_2 \, dy)^{1/2}$; here $|\cdot|$ denotes the Euclidean norm on $\mathbf{R}^3$. We emphasize that $L_2(Q)$ denotes a space of *vector* functions. The differential expression Δ_ε together with the conditions (2_ε) can be taken to define a self-adjoint operator on $L_2(Q)$, whose domain can be identified with the closure in the Sobolev space $H^2(Q)$ (of $\mathbf{R}^3$-valued functions) of those vector fields defined on $\bar{Q}$ which are C^∞-smooth when extended periodically to $\mathbf{R}^3$. We also introduce the spaces $L_p(Q)$ of $\mathbf{R}^3$-valued functions on Q with finite norm $\|u\|_p = (\int_Q |u(x_1, x_2, y)|^p \, dx_1 \, dx_2 \, dy)^{1/p}$; Δ_ε and conditions (2_ε) define a closed unbounded linear operator on $L_p(Q)$ whose domain can be identified with the closure of the smooth periodic vector fields in the Sobolev space $W^{2,p}(Q)$.

Having said all this, we recall that, using a well-known method (e.g., [1, 19, 28]), one can reduce the study of problem (1_ε)–(2_ε) to that of an abstract evolution equation on the space of divergence-free vector fields on Q. In fact, let

$$H_\varepsilon = \operatorname{cls} \left\{ u \in H^1_{\text{per}}(Q) \mid \nabla_\varepsilon \cdot u = 0 \text{ and } \int_Q u \, dx_1 \, dx_2 \, dx_3 = 0 \right\} \subset L_2(Q)$$

where the set $H^1_{\text{per}}(Q)$ of periodic vector fields in the Sobolev space $H^1(Q)$ is identified with a subset of $L_2(Q)$, and the closure is in $L_2(Q)$. Let $\mathbf{P}_\varepsilon \colon L_2(Q) \to H_\varepsilon$ be the orthogonal projection (the *Leray projector*). Define

$$\tilde{L}_\varepsilon = -\mathbf{P}_\varepsilon \Delta_\varepsilon$$

where as stated Δ_ε is defined using the periodic boundary conditions (2_ε). Further, define the bilinear operator B_ε on $H_\varepsilon \times H_\varepsilon$ by

$$B_\varepsilon(v, w) = \mathbf{P}_\varepsilon(v \cdot \nabla_\varepsilon) \, w$$

Then problem (1_ε)–(2_ε) is equivalent to the problem

$$\frac{\partial u}{\partial t} + \nu \tilde{L}_\varepsilon u + B_\varepsilon(u, u) = \mathbf{P}_\varepsilon F \qquad (3_\varepsilon)$$

The equivalence is to be understood in the following sense. If $u = u(t, x_1, x_2, y)$ is a solution of (3_ε), then there is a function $p = p(t, x_1, x_2, y)$, which defines an element of $H^1(Q)$ for each $t \in \mathbf{R}$, such that (u, p) is a solution of (1_ε)–(2_ε). Moreover, the pressure p is unique up to an additive constant. (See [1, Prop. 1.6].)

Remarks 2.1. 1. Note that, in the presence of the periodic boundary conditions (2_ε), we have

$$\mathbf{P}_\varepsilon \nabla_\varepsilon = \nabla_\varepsilon \mathbf{P}_\varepsilon \qquad \text{and} \qquad \mathbf{P}_\varepsilon \Delta_\varepsilon = \Delta_\varepsilon \mathbf{P}_\varepsilon$$

where all operators act on $L_2(Q)$. This can be proved by an elementary Fourier series argument; see [1, p. 43] and [23].

2. Note that 0 is an eigenvalue of $\tilde{L}_\varepsilon$ and that the rest of the spectrum of $\tilde{L}_\varepsilon$ lies in the positive real half-line and is discrete. In fact, for each $\gamma > 0$, $(\gamma + \tilde{L}_\varepsilon)^{-1}$ is a compact self- adjoint operator. The same remarks apply to the operator Δ_ε on $L_2(Q)$.

Equation (3_ε) is of parabolic type and can be written in the abstract form,

$$u' + \nu \tilde{L}_\varepsilon u = \tilde{f}(t, u, \varepsilon) \tag{4_ε}$$

where $\tilde{f}(t, u, \varepsilon) = -B_\varepsilon(u, u) + \mathbf{P}_\varepsilon F$, $(u \in H_\varepsilon)$. It is convenient to introduce a positive constant γ and add a term γI to both sides of (4_ε). Writing

$$L_\varepsilon = \nu \tilde{L}_\varepsilon + \gamma I, \qquad f = \tilde{f} + \gamma I$$

we transform (4_ε) into

$$u' + L_\varepsilon u = f(t, u, \varepsilon) \tag{5_ε}$$

The positive constant γ will be held fixed for the rest of the paper. It is also convenient to normalize the constant ν: from now on we set $\nu = 1$.

Roughly speaking, we propose to solve (5_ε) by "multiplying (5_ε) by $e^{L_\varepsilon t}$ and integrating." This will "work" because of two facts (among others). First, L_ε is positive and self-adjoint, hence $e^{-L_\varepsilon t}$ is an analytic semigroup. Second, the nonlinear term in (5_ε) is subordinate to a fractional power of L_ε. This basic observation has been exploited in [2–4, 27].

We pause for a brief review of the basic facts about analytic semigroups and fractional powers of operators which we will need. As a general reference for these results we give [18].

Let A be a closed linear operator in a Banach space E. Suppose that, for all complex numbers λ satisfying $\operatorname{Re} \lambda \geqslant \sigma$, the resolvent $(\lambda I + A)^{-1}$ exists and satisfies the following inequality:

$$\|(\lambda I + A)^{-1}\| \leqslant \frac{c}{1 + |\lambda|} \tag{6}$$

Then A generates an analytic semigroup e^{-At}, and moreover, the fractional powers A^z are defined for $\operatorname{Re} z \neq 0$ (see [18]). If A is unbounded, then A^z is unbounded for $\operatorname{Re} z > 0$.

If $w \in \mathbf{C}$ and $\tau \in (-\pi, \pi)$, define the ray

$$\Gamma(\tau, w) = \{\lambda \in \mathbf{C} \mid \lambda = w + \rho e^{i\tau}, \rho \in [0, \infty)\}$$

Let $\sigma \in \mathbf{R}$ and c be as above, and fix $\beta \in (\pi/2, \pi/2 + \arcsin(1/c))$. Set

$$\Gamma_1 = \Gamma(-\beta, \sigma), \qquad \Gamma_2 = \Gamma(\beta, \sigma)$$

and write $\Gamma_1 \cup \Gamma_2$ for the curve obtained by joining Γ_1 and Γ_2 at the vertex σ. We traverse this curve from bottom to top, i.e., so that the spectrum of $-A$ is to the left of $\Gamma_1 \cup \Gamma_2$. Then the following formula is valid:

$$e^{-At} = \frac{1}{2\pi i} \int_{\Gamma_1 \cup \Gamma_2} e^{\lambda t}(\lambda I + A)^{-1} \, d\lambda \tag{7}$$

One has, further, that, for $t > 0$ and $\alpha \geqslant 0$,

$$A^{\alpha} e^{-At} = \frac{1}{2\pi i} \int_{\Gamma_1 \cup \Gamma_2} \lambda^{\alpha} e^{\lambda t}(\lambda I + A)^{-1} \, d\lambda \tag{8}$$

For the negative fractional powers $-1 < -\alpha < 0$, one has

$$A^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^{\infty} t^{-\alpha}(\lambda I + A)^{-1} \, dt \tag{9}$$

We note, finally, the following estimate:

$$\|A^{\alpha} e^{-At}\| \leqslant \frac{C(\alpha, c, \sigma)}{t^{\alpha}} \qquad (t > 0, \alpha \geqslant 0) \tag{10}$$

where C depends on the quantities indicated but not on t.

For each fixed $\varepsilon > 0$, the operator L_{ε} satisfies condition (6) for $\operatorname{Re} \lambda \geqslant \sigma = 0$. (In Section 3 we will prove the basic fact that the constant $c(\varepsilon)$ in (6) can be fixed independently of ε.) We now *define* what we mean by a T-periodic solution of problem (3_{ε}): it is a solution of the equation

$$\begin{aligned} u(t) = {} & e^{-L_{\varepsilon} t}(I - e^{-L_{\varepsilon} T})^{-1} \int_0^T L_{\varepsilon}^{\alpha} e^{-L_{\varepsilon}(T-s)} f(s, L_{\varepsilon}^{-\alpha} u(s), \varepsilon) \, ds \\ & + \int_0^t L_{\varepsilon}^{\alpha} e^{-L_{\varepsilon}(t-s)} f(s, L_{\varepsilon}^{-\alpha} u(s), \varepsilon) \, ds \end{aligned} \tag{11}$$

Here $e^{-L_\varepsilon t}$ is the analytic semigroup on H_ε defined by the operator L_ε, and α is an appropriate positive number (we will later choose $\alpha = 3/4$). The relation between solutions of (11) and T-periodic couples $(u(t, x_1, x_2, y), p(t, x_1, x_2, y))$ which satisfy (1_ε)–(2_ε) will not be discussed here; we note only that a strong T-periodic solution of (1_ε)–(2_ε) defines a solution of (11). See [2, 27] for material related to this question.

It is convenient to reformulate (11) in terms of the search for a fixed point of a certain mapping. Let $C_T(H_\varepsilon)$ be the set of T-periodic, continuous mappings from $\mathbf{R}$ into H_ε; it is a Banach space with respect to the natural sup-norm. If $u = u(t)$ is an element of $C_T(H_\varepsilon)$, define

$$\tilde{\Phi}_\varepsilon(u)(t) = e^{-L_\varepsilon t}(I - e^{-L_\varepsilon T})^{-1} \int_0^T L_\varepsilon^\alpha e^{-L_\varepsilon (T-s)} f(s, L_\varepsilon^{-\alpha} u(s), \varepsilon)\, ds$$

$$+ \int_0^t L_\varepsilon^\alpha e^{-L_\varepsilon (t-s)} f(s, L_\varepsilon^{-\alpha} u(s), \varepsilon)\, ds \tag{12}$$

Clearly solutions of (11) are in 1–1 correspondence with fixed points of the map $\tilde{\Phi}_\varepsilon$.

Actually, it will be convenient in our later analysis to extend the domain of definition of $\tilde{\Phi}_\varepsilon$ to all of $C_T(L_2(Q))$. For this, let Δ_ε be defined as above by the differential expression $(\partial^2/\partial x_1^2) + (\partial^2/\partial x_2^2) + [(1/\varepsilon^2)(\partial^2/\partial y^2)]$ together with the periodic boundary conditions (2_ε). Let

$$D_\varepsilon = \gamma I - \Delta_\varepsilon$$

so that D_ε is a positive definite, self-adjoint operator on $L_2(Q)$. Define $\Phi_\varepsilon\colon C_T(L_2(Q)) \to C_T(L_2(Q))$ by

$$\Phi_\varepsilon(u)(t) = e^{-D_\varepsilon t}(I - e^{-D_\varepsilon T})^{-1} \int_0^T D_\varepsilon^\alpha e^{-D_\varepsilon (T-s)} f(s, D_\varepsilon^{-\alpha} \mathbf{P}_\varepsilon u(s), \varepsilon)\, ds$$

$$+ \int_0^t D_\varepsilon^\alpha e^{-D_\varepsilon (t-s)} f(s, D_\varepsilon^{-\alpha} \mathbf{P}_\varepsilon u(s), \varepsilon)\, ds \tag{13}$$

We pause to prove certain basic properties of Φ_ε.

Lemma 2.2. *Fix $\alpha = 3/4$. For each $\varepsilon > 0$, the map $\Phi_\varepsilon\colon C_T(L_2(Q)) \to C_T(L_2(Q))$ is well defined and completely continuous.*

Proof. Let $u(\cdot) \in C_T(L_2(Q))$, and fix $\varepsilon > 0$. Write $\partial_i = \partial/\partial x_i$ $(i = 1, 2)$, and $\partial_\varepsilon = (1/\varepsilon)(\partial/\partial y)$.

Note first that, if $u \in L_2(Q)$, then $D_\varepsilon^{-3/4} u \in L_p(Q)$ for $1 \leqslant p < \infty$; see [18, p. 336]. Second, for each $1 \leqslant i \leqslant 3$, we have $\partial_i D_\varepsilon^{-3/4} u \in L_q(Q)$ for $1 \leqslant q < 3$. Here we refer again to p. 336 of [18] and use the fact that, if $p = 2$ and $\alpha = 3/4$, then $(1/q) > (1/p) - [(2\alpha - 1)/3] = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$. These facts imply that

$$(D_\varepsilon^{-3/4} u \cdot \nabla_\varepsilon)\, D_\varepsilon^{-3/4} u \in L_2(Q)$$

Next, using [18, p. 336] still again, we see that the map $L_2(Q) \to L_2(Q)$: $u \to (D_\varepsilon^{-3/4} u \cdot \nabla_\varepsilon)\, D_\varepsilon^{-3/4} u$ is continuous, and maps bounded sets into precompact sets. The continuity implies that

$$f(\cdot, D_\varepsilon^{-3/4} u(\cdot), \varepsilon) \in C_T(L_2(Q))$$

Since the estimate (10) implies that

$$\| D_\varepsilon^{-3/4} e^{-D_\varepsilon t} \|_{L_2(q) \to L_2(Q)} \leqslant \frac{C}{t^{3/4}} \qquad (t > 0)$$

we see from formula (13) that Φ_ε carries $C_T(L_2(Q))$ into itself and is completely continuous. This completes the proof of Lemma 2.2. □

Observe that the image of Φ_ε actually lies in H_ε. This is because $\mathbf{P}_\varepsilon D_\varepsilon = D_\varepsilon \mathbf{P}_\varepsilon$. Moreover, $\Phi_\varepsilon(u) = \tilde{\Phi}_\varepsilon(u)$ whenever $u = u(\cdot)$ is in $C_T(H_\varepsilon)$. So the fixed points of Φ_ε coincide with those of $\tilde{\Phi}_\varepsilon$. From now on, we will study almost exclusively the map Φ_ε.

Next we introduce the "reduced" Navier–Stokes equations in Ω. These are obtained by passing to the limit as $\varepsilon \to 0$ in (5_ε). We state immediately that the relation between Eq. (5_ε) and the reduced equations is not trivial; indeed most of the analysis in Section 3 will be devoted to studying it. For now we simply formulate the reduced equations.

Write $\mathbf{P}_2$ for the Leray projector in $L_2(\Omega)$; thus $\mathbf{P}_2$ is the orthogonal projection onto the subspace

$$\left\{ v \in H^1_{\text{per}}(\Omega) \mid \nabla_2 \cdot v = 0 \text{ and } \int_\Omega v(x_1, x_2)\, dx_1\, dx_2 = 0 \right\}$$

Here ∇_2 is the two-dimensional divergence. We have written $L_2(\Omega)$ resp. $H^1(\Omega)$ to indicate the appropriate spaces of $\mathbf{R}^2$-valued functions.

Define M: $L_2(Q) \to L_2(Q)$ to be the projection obtained by integrating with respect to the y-variable:

$$(Mu)(x_1, x_2) = \int_0^1 u(x_1, x_2, y)\, dy$$

Thus we agree to identify the function Mu with the element $\tilde{u}$ of $L_2(Q)$ defined by $\tilde{u}(x_1, x_2, y) = Mu(x_1, x_2)$. We recall the following fact, proved in [23]: let $g \in L_2(Q)$ be a vector field which depends only on the variables $(x_1, x_2) \in \Omega$ and such that $\int_\Omega g_3(x_1, x_2)\, dx_1\, dx_2 = 0$. Then one has

$$\mathbf{P}_\varepsilon \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} \mathbf{P}_2 \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \\ g_3 \end{pmatrix} \tag{14}$$

Following [23, p. 513], we write $\bar{u} = Mu$ and $w = (I - M)\, u$. Applying M and $I - M$ to (3_ε), we get

$$\begin{aligned} &\frac{\partial \bar{u}}{\partial t} + \tilde{L}_\varepsilon \bar{u} + B_\varepsilon(\bar{u}, \bar{u}) \\ &\qquad = M\mathbf{P}_\varepsilon F - M[B_\varepsilon(\bar{u}, w) + B_\varepsilon(w, \bar{u}) + B_\varepsilon(w, w)] \\ &\frac{\partial w}{\partial t} + \tilde{L}_\varepsilon w \\ &\qquad = (I - M)\, \mathbf{P}_\varepsilon F - (I - M)[B_\varepsilon(\bar{u}, w) + B_\varepsilon(w, \bar{u}) + B_\varepsilon(w, w)] \end{aligned} \tag{15}$$

If $\mathbf{P}_\varepsilon F$ depends only on (t, x_1, x_2), and if the initial value $w(0) = 0$, then $w(t) = 0$ for all $t > 0$, and the first Eq. (15) becomes an equation for $\bar{u}$ alone. To obtain the reduced Navier–Stokes equations, we replace the forcing function F_ε by an appropriate limiting function F_0. Let us consider the two cases discussed earlier.

(i) If $F_\varepsilon(t, x_1, x_2, x_3) = F(t, x_1, x_2, x_3)$ for a fixed, T-periodic continuous function defined on $[0, \infty) \times [0, \ell_1] \times [0, \ell_2] \times [0, \varepsilon_0]$, we set $F_0(t, x_1, x_2) = F(t, x_1, x_2, 0)$.

(ii) If $F_\varepsilon(t, x_1, x_2, x_3) = \sum_{n=-\infty}^{\infty} F_n(t, x_1, x_2)\, e^{2\pi i n y}$, where $y = x_3/\varepsilon$ and where the series on the right defines a continuous, T-periodic function of $[0, \infty)$ into $L_2(Q)$, then we let $F_0(t, x_1, x_2)$ be the 0th Fourier coefficient in the expansion of F_ε.

In case (i), $\mathbf{P}_\varepsilon F_\varepsilon \to F_0$ strongly in $L_2(Q)$. In case (ii), $\mathbf{P}_\varepsilon F_\varepsilon \to F_0$ weakly in $L_2(Q)$ as $\varepsilon \to 0$. Our results will require only the weak convergence of $\mathbf{P}_\varepsilon F_\varepsilon$ to F_0 as $\varepsilon \to 0$.

The reduced Navier–Stokes equations are the equations for $\bar{u} = Mu$ in (15) when F_0 is substituted for F. We can rewrite the reduced Navier–

Stokes equations in the following form. Write $\bar{u} = Mu = (\bar{u}_1, \bar{u}_2, \bar{u}_3)$. Put $\bar{v} = (\bar{u}_1, \bar{u}_2)$. Then the equations for $\bar{u}$ in (15) are

$$\frac{\partial \bar{v}}{\partial t} - \mathbf{P}_2\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\bar{v} + \mathbf{P}_2(\bar{v} \cdot \nabla_2)\,\bar{v} = \mathbf{P}_2\begin{pmatrix} F_{01} \\ F_{02} \end{pmatrix}$$
$$\frac{\partial \bar{u}_3}{\partial t} - \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\bar{u}_3 + \left((\mathbf{P}_2\bar{v})_1\frac{\partial}{\partial x_1} + (\mathbf{P}_2\bar{v})_2\frac{\partial}{\partial x_2}\right)\bar{u}_3 = F_{03} \tag{16}$$

These equations are supplemented by the periodic boundary conditions

$$\bar{u}(t, 0, x_2) = \bar{u}(t, \ell_1, x_2) \tag{17a}$$
$$\bar{u}_{x_1}(t, 0, x_2) = \bar{u}_{x_1}(t, \ell_1, x_2) \tag{17b}$$
$$\bar{u}(t, x_1, 0) = \bar{u}(t, x_1, \ell_2) \tag{17c}$$
$$\bar{u}_{x_2}(t, x_1, 0) = \bar{u}_{x_2}(t, x_1, \ell_2) \tag{17d}$$

We also impose the mean value condition

$$\int_\Omega \bar{u}(0, x_1, x_2)\, dx_1\, dx_2 = 0 \tag{17e}$$

This last condition implies that $\int_\Omega \bar{u}(t, x_1, x_2)\, dx_1\, dx_2 = 0$ for all $t \geqslant 0$.

The exact form of the operator Φ_0 corresponding to these equations will be given in Section 3; see relation (53). The connection between Φ_ε and Φ_0 will also be discussed in Section 3.

We now formulate our main results. Recall that, if $\Phi: E \to E$ is a completely continuous map on a Banach space E, and if x is an isolated fixed point of Φ, then the topological index $\operatorname{ind}(x, \Phi)$ is equal to the topological degree at x of the mapping $I - \Phi$.

Theorem 1. *Suppose that F is T-periodic with respect to t. Suppose that the reduced Eqs. (16)–(17) admit a T-periodic solution $\bar{u}_0$ which has nonzero topological index with respect to the map Φ_0. Then for sufficiently small $\varepsilon \neq 0$, the divergence-free Navier–Stokes Eqs. (3_ε) in Q_ε with periodic boundary conditions admit a T-periodic solution u_ε, and one has*

$$\sup_t \left(\int_Q |u_\varepsilon(t, x_1, x_2, y) - \bar{u}_0(t, x_1, x_2)|^2\, dx_1\, dx_2\, dy\right) \to 0$$

as $\varepsilon \to 0$.

We remark that the mean value condition (17e) implies that, under reasonable conditions, a fixed point $\bar{u}_0$ of Φ_0 is necessarily isolated. For example, if λ_1 is the lowest nonzero eigenvalue of $-\Delta_2$ in $H_2 = \mathbf{P}_2 L_2(\Omega)$, and if $\frac{1}{2}\, |(\partial \bar{v}_{01}/\partial x_2) + (\partial \bar{v}_{02}/\partial x_1)| < \lambda_1$, then one can show that $\bar{u}_0$ is isolated.

As we will see in Section 3, $u_\varepsilon \in C_T(L_2(Q))$. Next we suppose that the forcing term F in (1) does not depend on t. We pose the problem of self-oscillations of the Navier–Stokes equations in Q_ε. We will apply the method of functionalization of a parameter to this problem [6, 17]. Suppose that the reduced Eqs. (16)–(17) admit a T_0-periodic solution $\bar{u}_0$. We introduce the time-change $t \to (T_0/T)\, t$ ($T > 0$ fixed) in Eqs. (1); they become

$$\frac{T_0}{T} \frac{\partial \hat{U}}{\partial t} - \Delta \hat{U} + (\hat{U} \cdot \nabla)\, \hat{U} + \nabla \hat{P} = F(x_1, x_2, x_3) \tag{18}$$

for the quantities $\hat{U}(t, x_1, x_2, x_3) = U((T/T_0)\, t, x_1, x_2, x_3)$ and $\hat{P}(t, x_1, x_2, x_3) = P((T/T_0)\, t, x_1, x_2, x_3)$. We look for T_0-periodic solutions of the equation (18). We will prove the following result.

Theorem 2. *Consider the autonomous reduced Navier–Stokes Eqs. (16)–(17). Let $\bar{u}_0$ be a T_0-periodic solution of these equations. Consider the linearization of the reduced equations along $\bar{u}_0$: writing $\bar{u}_0 = (\bar{u}_{01}, \bar{u}_{02}, \bar{u}_{03})$, $\bar{v}_0 = (\bar{u}_{01}, \bar{u}_{02})$, $u = (u_1, u_2, u_3)$, and $v = (u_1, u_2)$, we have*

$$\frac{\partial v}{\partial t} - \mathbf{P}_2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) v + \mathbf{P}_2(\bar{v}_0 \cdot \nabla_2)\, v + \mathbf{P}_2(v \cdot \nabla_2)\, \bar{v}_0 = 0$$

$$\frac{\partial u_3}{\partial t} - \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) u_3 + \left(\bar{u}_{01} \frac{\partial}{\partial x_1} + \bar{u}_{02} \frac{\partial}{\partial x_2} \right) u_3 + \left(u_1 \frac{\partial \bar{u}_{03}}{\partial x_1} + u_2 \frac{\partial \bar{u}_{03}}{\partial x_2} \right) = 0 \tag{19}$$

Suppose that Eqs. (19) admit no T_0-periodic solutions linearly independent of $\partial \bar{u}_0/\partial t$. Suppose further that Eqs. (19) admit no solution of the form $z(t, x_1, x_2) + (t/T_0)(\partial \bar{u}_0/\partial t)(t, x_1, x_2)$, where z is T_0-periodic.

Then for small $\varepsilon > 0$ the divergence-free Navier–Stokes equations with periodic boundary conditions in Q_ε admit a T_ε-periodic solution u_ε such that

$$T_\varepsilon \to T_0 \qquad as \quad \varepsilon \to 0$$

and

$$\sup_t \left(\int_Q \left| u_\varepsilon \left(\frac{T_0}{T_\varepsilon}\, t, x_1, x_2, y \right) - \bar{u}_0(t, x_1, x_2) \right|^2 dx_1\, dx_2\, dy \right) \to 0$$

as $\varepsilon \to 0$.

When we speak of solutions of Eqs. (19), we mean solutions in $C_T(L_2(Q))$ of the linearization at $\bar{u}_0$ of the operator equation $u = \Phi_0(u)$. The proofs of Theorem 1 and 2 will be carried out in Section 3.

Theorems 1 and 2 are quite general and easy to apply. Though we do not consider specific examples in this paper, we do include some remarks at the end of Section 3 concerning the verification of the hypotheses of our theorems.

3. PROOFS OF THEOREMS 1 AND 2

We begin by deriving an estimate which is basic in our proofs. Recall that, for each $\varepsilon > 0$, $D_\varepsilon = \gamma I - \Delta_\varepsilon$ is the positive definite, self-adjoint operator defined on $L_2(Q)$ by the periodic boundary conditions (2_ε); here γ is a fixed positive constant. It is easily seen that $\|(\lambda + D_\varepsilon)^{-1}\| \leqslant c(\varepsilon)/(1+|\lambda|)$ for $\operatorname{Re}\lambda \geqslant 0$, where the norm is that of bounded linear operators on $L_2(Q)$, and $c(\varepsilon)$ is a constant. We are going to show that $c(\varepsilon)$ can be chosen to be independent of ε. In fact, more generally we have:

Proposition 3.1. *For* $1 < p < \infty$,

$$\|(\lambda + D_\varepsilon)^{-1}\|_{L_p(Q) \to L_p(Q)} \leqslant \frac{\hat{c}}{1+|\lambda|} \qquad (\operatorname{Re}\lambda \geqslant 0) \tag{20}$$

and

$$\|\partial_i(\lambda + D_\varepsilon)^{-1}\|_{L_p(Q) \to L_p(Q)} \leqslant \frac{\hat{c}}{\sqrt{1+|\lambda|}} \qquad (\operatorname{Re}\lambda \geqslant 0;\ i = 1, 2, 3) \tag{21}$$

Here $\partial_1 = \partial/x_1$, $\partial_2 = \partial/\partial x_2$, $\partial_3 = (1/\varepsilon)(\partial/\partial y)$, *and* $\hat{c}$ *is a constant which does not depend on* ε (*nor on* λ).

Proof. The proof involves several steps. It is based on the fundamental formula of Grisvard [8]. Let us begin by recalling this formula. It was used by I. Gourova and P. Sobolevskii [7] for the construction of L-characteristics of difference operators.

Following [8], let E be a Banach space, and let $A_i: \mathscr{D}(A_i) \to E$ $(i = 1, 2)$ be two closed operators. The domains $\mathscr{D}(A_i)$ are subsets of E; it is supposed that their intersection $\mathscr{D}(A_1) \cap \mathscr{D}(A_2)$ is dense in E. Suppose that the two operators *commute* on a dense subset of $\mathscr{D}(A_1) \cap \mathscr{D}(A_2)$. Suppose further that $(z + A_i)^{-1}$ is defined for all z with $\operatorname{Re} z \geqslant 0$ and satisfies

$$\|(z + A_i)^{-1}\| \leqslant \frac{c}{1+|z|} \qquad (\operatorname{Re} z \geqslant 0;\ i = 1, 2) \tag{22}$$

here and below, c is a constant which may vary from line to line. Let $\psi \in (\pi/2, \pi/2 + \arcsin(1/c))$, and let $\Gamma_1 = \Gamma(-\psi, 0)$, $\Gamma_2 = \Gamma(\psi, 0)$ where the notation is as in Section 2: $\Gamma(\tau, w) = \{\lambda \in \mathbf{C} \mid \lambda = w + \rho e^{i\tau}, \rho \in [0, \infty)\}$. Let the curve $\Gamma_1 \cup \Gamma_2$ be traversed from bottom to top, keeping the spectra of $-A_1$ and $-A_2$ to the left. Then

$$(A_1 + A_2)^{-1} = \frac{1}{2\pi i} \int_\Gamma (A_1 - z)^{-1} (A_2 + z)^{-1} \, dz \tag{23}$$

Suppose that A_1 and A_2 are positive definite, self-adjoint operators whose spectrum is a subset of $[a, \infty)$, $a > 0$. In this case, (23) holds when Γ is replaced by $\Gamma_1(\psi_1, \sigma) \cup \Gamma_2(\psi_2, \sigma)$, where $-a < \sigma < a$ and $-\pi < \psi_1 < 0$, $0 < \psi_2 < \pi$. There is a similar freedom of choice when A_1 and A_2 are translates of positive, self-adjoint operators. It suffices that Γ be the boundary of a domain in $\mathbf{C}$ containing the spectra of $-A_1$ and $-A_2$.

We apply (23) to the following operators. Let A_1 be the operator on $L_p(Q)$ defined by $(\lambda/2) + (\gamma/2) - (\partial^2/\partial x_1^2) - (\partial^2/\partial x_2^2)$ with the boundary conditions $u(0, x_2, y) = u(\ell_1, x_2, y)$, $\nabla_2 u(0, x_2, y) = \nabla_2 u(\ell_1, x_2, y)$, $u(x_1, 0, y) = u(x_1, \ell_2, y)$, $\nabla_2 u(x_1, 0, y) = \nabla_2 u(x_1, \ell_2, y)$. (Thus A_1 is actually the tensor product of an appropriate identity operator with an unbounded closed operator.) Let A_2^ε be the operator on $L_p(Q)$ defined by $(\lambda/2) + (\gamma/2) - (1/\varepsilon^2)(\partial^2/\partial y^2)$ with the boundary conditions $u(x_1, x_2, 0) = u(x_1, x_2, 1)$, $(\partial u/\partial y)(x_1, x_2, 0) = (\partial u/\partial y)(x_1, x_2, 1)$. (Here again, A_2^ε is actually an appropriate tensor product...) It may be checked that the hypotheses listed above, in particular, (22), are valid for A_1 and A_2^ε, for all $1 < p < \infty$ and all $\varepsilon > 0$. In fact, for the operator A_1, (22) is well known and follows from the ellipticity of $(\gamma/2) - (\partial^2/\partial x_1^2) - (\partial^2/\partial x_2^2)$ with periodic boundary conditions; see [18, Sections 16.2 and 16.3]. For A_2^ε the estimate (22) holds, and for a similar reason. However, the constant c may a priori be a function of ε.

We will show that, in fact, the constant $c(\varepsilon)$ of (22) for A_2^ε can be fixed independently of ε. To do this, we estimate the Green's function of the ordinary differential operator $A_2^\varepsilon + z$. The Green's function $G(y, \eta)$ is obtained by studying the following boundary value problem:

$$\begin{gathered} -\frac{1}{\varepsilon^2} v'' + \left(z + \frac{\lambda}{2} + \frac{\gamma}{2} \right) v = -\delta(\eta) \\ v(0) = v(1), \qquad v'(0) = v'(1) \end{gathered} \tag{24}$$

where the prime $'$ denotes d/dy.

It will turn out later that, for all values z and λ of interest, the quantity $\omega = z + (\lambda/2) + (\gamma/2)$ lies outside the negative real axis $(-\infty, 0] \subset \mathbf{C}$. Choose $\sqrt{\omega}$ to be the square root of ω with positive real part. One finds that

$$G(y, \eta; \omega, \varepsilon) = \varepsilon \frac{e^{-\varepsilon\sqrt{\omega}\,|y-\eta|} + e^{-\varepsilon\sqrt{\omega}\,(1-|y-\eta|)}}{2\sqrt{\omega}\,(1-e^{-\varepsilon\sqrt{\omega}})} \tag{25}$$

The solution of the nonhomogeneous problem

$$-\frac{1}{\varepsilon^2} v'' + \omega v = \varphi(y)$$

$$v(0) = v(1), \qquad v'(0) = v'(1)$$

is

$$v(y) = \int_0^1 G(y, \eta; \omega, \varepsilon)\, \varphi(\eta)\, d\eta \tag{26}$$

One can verify that

$$\frac{1}{\varepsilon}\frac{\partial G}{\partial y}(y, \eta; \varepsilon, \omega) = \begin{cases} \varepsilon \dfrac{e^{-\varepsilon\sqrt{\omega}\,(1-y+\eta)} - e^{-\varepsilon\sqrt{\omega}\,(y-\eta)}}{2(1-e^{-\varepsilon\sqrt{\omega}})} & y > \eta \\[2ex] \varepsilon \dfrac{e^{-\varepsilon\sqrt{\omega}\,(y-\eta)} - e^{-\varepsilon\sqrt{\omega}\,(1-y+\eta)}}{2(1-e^{-\varepsilon\sqrt{\omega}})} & y < \eta \end{cases} \tag{27}$$

Finally, we note that, in further developments, z and λ will take values such that ω lies in a closed sector in $\mathbf{C}$ which is disjoint from the negative real axis. In this case there is a positive constant d such that

$$\frac{1}{d}\operatorname{Re}\sqrt{\omega} \leqslant |\sqrt{\omega}| \leqslant d \operatorname{Re}\sqrt{\omega}$$

for all relevant values of ω.

We now prove the following.

Lemma 3.2. *There is a positive constant c with the following properties.*

(a) If $\varphi \in L_p[0, 1]$, then

$$\left\| y \mapsto \int_0^1 G(y, \eta; \omega, \varepsilon)\, \varphi(\eta)\, d\eta \right\|_p \leqslant \frac{c}{|\omega|} \|\varphi\|_p$$

(b) If $\varphi \in L_p[0, 1]$, *then*

$$\left\| y \mapsto \int_0^1 \frac{1}{\varepsilon} \frac{\partial G}{\partial y} (y, \eta; \omega, \varepsilon)\, \varphi(\eta)\, d\eta \right\|_p \leqslant \frac{c}{|\sqrt{\omega}|} \|\varphi\|_p$$

for any $p \geqslant 1$.

Proof. Recall that $\partial_3 = (1/\varepsilon)(\partial/\partial y)$, so that the integrand in (b) is $(\partial_3 G)\, \varphi$.

Observe first that G and $\partial_3 G$ can be estimated as follows:

$$|G(y, \eta; \omega, \varepsilon)| \leqslant \begin{cases} \dfrac{c}{|\omega|} & \text{for} \quad |\varepsilon \sqrt{\omega}| \leqslant 1 \\ c\varepsilon \dfrac{e^{-\varepsilon \operatorname{Re} \sqrt{\omega}\, |y-\eta|} + e^{-\varepsilon \operatorname{Re} \sqrt{\omega}\, (1-|y-\eta|)}}{|\sqrt{\omega}|} & \text{for} \quad |\varepsilon \sqrt{\omega}| \geqslant 1 \end{cases} \tag{28a}$$

$$|\partial_3 G(y, \eta; \omega, \varepsilon)| \leqslant \begin{cases} \dfrac{c}{|\sqrt{\omega}|} & \text{for} \quad |\varepsilon \sqrt{\omega}| \leqslant 1 \\ c\varepsilon [e^{-\varepsilon \operatorname{Re} \sqrt{\omega}\, |y-\eta|} + e^{-\varepsilon \operatorname{Re} \sqrt{\omega}\, (1-|y-\eta|)}] & \text{for} \quad |\varepsilon \sqrt{\omega}| \geqslant 1 \end{cases} \tag{28b}$$

where c is a constant independent of ω and ε.

Suppose that $|\varepsilon \sqrt{\omega}| \leqslant 1$. We have

$$\left(\int_0^1 \left| \int_0^1 G(y, \eta; \omega, \varepsilon)\, \varphi(\eta)\, d\eta \right|^p dy \right)^{1/p}$$

$$\leqslant \left[\int_0^1 \left(\int_0^1 \frac{c}{|\omega|} |\varphi(\eta)|\, d\eta \right)^p dy \right]^{1/p}$$

$$\leqslant \frac{c}{|\omega|} \left[\int_0^1 \int_0^1 |\varphi(\eta)|^p\, d\eta\, dy \right]^{1/p}$$

$$= \frac{c}{|\omega|} \|\varphi\|_p$$

On the other hand, if $|\varepsilon\sqrt{\omega}| \geqslant 1$, we have

$$\left(\int_0^1 \left|\int_0^1 G(y, \eta; \omega, \varepsilon)\, \varphi(\eta)\, d\eta\right|^p dy\right)^{1/p}$$

$$\leqslant c\left[\left(\int_0^1 \left|\int_0^1 \frac{\varepsilon e^{-\operatorname{Re}\sqrt{\omega}\,|y-\eta|}}{|\sqrt{\omega}|}\, |\varphi(\eta)|\, d\eta\right|^p dy\right)^{1/p}\right.$$

$$\left. + \left(\int_0^1 \left|\int_0^1 \frac{\varepsilon e^{-\operatorname{Re}\sqrt{\omega}\,(1-|y-\eta|)}}{|\sqrt{\omega}|}\, |\varphi(\eta)|\, d\eta\right|^p dy\right)^{1/p}\right]$$

$$= c[I_1(\omega) + I_2(\omega)]$$

We will estimate $I_1(\omega)$ and $I_2(\omega)$ separately. First, we have

$$I_1(\omega) = \frac{1}{|\sqrt{\omega}|}\left(\int_0^1 \left|\int_{-\infty}^{\infty} \varepsilon e^{-\varepsilon \operatorname{Re}\sqrt{\omega}\,|y-\eta|}\tilde{\varphi}(\eta)\, d\eta\right|^p dy\right)^{1/p}$$

where $\tilde{\varphi}$ is the extension of φ to $\mathbf{R}$ obtained by setting $\tilde{\varphi}(\eta) = 0$ for $\eta \notin [0, 1]$, $\tilde{\varphi}(\eta) = \varphi(\eta)$ for $\eta \in [0, 1]$. Setting $y - \eta = \tau$ and using the generalized Minkowski inequality of [18, p. 46], we see that

$$I_1(\omega) = \frac{1}{|\sqrt{\omega}|}\left(\int_0^1 \left|\int_{-\infty}^{\infty} \varepsilon e^{-\varepsilon \operatorname{Re}\sqrt{\omega}\,|\tau|}|\tilde{\varphi}(y-\tau)\, d\tau\right|^p dy\right)^{1/p}$$

$$\leqslant \frac{1}{|\sqrt{\omega}|}\left\{\int_{-\infty}^{\infty} \varepsilon e^{-\varepsilon \operatorname{Re}\sqrt{\omega}\,|\tau|}\left[\int_0^1 |\tilde{\varphi}(y-\tau)|^p\, dy\right]^{1/p} d\tau\right\}^{p \cdot 1/p}$$

$$\leqslant \frac{1}{|\sqrt{\omega}|} \cdot \frac{2}{\operatorname{Re}\sqrt{\omega}}\, \|\varphi\|_p \leqslant \frac{c}{|\omega|}\, \|\varphi\|_p$$

where the constant c is independent of ε, ω and p.

Turning to $I_2(\omega)$, we have

$$I_2(\omega) = \frac{1}{|\sqrt{\omega}|}\left(\int_0^1 \left|\int_0^y \varepsilon e^{-\varepsilon \operatorname{Re}\sqrt{\omega}\,(1-|y-\eta|)}\varphi(\eta)\, d\eta\right.\right.$$

$$\left.\left. + \int_y^1 \varepsilon e^{-\varepsilon \operatorname{Re}\sqrt{\omega}\,(1-|y-\eta|)}\varphi(\eta)\, d\eta\right|^p dy\right)^{1/p}$$

$$\leqslant \frac{1}{\sqrt{\omega}} \left[\left(\int_0^1 \left| \int_0^y \varepsilon e^{-\varepsilon \operatorname{Re} \sqrt{\omega}\,(1-y+\eta)} \varphi(\eta)\, d\eta \right|^p dy \right)^{1/p} \right.$$

$$\left. + \left(\int_0^1 \left| \int_y^1 \varepsilon e^{-\varepsilon \operatorname{Re} \sqrt{\omega}\,(1-y+\eta)} \varphi(\eta)\, d\eta \right|^p dy \right)^{1/p} \right]$$

$$= \frac{1}{|\sqrt{\omega}|} \left[\left(\int_0^1 \left| \int_{1-y}^1 \varepsilon e^{-\varepsilon \operatorname{Re} \sqrt{\omega}\,\tau} \varphi(y+\tau-1)\, d\tau \right|^p dy \right)^{1/p} \right.$$

$$\left. + \left(\int_0^1 \left| \int_y^1 \varepsilon e^{-\varepsilon \operatorname{Re} \sqrt{\omega}\,\tau} \varphi(y-\tau+1)\, d\tau \right|^p dy \right)^{1/p} \right]$$

$$\leqslant \frac{1}{|\sqrt{\omega}|} \left[\int_{-\infty}^{\infty} \varepsilon e^{-\varepsilon \operatorname{Re} \sqrt{\omega}\,|\tau|} \left(\int_0^1 |\tilde{\varphi}(y+\tau-1)|^p\, dy \right)^{1/p} d\tau \right.$$

$$\left. + \int_{-\infty}^{\infty} \varepsilon e^{-\varepsilon \operatorname{Re} \sqrt{\omega}\,|\tau|} \int_0^1 |\tilde{\varphi}(y-\tau+1)|^p\, dy \right)^{1/p} d\tau \right]$$

Here $\tilde{\varphi}$ is the extension of φ to **R** introduced previously, and we have used the generalized Minkowski inequality again. So we have

$$I_2(\omega) \leqslant \frac{c}{|\omega|} \|\varphi\|_p$$

where c is independent of ω, ε and p. Putting together the above estimates, we obtain (a) of the lemma for a constant c which is independent of ω, ε and p.

The estimate (28b) for $|\partial_3 G|$ differs from the estimate (28a) for $|G|$ only by a factor of $|\sqrt{\omega}|$. So going through the above calculations again, one has

$$\left(\int_0^1 \left| \int_0^1 \partial_3 G(y, \eta; \omega, \varepsilon)\, \varphi(\eta)\, d\eta \right|^p dy \right)^{1/p} \leqslant \frac{c}{|\sqrt{\omega}|} \|\varphi\|_p$$

for a constant c independent of ω, ε and p. This complete the proof of Lemma 3.2. □

Lemma 3.2 is actually the key step in our analysis of Problem (3_ε). The regular behavior of the Green's function G as $\varepsilon \to 0$ does not seen to have been noted explicitly in previous papers.

Returning to the proof of Proposition 3.1, let c be a constant so that (22) is valid for $A_1 = (\lambda/2) + (\gamma/2) - (\partial^2/\partial x_1^2) - (\partial^2/\partial x_2^2)$ and $A_2^\varepsilon = (\lambda/2) + (\gamma/2) - [(1/\varepsilon^2)(\partial^2/\partial y^2)]$. We wish to apply the Grisvard formula (23) to compute $(A_1+A_2)^{-1}=(\lambda+D_\varepsilon)^{-1}$. It is convenient to choose Γ in the following way. Let $\psi\in(\pi/2, \pi/2+\arcsin(1/c))$. If $\operatorname{Im}\lambda\geqslant 0$, let Γ be the straight line in $\mathbf{C}$ passing through $z=0$ which makes angle ψ with the positive real axis, trasversed from bottom to top. Explicitly, $\Gamma=\Gamma(\psi-\pi, 0)\cup\Gamma(\psi, 0)$. If $\operatorname{Im}\lambda<0$, let Γ be the straight line passing through $z=0$ which makes angle $\pi-\psi$ with the positive real axis. Choosing Γ in this way, one can determine a constant $c>0$ so that

$$|\lambda\pm z|\geqslant c(|\lambda|+|z|) \tag{29}$$

for all $\lambda\in\mathbf{C}$ with $\operatorname{Re}\lambda\geqslant 0$ and all $z\in\Gamma$. The constant is independent of λ and z.

Now (22) implies that, if c is increased appropriately, then

$$\|(A_1-z)^{-1}\|_{L_p(Q)\to L_p(Q)}\leqslant\frac{c}{1+|\lambda-z|}$$

for $\operatorname{Re}\lambda\geqslant 0$ and $z\in\Gamma$. Using Lemma 3.2(a) and (29), we conclude that

$$\|(A_1-z)^{-1}\,(A_2^\varepsilon+z)^{-1}\|_{L_p(Q)\to L_p(Q)}\leqslant\frac{c}{(1+|\lambda|+|z|)^2} \tag{30}$$

where c is independent of λ, z and ε. Hence

$$\left\|\int_\Gamma(A_1-z)^{-1}\,(A_2^\varepsilon+z)^{-1}\,dz\right\|_{L_p(Q)\to L_p(Q)}\leqslant c\int_0^\infty\frac{d\,|z|}{(1+|\lambda|+|z|)^2}\leqslant\frac{\hat{c}}{1+|\lambda|}$$

for all λ with $\operatorname{Re}\lambda\geqslant 0$; $\hat{c}$ is independent of ε and λ. This proves (20).

In a similar way, using the estimate (22) for A_1, (29), and Lemma 3.2(b), we see that

$$\|(A_1-z)^{-1}\,\partial_3(A_2^\varepsilon+z)^{-1}\|_{L_p(Q)\to L_p(Q)}\leqslant\frac{c}{(1+|\lambda|+|z|)^{3/2}} \tag{31}$$

where c is independent of λ, z and ε. It follows that

$$\left\|\int_\Gamma(A_1-z)^{-1}\,\partial_3(A_2^\varepsilon+z)^{-1}\right\|_{L_p(Q)\to L_p(Q)}$$
$$\leqslant c\int_0^\infty\frac{d\,|z|}{(1+|\lambda|+|z|)^{3/2}}\leqslant\frac{\hat{c}}{(1+|\lambda|)^{1/2}} \tag{32}$$

where, again, $\hat{c}$ does not depend on ε and λ.

One can verify that

$$\|\partial_i (A_1 - z)^{-1} (A_2^\varepsilon + z)^{-1}\|_{L_p(Q) \to L_p(Q)} \leqslant \frac{c}{(1 + |\lambda| + |z|)^{3/2}} \tag{33}$$

for $i = 1, 2$, Re $\lambda \geqslant 0$ and $z \in \Gamma$. This gives

$$\left\| \int_\Gamma \partial_i (A_1 - z)^{-1} (A_2^\varepsilon + z)^{-1} \, dz \right\| \leqslant c \int_0^\infty \frac{d\,|z|}{(1 + |\lambda| + |z|)^{3/2}} \leqslant \frac{\hat{c}}{(1 + |\lambda|)^{1/2}} \tag{34}$$

for $i = 1, 2$. So the Grisvard formula together with (30), (32), and (34) implies (21). This completes the proof of Proposition 3.1. □

We will use Proposition 3.1 to study the behavior as $\varepsilon \to 0$ of the operators

$$D_\varepsilon^{-\alpha}, \qquad e^{-D_\varepsilon t}, \qquad D_\varepsilon^\alpha e^{-D_\varepsilon t}$$

Here $\alpha \in (0, 1)$ is a number which will later be fixed at the value 3/4. To study the limits as $\varepsilon \to 0$ of these operators, we use the estimate (20) together with formulas (7)–(9): it is sufficient to determine that $(\lambda + D_\varepsilon)^{-1}$ has a limit as $\varepsilon \to 0$, and that the limit is uniform in each compact subset of $\{\lambda \in \mathbf{C} \mid \lambda$ lies on or to the right of the curve $\Gamma = \Gamma(-\beta, 0) \cup \Gamma(\beta, 0)\}$ for some fixed $\beta \in (\pi/2, \pi/2 + \arcsin(1/\hat{c}))$. A simple calculation shows that it is actually sufficient that $(\lambda + D_\varepsilon)^{-1}$ tend to a limit as $\varepsilon \to 0$ uniformly in each compact subset K of $\{\lambda \in \mathbf{C} \mid \text{Re}\, \lambda \geqslant 0\}$ (use estimate (20)).

In the following proposition, we determine the limits as $\varepsilon \to 0$ of $(\lambda + D_\varepsilon)^{-1}$ and also of $\partial_i (\lambda + D_\varepsilon)^{-1}$ $(1 \leqslant i \leqslant 3)$. Before stating the result, recall that M: $L_p(Q) \to L_p(Q)$ is defined by $(Mu)(x_1, x_2) = \int_0^1 u(x_1, x_2, y)\, dy$. We regard Mu as an element of $L_p(Q)$ which does not depend on y. For $1 < p < \infty$, let D_0 denote the closed operator on $L_p(Q)$ defined by $\gamma I - (\partial^2/\partial x_1^2) - (\partial^2/\partial x_2^2)$ with periodic boundary conditions.

Proposition 3.3. *As $\varepsilon \to 0$ we have the following convergences, in the operator norm, for $1 < p < \infty$:*

$$(\lambda + D_\varepsilon)^{-1} \to (\lambda + D_0)^{-1} M \tag{35}$$

$$\partial_i (\lambda + D_\varepsilon)^{-1} \to \partial_i (\lambda + D_0)^{-1} M \qquad (i = 1, 2) \tag{36}$$

$$\partial_3 (\lambda + D_\varepsilon)^{-1} \to 0 \tag{37}$$

The convergences are uniform in each compact subset K of $\{\lambda \in \mathbf{C} \mid \text{Re}\, \lambda \geqslant 0\}$.

Proof. First, let c be the constant of relations (30), (31), and (33), and choose $\psi \in (\pi/2, \pi/2 + \arcsin(1/c))$. Choose ψ close enough to $\pi/2$ so that $-(\lambda/2) - (\gamma/2)$ lies to the left of $\Gamma_\psi = \Gamma(-\psi, 0) \cup \Gamma(\psi, 0)$ for all $\lambda \in K$.

Consider first $(\lambda + D_\varepsilon)^{-1}$. Observe that, for all $\omega = z + (\lambda/2) + (\gamma/2)$ with $\operatorname{Re} \lambda \geqslant 0$ and $z \in \Gamma_\psi$, one has

$$G(y, \eta; \omega, \varepsilon) \to \frac{1}{\omega} \qquad (\varepsilon \to 0)$$

uniformly with respect to $(y, \eta) \in [0, 1] \times [0, 1]$, $\lambda \in K$, and z in a bounded subset of Γ_ψ. Using the Grisvard formula (23), we get

$$(\lambda + D_\varepsilon)^{-1} = \frac{1}{2\pi i} \int_{\Gamma_\psi} (A_1 - z)^{-1} (A_2^\varepsilon + z)^{-1} \, dz$$

$$\xrightarrow[\varepsilon \to 0]{} \frac{1}{2\pi i} \int_{\Gamma_\psi} (A_1 - z)^{-1} \frac{1}{\omega} M \, dz$$

Integrating using the Cauchy formula, we find that $\lim_{\varepsilon \to 0} (\lambda + D_\varepsilon)^{-1} = (\lambda + D_0)^{-1} M$, and the limit is uniform for $\lambda \in K$. This proves (35).

The relations (36) follow from (35). To prove (37), we use (31) to see that $\partial_3 (\lambda + D_\varepsilon)^{-1} = (1/2\pi i) \int_{\Gamma_\psi} (A_1 - z)^{-1} \, \partial_3 (A_2^\varepsilon + z)^{-1} \, dz$. But

$$\partial_3 G(y, \eta; \omega, \varepsilon) \to 0 \qquad (\varepsilon \to 0)$$

uniformly with respect to $(y, \eta) \in [0, 1] \times [0, 1]$, $\lambda \in K$, and z in a bounded subset of Γ_ψ. This proves (37). □

Using formulas (7), (8), (9), and (10), we have

Corollary 3.4. *As $\varepsilon \to 0$, we have for each $\alpha \in (0, 1)$ and $1 < p < \infty$,*

$$D_\varepsilon^{-\alpha} \to D_0^{-\alpha} M$$

$$e^{-D_\varepsilon t} \to e^{-D_0 t} M$$

$$D_\varepsilon^{\alpha} e^{-D_\varepsilon t} \to D_0^{\alpha} e^{-D_0 t} M \qquad (t > 0)$$

The convergences are in the operator norm on the space of bounded linear operators from $L_p(Q)$ to $L_p(Q)$. Moreover $\|D_\varepsilon^\alpha e^{-D_\varepsilon t}\|_{L_p(Q) \to L_p(Q)}$ and $\|D_0^\alpha e^{-D_0 t} M\|_{L_p(Q) \to L_p(Q)} \leqslant (C/t^\alpha)$, where constant C is independent of $t > 0$ and of ε.

Let us now study the convergence as $\varepsilon \to 0$ of the operators $D_\varepsilon^{-3/4}$ and $\partial_i D_\varepsilon^{-3/4}$ $(1 \leqslant i \leqslant 3)$.

Proposition 3.5. *As $\varepsilon \to 0$ we have the following convergences:*

$$D_\varepsilon^{-3/4} \to D_0^{-3/4} M \tag{38}$$

$$\partial_i D_\varepsilon^{-3/4} \to \partial_i D_0^{-3/4} M \qquad (i=1,2) \tag{39}$$

$$\partial_3 D_\varepsilon^{-3/4} \to 0 \tag{40}$$

Here the convergence in (38) is in the operator norm on the space of bounded linear operators from $L_2(Q)$ to $L_q(Q)$ $(1<q<\infty)$. The convergences in (39) and (40) are with respect to the operator norm on the space of bounded linear operators from $L_2(Q)$ to $L_q(Q)$, $1<q<3$.

Proof. We begin with the formula (9) for $D_\varepsilon^{-\alpha}$: if $0<\alpha<1$, then

$$D_\varepsilon^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty t^{-\alpha}(t+D_\varepsilon)^{-1}\, dt$$

which is valid when $D_\varepsilon^{-\alpha}$ is viewed as a bounded linear operator on $L_p(Q)$, $1<p<\infty$. We will need an estimate on the norm of $D_\varepsilon^{-\alpha}$ as a linear transformation from $L_p(Q)$ to $L_q(Q)$ for $1<p\leqslant q<\infty$. For this we have

Lemma 3.6. *If $t \geqslant 0$, we have for $1<p\leqslant q<\infty$,*

$$\|(t+D_\varepsilon)^{-1}\|_{L_p(Q)\to L_q(Q)} \leqslant c_*(1+t)^{-1+3/2(1/p-1/q)} \tag{41}$$

where c_* is a constant independent of $t \geqslant 0$ and ε.

Proof. Fix $t \geqslant 0$. As above, we will write c to indicate a constant whose value may vary from line to line.

First observe that the operator $D_\varepsilon = \gamma I - \Delta_\varepsilon$ acts componentwise on elements $u=(u_1, u_2, u_3)$ of $\mathscr{D}(\Delta_\varepsilon) \subset L_2(Q)$: $D_\varepsilon u = ((\gamma I - \Delta_\varepsilon)\, u_1, (\gamma I - \Delta_\varepsilon)\, u_2, (\gamma I - \Delta_\varepsilon)\, u_3)$. It will be convenient to abuse notation in the proof of Lemma 3.6 (and only in the proof of Lemma 3.6). Namely, we will let $L_p(Q)$ denote the usual space of **R**-valued, p-integrable functions on Q. Similarly, $L_p(\Omega)$ will denote the space of **R**-valued, p-integrable functions on Ω, $L_p[0, \ell_1]$ will denote ... etc. We let D_ε denote the operator defined on $L_p(Q)$ by $\gamma I - ((\partial^2/\partial x_1^2) + (\partial^2/\partial x_2^2) + [(1/\varepsilon^2)(\partial^2/\partial y^2)])$ with periodic boundary conditions. All other differential operators to be encountered will also act on spaces of scalar functions. We emphasize that this abuse of notation will be made only in the proof of Lemma 3.6.

Let $\Gamma = \Gamma(-3\pi/4, 0) \cup \Gamma(3\pi/4, 0)$, traversed from bottom to top. For the rest of the proof of Lemma 3.6, z will denote a generic point of Γ. Observe that there is a positive constant c such that

$$\left| \frac{t}{2} \pm z \right| \geqslant c(t + |z|)$$

for all $t \geqslant 0$, $z \in \Gamma$.

Consider points $\omega = z + (t/2) + (\gamma/2)$. Using the basic estimate (28a), we see that, if $\varphi \in L_1[0, 1]$, then

$$\left\| y \mapsto \int_0^1 G(y, \eta; \omega, \varepsilon)\, \varphi(\eta)\, d\eta \right\|_\infty \leqslant \frac{c}{|\sqrt{\omega}|} \|\varphi\|_1 \tag{42}$$

We apply the Riesz interpolation theory [18, Sect. 2], Lemma 3.2(a) and (42) to the operator $A_2^\varepsilon = (t/2) + (\gamma/2) - [(1/\varepsilon^2)(\partial^2/\partial y^2)]$ to conclude that, if $1 < p \leqslant q < \infty$, then

$$\|(A_2^\varepsilon + z)^{-1}\|_{L_p[0,1] \to L_q[0,1]} \leqslant c(1 + t + |z|)^{-1 + (1/2)(1/p - 1/q)} \tag{43}$$

Next we consider the operator $A_1 = (t/2) + (\gamma/2) - (\partial^2/\partial x_1^2) - (\partial^2/\partial x_2^2)$. Set $A_1^{x_1} = (t/4) + (\gamma/4) - (\partial^2/\partial x_1^2)$, $A_1^{x_2} = (t/4) + (\gamma/4) - (\partial^2/\partial x_2^2)$ with the appropriate boundary conditions. When convenient, we will regard $A_1^{x_1}$ resp. $A_1^{x_2}$ as an operator on $L_p[0, \ell_1]$ resp. $L_p[0, \ell_2]$.

We wish to apply the Grisvard formula to the operator $A_1 - z = (A_1^{x_1} - z/2) + (A_1^{x_2} - z/2)$. For this it will be convenient to determine a curve of integration Γ' which depends on z. We proceed as in the proof of Proposition 3.1. Fix $1 < p < \infty$. Since $z \in \Gamma$, $-z$ has nonnegative real part. There is a constant c' such that

$$\|A_1^{x_i} - z/2 + z'\|_{L_p(Q) \to L_p(Q)} \leqslant \frac{c'}{1 + |z'|} \qquad (\operatorname{Re} z' \geqslant 0,\ i = 1, 2)$$

c' depends neither on $z \in \Gamma$ nor on z' in the right-hand plane. Choose $\psi' \in (\pi/2, \pi/2 + \arcsin(1/c'))$, then choose a straight line Γ' passing though the origin in $\mathbf{C}$ as follows. If $\operatorname{Im}(-z) \geqslant 0$, then Γ' makes angle ψ with the positive real axis, while if $\operatorname{Im}(-z) < 0$, then Γ' makes angle $\pi - \psi'$ with the positive real axis.

The following observation will be important later on. Note that there is a positive constant c such that, for all $z \in \Gamma$ and $z' \in \Gamma'$

$$\left| -\frac{z}{2} \pm z' \right| \geqslant c(|z| + |z'|)$$

It follows that there is a positive constant c for which

$$\left|\frac{t}{4}-\frac{z}{2}\pm z'\right| \geqslant c(t+|z|+|z'|) \tag{44}$$

for all $t \geqslant 0$, $z \in \Gamma$, and $z' \in \Gamma'$.

Applying the Grisvard formula (23), we obtain

$$(A_1 - z)^{-1} = \frac{1}{2\pi i}\int_{\Gamma'}\left(A_1^{x_1}-\frac{z}{2}-z'\right)^{-1}\left(A_1^{x_2}-\frac{z}{2}+z'\right)dz' \tag{45}$$

This formula holds in the space of bounded linear operators from $L_p(Q)$ to $L_p(Q)$. We use it to study $(A_1 - z)^{-1}$ as a map from p-integrable functions to q-integrable functions, $q \geqslant p$.

Note first that the Green's function of $A_1^{x_1}-(z/2)-z'$ is of the form $G(x_1, \eta; \omega_1, 1)$, where $\omega_1 = -(z/2) - z' + (t/4) + (\gamma/4)$ and G is as in (25). Similarly, the Green's function of $A_1^{x_2}-(z/2)+z'$ is of the form $G(x_2, \eta; \omega_2, 1)$, where $\omega_2 = -(z/2) + z' + (t/4) + (\gamma/4)$. Estimates analogous to (42) hold for these Green's functions. Using (28a), the Riesz interpolation theory, and (44), we conclude that

$$\begin{aligned}\left\|\left(A_1^{x_1}-\frac{z}{2}-z'\right)^{-1}\right\|_{L_p[0,\ell_1]\to L_q[0,\ell_1]} &\leqslant c(1+t+|z|+|z'|)^{-1+(1/2)(1/p-1/q)}\\ \left\|\left(A_1^{x_2}-\frac{z}{2}-z'\right)^{-1}\right\|_{L_p[0,\ell_2]\to L_q[0,\ell_2]} &\leqslant c(1+t+|z|+|z'|)^{-1+(1/2)(1/p-1/q)}\end{aligned} \tag{46}$$

Consider now a function $\varphi \in L_p(Q)$ of the form $\varphi(x_1, x_2, y) = \varphi_1(x_1)\,\varphi_2(x_2)$, where $\varphi_i \in L_p[0, \ell_i]$ $(i = 1, 2)$. Using (45) and (46), we find that

$$\begin{aligned}\|(A_1 - z)^{-1}\varphi\|_q &\leqslant \frac{1}{2\pi}\int_{\Gamma'}\left\|\left(A_1^{x_1}-\frac{z}{2}-z'\right)^{-1}\varphi_1\right\|_q\\ &\quad\times\left\|\left(A_1^{x_2}-\frac{z}{2}+z'\right)^{-1}\varphi_2\right\|_q d|z'|\\ &\leqslant c\,\|\varphi\|_p\int_0^\infty (1+t+|z|+|z'|)^{-2+(1/p-1/q)}\,d|z'|\\ &\leqslant c\,\|\varphi\|_p\,(1+t+|z|)^{-1+(1/p-1/q)}\end{aligned} \tag{47}$$

We can interpret all norms as being in $L_p(Q)$ resp. $L_q(Q)$.

Next, express $L_p(Q)$ as the tensor product $L_p[0, \ell_1] \otimes L_p[0, \ell_2] \otimes L_p[0, 1]$. Let $\varphi \in L_p(Q)$ be a function of the form $\varphi(x_1, x_2, y) = \varphi_1(x_1)\, \varphi_2(x_2)\, \varphi_3(y)$. Using the Grisvard formula, we have

$$(t + D_\varepsilon)^{-1} = \frac{1}{2\pi i} \int_\Gamma (A_1 - z)^{-1} (A_2^\varepsilon + z)^{-1}\, dz$$

where the formula holds in $L_p(Q)$. Using (43) and (47), we have

$$\begin{aligned}\|(t + D_\varepsilon)^{-1}\, \varphi\|_q &\leqslant \frac{1}{2\pi} \int_\Gamma \|(A_1 - z)^{-1}\, \varphi_1 \varphi_2\|_q\, \|(A_2^\varepsilon + z)^{-1}\, \varphi_3\|_q\, d|z| \\ &\leqslant c \int_\Gamma (1 + t + |z|)^{-2 + (3/2)(1/p - 1/q)}\, d|z|\, \|\varphi\|_p \\ &\leqslant c_*(1 + t)^{-1 + (3/2)(1/p - 1/q)}\, \|\varphi\|_p\end{aligned}$$

This complete the proof of Lemma 3.6. □

Return now to the formula (9) for $D_\varepsilon^{-\alpha}$. Suppose that

$$\frac{1}{q} > \frac{1}{p} - \frac{2\alpha}{3}$$

Then the integral in (9) converges, uniformly as $\varepsilon \to 0$, in the norm of linear operators acting from $L_p(Q)$ to $L_q(Q)$: this follows from Lemma 3.6. Let $\alpha = 3/4$, $p = 2$, $2 \leqslant q < \infty$. Then the preceding relation is valid, and we can pass to the limit in (9). We obtain

$$D_\varepsilon^{-3/4} \to D_0^{-3/4} M$$

as operators from $L_2(Q)$ to $L_q(Q)$ for $2 \leqslant q < \infty$. This proves (38).

To prove (39) and (40), we use the formula

$$D_\varepsilon^{-\alpha} = \frac{\sin \pi\alpha}{\pi(1 - \alpha)} \int_0^\infty t^{1-\alpha} (t + D_\varepsilon)^{-2}\, dt$$

valid for $0 < \alpha < 1$ (see [18, p. 281]). We obtain

$$\partial_i D_\varepsilon^{-\alpha} = \frac{\sin \pi\alpha}{\pi(1 - \alpha)} \int_0^\infty t^{1-\alpha}\, \partial_i (t + D_\varepsilon)^{-1}\, (t + D_\varepsilon)^{-1}\, dt \tag{48}$$

where the integrand on the right is at first viewed as an operator from $L_p(Q)$ to $L_q(Q)$. To estimate the integrand, we use Lemma 3.6 to obtain

$$\begin{aligned}
&\|\partial_i(t+D_\varepsilon)^{-1}\,(t+D_\varepsilon)^{-1}\|_{L_p(Q)\to L_q(Q)}\\
&\quad\leqslant \|\partial_i(t+D_\varepsilon)^{-1}\|_{L_q(Q)\to L_q(Q)}\,\|(t+D_\varepsilon)^{-1}\|_{L_p(Q)\to L_q(Q)}\\
&\quad\leqslant \frac{c}{(1+t)^{1/2}}\cdot(1+t)^{-1+(3/2)(1/p-1/q)}
\end{aligned}$$

We have also used relation (21); the estimate is uniform in ε. If $p\geqslant 2$ and

$$\frac{1}{q}>\frac{1}{p}-\frac{2\alpha-1}{3} \tag{49}$$

then the integral in (48) converges and defines a bounded linear operator from $L_p(Q)$ to $L_q(Q)$.

Now set $\alpha=3/4$, $p=2$, $2\leqslant q<3$. Then (49) is valid, and using Proposition 3.3 we get

$$\begin{aligned}
&\partial_i D_\varepsilon^{-3/4}\to\partial_i D_0^{-3/4}M \qquad (i=1,2)\\
&\partial_3 D_\varepsilon^{-3/4}\to 0
\end{aligned}$$

The convergences are in the operator norm on bounded linear operators from $L_p(Q)$ to $L_q(Q)$. This completes the proof of Proposition 3.5. □

Return now to the nonlinear operator $\Phi_\varepsilon\colon C_T(L_2(Q))\to C_T(L_2(Q))$ defined in (13). We showed in Lemma 2.2 that, for each $\varepsilon>0$, Φ_ε is completely continuous. We now study the behavior of Φ_ε as $\varepsilon\to 0$.

Let $u\in L_2(Q)$. We can write

$$\begin{aligned}
\mathbf{P}_\varepsilon(D_\varepsilon^{-\alpha}u\cdot\nabla_\varepsilon)\,D_\varepsilon^{-\alpha}u&=\mathbf{P}_\varepsilon(D_0^{-\alpha}Mu\cdot\nabla_2)\,D_0^{-\alpha}Mu\\
&\quad+\mathbf{P}_\varepsilon[(D_\varepsilon^{-\alpha}u\cdot\nabla_\varepsilon)\,D_\varepsilon^{-\alpha}u-(D_0^{-\alpha}Mu\cdot\nabla_2)\,D_0^{-\alpha}Mu]
\end{aligned} \tag{50}$$

Here we have again abused notation; we have identified ∇_2 with $(\nabla_2, 0)^{tr}$. Setting $\alpha=3/4$, noting that $\|\mathbf{P}_\varepsilon\|=1$, and using Proposition 3.5, we see that the second term on the right-hand side of (50) tends to zero as $\varepsilon\to 0$. As for the first term in (50), note that the functions $D_0^{-\alpha}Mu$ and $\partial_i D_0^{-\alpha}Mu$

do not depend on y $(i=1, 2)$. Using the relation (14) between the Leray projectors $\mathbf{P}_\varepsilon$ and $\mathbf{P}_2$, we obtain

$$\mathbf{P}_\varepsilon(D_0^{-\alpha} Mu \cdot \nabla_2)\, D_0^{-\alpha} Mu = \begin{pmatrix} \mathbf{P}_2 \begin{pmatrix} D_0^{-\alpha} Mu_1 \partial_1 D_0^{-\alpha} Mu_1 + D_0^{-\alpha} Mu_2 \partial_2 D_0^{-\alpha} Mu_1 \\ D_0^{-\alpha} Mu_1 \partial_1 D_0^{-\alpha} Mu_2 + D_0^{-\alpha} Mu_2 \partial_2 D_0^{-\alpha} Mu_2 \end{pmatrix} \\ D_0^{-\alpha} Mu_1 \partial_1 D_0^{-\alpha} Mu_3 + D_0^{-\alpha} Mu_2 \partial_2 D_0^{-\alpha} Mu_3 \end{pmatrix} \tag{51}$$

Note that (51) does not depend on ε.

Next consider the function $F(t, x_1, x_2, x_3)$ of type (i). For fixed $t \in \mathbf{R}$ and for $\varepsilon > 0$, write $\varphi_\varepsilon(x_1, x_2, y) = F(t, x_1, x_2, \varepsilon y)$ and $\varphi_0(x_1, x_2) = F(t, x_1, x_2, 0)$. Then $\|\varphi - \varphi_0\|_{L_2(Q)} \to 0$ as $\varepsilon \to 0$, and so $\|\mathbf{P}_\varepsilon(\varphi_\varepsilon - \varphi_0)\|_{L_2(Q)} \to 0$ as $\varepsilon \to 0$. Writing

$$F(t, x_1, x_2, 0) = \begin{pmatrix} F_{01}(t, x_1, x_2) \\ F_{02}(t, x_1, x_2) \\ F_{03}(t, x_1, x_2) \end{pmatrix}$$

and using (14) again, we see that, in $L_2(Q)$,

$$\lim_{\varepsilon \to 0} \mathbf{P}_\varepsilon F = \begin{pmatrix} \mathbf{P}_2 \begin{pmatrix} F_{01} \\ F_{02} \end{pmatrix} \\ F_{03} \end{pmatrix} \tag{52}$$

We now write

$$\hat{f}(t, u, 0) = \gamma \mathbf{P}_\varepsilon Mu - \mathbf{P}_\varepsilon(D_0^{-\alpha} Mu \cdot \nabla_2)(D_0^{-\alpha} Mu) + \begin{pmatrix} \mathbf{P}_2 \begin{pmatrix} F_{01} \\ F_{02} \end{pmatrix} \\ F_{03} \end{pmatrix}$$

and

$$\Phi_0(u)(t) = e^{-D_0 t}(I - e^{-D_0 T})^{-1} \int_0^T D_0^\alpha e^{-D_0(T-s)} \hat{f}(s, u(s), 0)\, ds + \int_0^t D_0^\alpha e^{-D_0(t-s)} \hat{f}(s, u(s), 0)\, ds \tag{53}$$

Using (50), (51), and (52) together with Proposition 3.5, we see that the family of completely continuous operators $\{\Phi_\varepsilon \mid \varepsilon \geqslant 0\}$ from $C_T(L_2(Q))$ to

$C_T(L_2(Q))$ is also continuous with respect to ε in the sense that $(\varepsilon, u) \to \Phi_\varepsilon(u)$: $[0, \varepsilon_0] \times C_T(L_2(Q)) \to C_T(L_2(Q))$ is a completely continuous function. Comparing (53) and (16), we see, in addition, that Φ_0 is the fixed-point operator corresponding to the reduced Navier–Stokes equations. Similar arguments apply when F is of type (ii).

We can now prove Theorems 1 and 2. The hypothesis of Theorem 1 means that the equation $u = \Phi_0(u)$ has a solution $\bar{u}_0$ in $C_T(L_2(Q))$ such that the topological index $\operatorname{ind}(\bar{u}_0, \Phi_0) \neq 0$. Recall that the term "$T$-periodic solution of (3_ε)" means a solution $u_\varepsilon \in C_T(L_2(Q))$ of the equation $u = \Phi_\varepsilon(u)$.

Proof of Theorem 1. The proof is immediate from the complete continuity of $(\varepsilon, u) \to \Phi_\varepsilon(u)$ and a basic property of the topological index ([17, Theorem 54.1]). □

Before proving Theorem 2, we write explicitly the linearization at $\bar{u}_0$ of the operator equation $u = \Phi_0(u)$. Noting that $M\bar{u}_0 = \bar{u}_0$, we set

$$g(t, v) = \gamma \mathbf{P}_\varepsilon M v - \mathbf{P}_\varepsilon(D_0^{-\alpha} \bar{u}_0 \cdot \nabla_2)(D_0^{-\alpha} M v) - \mathbf{P}_\varepsilon(D_0^{-\alpha} M v \cdot \nabla_2)(D_0^{-\alpha} \bar{u}_0)$$

then write

$$v = e^{-D_0 t}(I - e^{-D_0 T})^{-1} \int_0^T D_0^\alpha e^{-D_0(T-s)} g(s, v(s))\, ds$$
$$+ \int_0^t D_0^\alpha e^{-D_0(t-s)} g(s, v(s))\, ds$$

This equation is the linearization of $u = \Phi_0(u)$ at $\bar{u}_0$.

Proof of Theorem 2. The proof uses the method of functionalization of a parameter ([6, 17]); we sketch the details. By assumption, the operator Φ_0 (with $T = T_0$) admits a fixed point $\bar{u}_0$. However, because the function F is autonomous, we cannot immediately use the topological degree to derive the existence of a fixed point of Φ_ε for $\varepsilon \neq 0$.

As in Section 2, we introduce the time-scaling $t \to (T_0/T)\, t$ and rewrite the Navier–Stokes equations in the form (18). Carrying the time-change through the definitions, one obtains for $\varepsilon \geqslant 0$ the operators $\hat{\Phi}_\varepsilon(T, \cdot)$: $C_{T_0}(L_2(Q)) \to C_{T_0}(L_2(Q))$ defined by replacing ∇_ε by $(T_0/T)((\partial/\partial x_1)$, $(\partial/\partial x_2)$, $[(1/\varepsilon)(\partial/\partial y)])$ and D_ε by $\gamma I - (T_0/T)((\partial^2/\partial x_1^2) + (\partial^2/\partial x_2^2) + [(1/\varepsilon^2)$ $(\partial^2/\partial x_3^2)])$ in the formula (13) for Φ_ε.

Now we repeat the proof of Lemma 1 of [13] to conclude that, under the hypotheses of Theorem 2, there exists a continuous affine functional

$T\colon C_{T_0}(L_2(Q)) \to \mathbf{R}$ such that $T(\bar{u}_0) = T_0$ and such that the fixed point $\bar{u}_0$ of the operator $\Gamma_0\colon C_{T_0}(L_2(Q)) \to C_{T_0}(L_2(Q))$ defined by

$$\Gamma_0(u) = \hat{\Phi}_0(T(u), u)$$

is isolated with $|\mathrm{ind}(\bar{u}_0, \Gamma_0)| = 1$. We define

$$\Gamma_\varepsilon(u) = \hat{\Phi}_\varepsilon(T(u), u)$$

Then $\{\Gamma_\varepsilon \mid \varepsilon \geqslant 0\}$ is completely continuous on $[0, \varepsilon_0] \times C_{T_0}(L_2(Q))$. So using the theory of the topological degree again [17], we find that, for small ε, $\Gamma_\varepsilon(u) = u$ admits a solution u_ε near $\bar{u}_0$ in $C_{T_0}(L_2(Q))$. Clearly $T_\varepsilon = T(u_\varepsilon) \to T_0$ as $\varepsilon \to 0$. This completes the proof of Theorem 2. □

We finish the paper with some remarks about the ease and range of application of these results. The main point is the verification of the hypothesis concerning the topological index $\mathrm{ind}(\bar{u}_0, \Phi_0)$ in Theorem 1 resp. the conditions placed on Eq. (19) in Theorem 2.

First, the map Φ_0 has values in the space $X = \mathrm{Im}\, M \subset L_2(Q)$ of vector functions depending only on (x_1, x_2). Hence the index depends only on the properties of Φ_0 when restricted to X.

Second, note that the first two equations in the reduced equations (16) are just the incompressible Navier–Stokes equations in the two-dimensional domain Ω with forcing term $\binom{F_{01}}{F_{02}}$. Let us write the operator Φ_0 in the form $\begin{pmatrix} \Phi_{01} \\ \Phi_{02} \\ \Phi_{03} \end{pmatrix}$, where $\binom{\Phi_{01}}{\Phi_{02}}$ is the fixed-point operator corresponding to T-periodic solutions of the (incompressible) Navier–Stokes equations in Ω.

From the form of Φ_0, one sees that

$$\Phi_0(C_T(L_2(Q))) \subset \begin{pmatrix} \mathbf{P}_2 & 0 \\ 0 & I \end{pmatrix} (C_T(L_2(Q)))$$

where $\mathbf{P}_2$ is the Leray projector in the first two components. Recall that $\int_\Omega F_{30}\, dx_1\, dx_2 = 0$. Then [23] Φ_0 maps $C_T(ML_2(Q))$ into the subspace of $C_T(ML_2(Q))$ consisting of T-periodic functions taking values in the set of velocity fields of the type

$$\left\{ \mathbf{P}_2(u_1, u_2), \int_\Omega u_3\, dx_1\, dx_2 = 0 \right\}$$

Now we can apply the restriction theorem [17] to obtain the following results. Let $\bar{u}_0$ be a periodic solution of the restricted equations (16)–(17).

Write $\bar{v}_0$ for the first two components of $\bar{u}_0$, so that $\bar{v}_0$ is a T-periodic solution of the two-dimensional, incompressible Navier–Stokes equations in Ω.

Consider first the nonautonomous case. Suppose that the topological index of $\bar{v}_0$ with respect to the operator $\binom{\Phi_{01}}{\Phi_{02}}$ can be calculated by linearization ([17], pp. 108–109, 123–135). Suppose that this index is nonzero. Then the topological index of $\bar{u}_0$ with respect to Φ_0 is nonzero, and Theorem 1 is applicable.

In the autonomous case, let the hypotheses of Theorem 2 be satisfied with respect to the periodic solution $\bar{v}_0$ of the two-dimensional incompressible Navier–Stokes equations in Ω. This is true if, for example, $\bar{v}_0$ is orbitally stable. Then the hypotheses of Theorem 2 are also satisfied with respect to $\bar{u}_0$, and we can conclude the existence of a T_ε-periodic solution of the incompressible Navier–Stokes equations in Q_ε if ε is small.

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