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Positive solutions of a non-linear eigenvalue problem with discontinuous non-linearity†

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Synopsis

We seek non-trivial solutions $(u, \lambda) \in C^1([0,1]) \times [0, \infty)$, with $u(x) \geqq 0$ for all $x \in [0,1]$, of the non-linear eigenvalue problem $-u''(x) = \lambda f(u(x))$ for $x \in (0,1)$ and $u(0) = u(1) = 0$, where $f: [0,\infty) \to [0,\infty)$ is such that $f(p) = 0$, *for* $p \in [0,1)$, and $f(p) = K(p)$, for $p \in (1, \infty)$, and $K: [1,\infty) \to (0,\infty)$ is assumed to be twice continuously differentiable. (The value $f(1)$ is only required to be positive.)

Existence and multiplicity theorems are given in the cases where f is asymptotically sub-linear and f is asymptotically super-linear. Moreover if strengthened assumptions are made on the growth of the non-linear term f we obtain the precise number of non-trivial solutions for given values of $\lambda \in [0, \infty)$.

Introduction

We consider the non-linear two point boundary value problem

$$\begin{cases} -u''(x) = \lambda f(u(x)) & \text{for} \quad 0 < x < 1 \\ u(0) = u(1) = 0. \end{cases} \tag{1.0}$$

where $f: [0,\infty) \to [0,\infty)$ has the following form

$$f(p) = \begin{cases} 0 & \text{for} \quad 0 \leqq p < 1 \\ K(p) & \text{for} \quad 1 < p < \infty \end{cases}.$$

The function $K: [1,\infty) \to (0,\infty)$ is assumed to be twice continuously differentiable. We suppose that $f(1)$ is positive ($f(1)$ need not be related to K).

The function u and the number λ are unknown quantities in the problem (1.0) and so we consider a solution of (1.0) to be an ordered pair (u, λ).

Since f is discontinuous at 1, we cannot expect to find twice continuously differentiable solutions u of (1.0) with $\|u\| > 1$, where $\|u\| = \max\{|u(x)|: 0 \leqq x \leqq 1\}$. We give the following:

Definition. A solution of (1.0) is a pair $(u, \lambda) \in C^1([0,1]) \times [0, \infty)$ such that

$$u(x) \geqq 0 \quad \text{for all} \quad x \in [0,1], \quad u(0) = u(1) = 0$$

u' is absolutely continuous on $[0,1]$ and $-u''(x) = \lambda f(u(x))$ for almost all $x \in [0,1]$.

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We denote by S the subset of $C^1([0,1])\times[0,\infty)$ consisting of all the solutions (u,λ) of (1.0). Since $f(0)=0$, we have that $\{(0,\lambda): \lambda\in[0,\infty)\}\subset S$ and we call this the set of trivial solutions of (1.0).

The main result in section 1 is the following Theorem:

THEOREM 1.2. *For each $\rho\in(1,\infty)$ there exists exactly one solution (u,λ) of* (1.0) *such that $\|u\|=\rho$. Moreover, $S\cap\{(u,\lambda)\in S: \|u\|\leqq 1\}=\{(0,\lambda): \lambda\in[0,\infty)\}$.*

In order to prove this result we consider, for all $n\in\mathbb{N}$, the following problem

$$\begin{cases} -u''(x)=\lambda f_n(u(x)) & \text{for} \quad 0<x<1 \\ \quad u(0)=u(1)=0 \end{cases} \tag{1.n}$$

where $f_n:[0,\infty)\to(0,\infty)$ is defined by

$$f_n(p)=\begin{cases} \dfrac{1}{n} & \text{for} \quad 0\leqq p<1 \\ K(p) & \text{for} \quad 1<p<\infty. \end{cases}$$

The value $f_n(1)$ is only required to be positive. For all $\rho\in(1,\infty)$ and $n\in\mathbb{N}$, the existence and uniqueness of the solution (u_n,λ_n) of $(1.n)$ such that $\|u_n\|=\rho$ follow from [**3**, Theorem 2.2], where f on $[0,1)$ is defined by a twice continuously differentiable function $H:[0,1]\to(0,\infty)$.

Then passing to a limit as $n\to\infty$ we get that $(u_n,\lambda_n)\to(u_0,\lambda_0)$ in $C^1([0,1])\times[0,\infty)$ and that (u_0,λ_0) is the unique solution of (1.0) such that $\|u_0\|=\rho$. The second part of Theorem 1.2 is proved directly.

Let us note that, from the results in [**1**] and [**2**], [**3**, Theorem 2.2] remains true if $H:[0,1]\to(0,\infty)$ and $K:[1,\infty)\to(0,\infty)$ are only Lipschitz continuous functions. Therefore our Theorem 1.2 holds true also in such a case.

In [**3**] the number of solutions for given values of λ is also determined. Analogously in this paper, Theorem 1.7 and 1.8 state results concerning the number of non-trivial solutions of (1.0) corresponding to given values of λ, in the case where f is asymptotically sub-linear and in the case where f is asymptotically super-linear, respectively.

These results and Theorem 1.9 lead us to search for strengthened conditions on K which imply that S has the structure illustrated in Figure 1 for the sub-linear case, and the structure illustrated in Figure 2 (where $0<A\leqq\infty$) for the super-linear one.

Actually in Section 2, Theorem 2.2 and 2.4 state that Figure 1 applies provided that either $K'(p)\leqq 0$ and $[K'(p)/K^3(p)]'\leqq 0$ for all $p\geqq 1$ or $\frac{3}{2}K(1)>K'(1)$ and $[p^{\frac{3}{2}}K'(p)]'\leqq 0$ for all $p\geqq 1$ respectively.

Finally, Theorem 2.5 asserts that Figure 2 applies provided that $p^{-1}K(p)$ is a non-decreasing function on $(1,\infty)$ and $\lim_{p\to\infty} p^{-1}K(p)=A$ $(0<A\leqq\infty)$.

§1. GENERAL PROPERTIES OF S

We will study the structure of S by the relationship stated in the next Lemma 1.1, between λ and $\|u\|$ for any solution (u,λ) of (1.0) such that $\|u\|>1$.

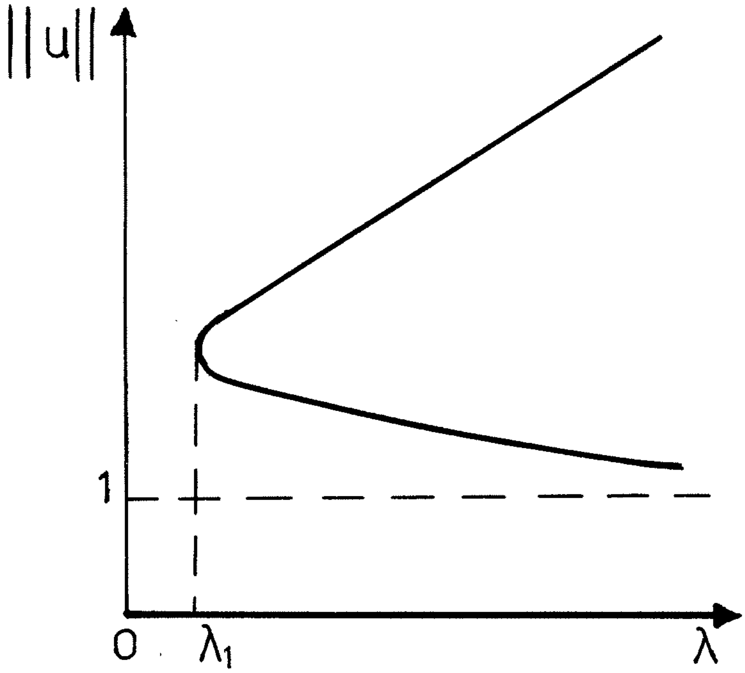

Fig. 1

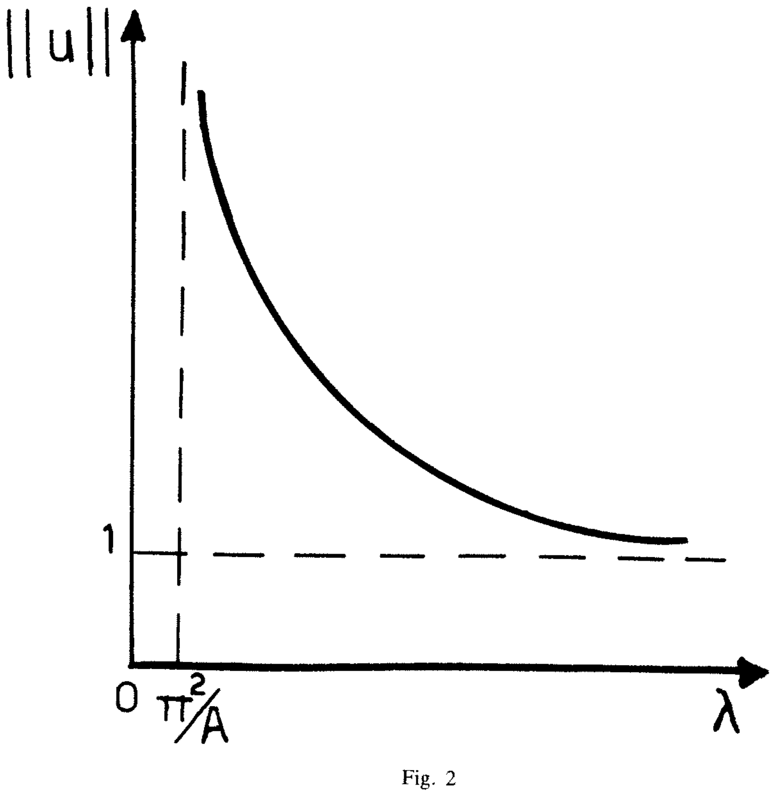

Fig. 2

For this, for every problem $(1.n)$ we put

$$F_n(\omega)=\int_0^\omega f_n(s)\,ds \quad \text{for} \quad \omega>0$$

and

$$g_n(\rho)=\sqrt{2}\int_0^\rho \{F_n(\rho)-F_n(\omega)\}^{-\frac{1}{2}}\,d\omega \quad \text{for} \quad \rho>0.$$

In the case of problem (1.0) we have that

$$f(p)=\begin{cases}0 & 0\leqq p<1\\ K(p) & p>1\end{cases}$$

so we put

$$F(\omega)=\int_1^\omega K(s)\,ds \quad \text{for} \quad \omega>1 \quad \text{and} \quad F(\omega)=0 \quad \text{for} \quad 0\leqq\omega\leqq 1$$

and

$$g(\rho)=\sqrt{2}F(\rho)^{-\frac{1}{2}}+\sqrt{2}\int_1^\rho \{F(\rho)-F(\omega)\}^{-\frac{1}{2}}\,d\omega.$$

We note that, for all n, $g_n(\rho)<\infty$ for $\rho>0$ (since $f_n(\rho)>0$), and that $g_n(\rho)$ is a continuous function on $[0,\infty)$ for all n, while g is a continuous function on $(1,\infty)$.

We have the following:

LEMMA 1.1 *Suppose that $(u,\lambda)\in S$, with $\|u\|=\rho>1$. Then $\lambda^{\frac{1}{2}}=g(\|u\|)$.*

Proof. Let $(u,\lambda)\in S$ with $\|u\|=\rho>1$ and $\lambda>0$. Put $t_0=\inf I(u)=\inf\{x\in[0,1]: u(x)=1\}$. We have that $u(x)=u'(0)x$ for all $x\in[0,t_0]$ and $u'(x)=u'(0)=t_0^{-1}>0$ for all $x\in(0,t_0]$. It follows from the results in [**1**] that $u(x)=u(1-x)$ for all $x\in[0,1]$ and u is monotonically increasing on $[0,\frac{1}{2}]$.

Furthermore u is twice continuously differentiable on $(t_0,\frac{1}{2}]$ and $-\frac{1}{2}[u'(x)^2]'=\lambda F'(u(x))$.

For $t_0\leqq y\leqq\frac{1}{2}$ we obtain

$$(*) \qquad u'(y)^2=2\lambda\{F(\|u\|)-F(u(y))\}$$

and

$$u'(t_0)^2=2\lambda F(\|u\|).$$

Hence $t_0^{-2}=u'(y)^2=2\lambda F(\|u\|)$ for all $y\in(0,t_0]$, and

$$\int_0^{t_0}\sqrt{(2\lambda)}\,dy=F(\|u\|)^{-\frac{1}{2}}$$

Consequently from $(*)$ we have

$$\int_1^{\|u\|}\{F(\|u\|)-F(\omega)\}^{-\frac{1}{2}}\,d\omega=\int_{t_0}^{\frac{1}{2}}\sqrt{(2\lambda)}\,dy=\int_0^{\frac{1}{2}}\sqrt{(2\lambda)}\,dy-\int_0^{t_0}$$

$$=2^{-\frac{1}{2}}\lambda^{\frac{1}{2}}-F(\|u\|)^{-\frac{1}{2}} \quad \text{and so} \quad \lambda^{\frac{1}{2}}=g(\|u\|).$$

This relationship was introduced in [4] to obtain some results on the number of solutions u for fixed λ in the case where f is continuous on $[0, \infty)$.

THEOREM 1.2. *For each $\rho \in (1, \infty)$ there exists exactly one solution (u, λ) of (1.0) such that $\|u\| = \rho$. Moreover,*

$$S \cap \{(u, \lambda) \in S : \|u\| \leqq 1\} = \{(0, \lambda) : \lambda \in [0, \infty)\}.$$

Proof. If $\|u\| = \rho \in [0, 1)$ we are led to consider the boundary value problem

$$\begin{cases} -u''(x) = 0 & 0 < x < 1. \\ u(0) = u(1) = 0 \end{cases}$$

If $\|u\| = \rho = 1$, since meas $\{x \in (0, 1): u(x) = 1\} = 0$, we have to consider

$$\begin{cases} -u''(x) = 0 & \text{for almost all} \quad x \in (0, 1). \\ u(0) = u(1) = 0 \end{cases}$$

It is easily seen that for each $\rho \in [0, 1]$ the only solutions of (1.0) of the type stated in the Introduction are $(0, \lambda)$ for all $\lambda \in [0, \infty)$.

Let $\rho > 1$, from the results in [3] there exists exactly one solution (u_n, λ_n) of the problem $(1.n)$ such that $\|u_n\| = \rho$ and $\lambda_n^{\frac{1}{2}} = g_n(\rho)$. We want to prove that $(u_n, \lambda_n) \to (u_0, \lambda_0)$ in $C^1([0, 1]) \times [0, \infty)$ and that (u_0, λ_0) is the unique solution of (1.0) such that $\|u_0\| = \rho$.

In order to do this we first note that, for any $\rho > 1$, it follows from the definition that

$$g_n(\rho) \to g(\rho) \quad \text{as} \quad n \to \infty.$$

Now, for any $n \in \mathbb{N}$, we have

$$\|u_n''\| \leqq \Lambda \max\{1, \sup_{p \in [1, \rho]} K(p)\} = \Lambda M(\rho)$$

where $\Lambda = \sup_n \{\lambda_n\} < +\infty$ and $M(\rho) = \max\{1, \sup_{p \in [1, \rho]} K(p)\} < +\infty$ for every $\rho \in (1, \infty)$. Hence

$$|u_n'(x) - u_n'(y)| \leqq \left| \int_x^y \lambda_n f_n(u_n(t))\, dt \right| \leqq \Lambda M(\rho)\, |x - y| \quad \text{for all} \quad x, y \in [0, 1]$$

and if we put $y = \frac{1}{2}$ we have $\|u_n'\| \leqq \frac{1}{2} \Lambda M(\rho)$ for any n, since $u_n'(\frac{1}{2}) = 0$ for all $n \geqq 1$ [see **1**].

Thus we see that $\{u_n'\}$ is equicontinuous on $[0, 1]$ and uniformly bounded. Therefore, by the Ascoli Arzelà theorem, by passing to a subsequence if necessary, we have that

$$u_n \to u_0 \quad \text{in} \quad C^1([0, 1]) \qquad \lambda_n \to \lambda_0 = g^2(\rho)$$

and

$$u_0(0) = u_0(1) = 0.$$

For every $n \in \mathbb{N}$, we set

$$w_n(x) = \{f_n(u_n(x)) - h_n(u_n(x))\} / \{K(1) - n^{-1}\}$$

where

$$h_n(p)=\begin{cases}\dfrac{1}{n} & \text{if} \quad 0\leqq p\leqq 1\\[2ex] K(p)-K(1)+\dfrac{1}{n} & \text{if} \quad p>1\end{cases}$$

and

$$h_n(p)\to h(p)=\begin{cases}0 & \text{if} \quad 0\leqq p\leqq 1\\ K(p)-K(1) & \text{if} \quad p>1\end{cases} \quad \text{uniformly on } [0,\infty).$$

Therefore

$$w_n(x)=\begin{cases}0 & \text{if} \quad u_n(x)<1\\ 1 & \text{if} \quad u_n(x)>1\end{cases}$$

and $0\leqq w_n(x)\leqq 1$ for almost all $x\in[0,1]$.

Furthermore

$$u_n'(x)=u_n'(0)-\lambda_n\int_0^x\left[h_n(u_n(t))+\left\{K(1)-\frac{1}{n}\right\}w_n(t)\right]dt \quad \text{for all} \quad x\in[0,1].$$

Since $\{w_n\}$ is bounded in $L^2(0,1)$, by passing to a subsequence if necessary, we can assume that $\{w_n\}$ converges weakly to an element w in $L^2(0,1)$. Setting

$$v_n(x)=\int_0^x w_n(t)\,dt \quad \text{for all} \quad x\in[0,1]$$

we see that $\{v_n\}$ is uniformly bounded and equicontinuous on $[0,1]$. Thus, by passing to a further subsequence if necessary, we may assume that v_n converges in $C([0,1])$ to an element v. It is easily seen that

$$v(x)=\int_0^x w(t)\,dt \quad \text{for all} \quad x\in[0,1].$$

From the properties of $\{w_n\}$ we have

$$\begin{aligned}0\leqq w(x)\leqq 1 &\quad \text{for almost all} \quad x\in[0,1]\\ w(x)=0 &\quad \text{if} \quad u_0(x)<1\\ w(x)=1 &\quad \text{if} \quad u_0(x)>1.\end{aligned}$$

And so, by the uniform convergence of $\{h_n\}$ on $[0,\infty)$, we obtain

$$u_0'(x)=u_0'(0)-\lambda_0\left(\int_0^x h(u_0(t))\,dt+K(1)\int_0^x w(t)\,dt\right).$$

Therefore u_0' is absolutely continuous on $[0,1]$ and

$$-u_0''(x)=\lambda_0(h(u_0(x))+K(1)w(x)) \quad \text{for almost all} \quad x\in(0,1).$$

Thus

$$-u_0''(x)\in\lambda_0\tilde{f}(u_0(x)) \quad \text{for almost all} \quad x\in(0,1)$$

where

$$\tilde{f}(u_0(x)) = \begin{cases} f(u_0(x)) & \text{if} \quad u_0(x) \neq 1 \\ \text{the interval } [0, K(1)] & \text{if} \quad u_0(x) = 1 \end{cases}.$$

By virtue of the properties of the solutions of $(1.n)$, [see **1**], it is easily seen that the function u_0 is such that $u_0(x) = u_0(1-x)$ for all $x \in [0, 1]$, $u_0'(x) \geqq 0$ for all $x \in (0, \frac{1}{2}]$ and $u_0(\frac{1}{2}) = \|u_0\|$.

Put $t_0 = \inf I(u_0) = \inf \{x \in [0, 1]: u_0(x) = 1\}$, $0 < t_0 < \frac{1}{2}$. Since $u'(t_0) = t_0^{-1} > 0$, then $I(u_0) = \{t_0, 1 - t_0\}$ and consequently $-u_0'' = \lambda_0 f(u_0(x))$ for almost all $x \in (0, 1)$, that is $(u_0, \lambda_0) \in S$ with $\|u_0\| = \rho > 1$ and $\lambda_0^{\frac{1}{2}} = g(\|u_0\|)$.

Let (u_1, λ_1), $\lambda_1 > 0$, be any solution of (1.0) with $\|u_1\| = \rho > 1$. From Lemma 1.1 we have that $\lambda_1 = \lambda_0$. Consider the initial value problem

$$\begin{cases} -w''(x) = \lambda_0 f(w(x)) & \text{for almost all} \quad x \in (0, \frac{1}{2}) \\ w(\frac{1}{2}) = \rho \quad w'(\frac{1}{2}) = 0. \end{cases}$$

It is easily seen that this problem has an unique solution and so $u_1 = u_0$. Therefore (u_0, λ_0) is the unique solution of (1.0) with $\|u_0\| = \rho > 1$. This completes the proof.

We have just proved that for each value of $\rho \in (1, \infty)$ there exists exactly one non-trivial solution (u, λ) of (1, 0), with $\|u\| = \rho$ and $\lambda = g^2(\rho)$. Therefore the set $S_1 = S \cap \{(u, \lambda) \in S: \|u\| = \rho > 1\}$ can be parametrized by $\rho \in (1, \infty)$.

Indeed $S_1 = \{(\varphi(\rho), g^2(\rho)): \rho > 1\}$, where $\varphi: (1, \infty) \to C^1([0, 1])$ is the function defined by

$\varphi(\rho) := $ is the solution of the problem

$$\begin{cases} -v''(x) = g^2(\rho) f(v(x)) & \text{for almost all} \quad x \in (0, 1) \\ \quad v(0) = v(1) = 0 \end{cases}$$

with $\|\varphi(\rho)\| = \rho$.

Similar techniques to the ones used in Theorem 1.2 allow us to prove that φ is continuous on $(1, \infty)$ and so the validity of the following:

LEMMA 1.3. *The set S_1 is a continuous curve in $C^1([0, 1]) \times [0, \infty)$.*

In order to study the structure of S_1 we seek non-trivial solutions corresponding to fixed $\lambda > 0$.

Let us first note that $\lim_{\rho \to 1+} g(\rho) = +\infty$ and so there exists $\bar{\lambda} > 0$ such that for each $\lambda > \bar{\lambda}$ there is at least one non-trivial solution of (1.0). We give now some preliminary Lemmas.

LEMMA 1.4. *The function $g: (1, \infty) \to (0, \infty)$ is continuously differentiable and*

$$g'(\rho) = \sqrt{2} K(\rho) \left\{ -\tfrac{1}{2} F(\rho)^{-\frac{3}{2}} + \frac{F(\rho)^{-\frac{1}{2}}}{K(1)} - \int_1^\rho \{F(\rho) - F(\omega)\}^{-\frac{1}{2}} \frac{K'(\omega)}{K^2(\omega)} \, d\omega \right\}$$

Proof. Let $\rho > 1$. Then

$$\int_1^\rho \{F(\rho)-F(\omega)\}^{-\frac{1}{2}}\,d\omega = -2\int_1^\rho [\{F(\rho)-F(\omega)\}^{\frac{1}{2}}]' \frac{d\omega}{K(\omega)}$$
$$= 2\frac{F(\rho)^{\frac{1}{2}}}{K(1)} - 2\int_1^\rho \{F(\rho)-F(\omega)\}^{\frac{1}{2}} \frac{K'(\omega)}{K^2(\omega)}\,d\omega.$$

The result follows by differentiation.

LEMMA 1.5. *Suppose that* $\limsup_{p\to+\infty} f(p)/p \leqq \alpha$, *where* $\alpha > 0$, *then* $\liminf_{\rho\to+\infty} g(\rho) \geqq \alpha^{-\frac{1}{2}}\pi$

Proof. By our hypothesis for all $r > \alpha$ there exists $R > 0$ such that $f(p) \leqq R + rp$, for all $p \geqq 0$. Thus we have

$$\frac{g(\rho)}{\sqrt{2}} = F(\rho)^{-\frac{1}{2}} + \int_1^\rho \left\{\int_\omega^\rho K(s)\,ds\right\}^{-\frac{1}{2}} d\omega$$
$$\geqq \int_1^\rho \frac{d\omega}{(\rho-\omega)^{\frac{1}{2}}\{R+\frac{1}{2}r(\rho+\omega)\}^{\frac{1}{2}}}.$$

First we put $\rho - \omega = x$ and then $x^{\frac{1}{2}} = C^{\frac{1}{2}}\sin\theta$, where $C = 2Rr^{-1} + 2\rho$, and so we get

$$\frac{g(\rho)}{\sqrt{2}} \geqq \sqrt{\left(\frac{2}{r}\right)}\int_0^{\rho-1} \frac{dx}{x^{\frac{1}{2}}\{C-x\}^{\frac{1}{2}}} = 2\sqrt{\left(\frac{2}{r}\right)}\,\text{arc sin}\,\sqrt{\left(\frac{\rho-1}{C}\right)}.$$

Hence $\liminf_{\rho\to+\infty} g(\rho) \geqq r^{-\frac{1}{2}}\pi$, for all $r > \alpha$, and the assertion is proved.

LEMMA 1.6. *Suppose that* $\liminf_{p\to+\infty} f(p)/p \geqq \beta > 0$ *then* $\limsup_{\rho\to+\infty} g(\rho) \leqq \beta^{-\frac{1}{2}}\pi$.

Proof. By our hypothesis for all $0 < m < \beta$ there exists $M > 0$ such that $f(p) \geqq mp$ for $p > M$.

Let $\rho \geqq 2M$, then

$$\int_\omega^\rho K(s)\,ds \geqq \frac{m}{2}(\rho^2-\omega^2) \quad \text{for} \quad M \leqq \omega < \rho$$

and

$$\int_\omega^\rho K(s)\,ds = \int_\omega^M + \int_M^\rho K(s)\,ds \geqq \frac{m}{2}(\rho^2 - M^2) \quad \text{for} \quad 1 \leqq \omega \leqq M.$$

Therefore, whenever $\rho \geqq 2M$, we get

$$\frac{g(\rho)}{\sqrt{2}} = \left\{\int_1^\rho K(s)\,ds\right\}^{-\frac{1}{2}} + \int_1^\rho \left\{\int_\omega^\rho K(s)\,ds\right\}^{-\frac{1}{2}} d\omega$$
$$\leqq \left\{\frac{2}{m(\rho^2-M^2)}\right\}^{\frac{1}{2}} + \int_1^M \left\{\frac{2}{m(\rho^2-M^2)}\right\}^{\frac{1}{2}} d\omega + \int_M^\rho \left\{\frac{2}{m(\rho^2-\omega^2)}\right\}^{\frac{1}{2}} d\omega$$
$$\leqq \left\{\frac{2}{m(\rho^2-M^2)}\right\}^{\frac{1}{2}} + \left\{\frac{2M^2}{m(\rho^2-M^2)}\right\}^{\frac{1}{2}} + \left(\frac{2}{m}\right)^{\frac{1}{2}}\left[\text{arc sin}\,1 - \text{arc sin}\,\frac{M}{\rho}\right].$$

Hence $\limsup_{\rho\to+\infty} g(\rho) \leqq m^{-\frac{1}{2}}\pi$ for all $0<m<\beta$. This completes the proof.

We show now that the number of non-trivial solutions corresponding to $\lambda>0$ is determined by the asymptotic behaviour of f at infinity. We say that f is *asymptotically sub-linear* if $f(p)/p \to 0$ as $p\to+\infty$ and that f is *asymptotically super-linear* if $\liminf_{p\to+\infty} f(p)/p>0$.

THEOREM 1.7. *Suppose that f is asymptotically sub-linear. Let* $\lambda_1 = \inf\{\lambda>0\colon (u,\lambda)\in S_1\}$. *Then* $\lambda_1>0$ *and for each* $\lambda>\lambda_1$ *there are at least two non-trivial solutions of* (1.0)

Proof. Since f is asymptotically sub-linear, Lemma 1.5 applies for all $\alpha>0$ and so $\lim_{\rho\to+\infty} g(\rho)=+\infty$. Furthermore $\lim_{\rho\to1+} g(\rho)=+\infty$ and g is continuous on $(1,\infty)$. Then for each $\lambda>\lambda_1$ there are at least two non-trivial solutions of (1.0).

Finally $\lambda_1>0$. In fact suppose that $\lambda_1=0$ then, by the above properties of g, there exists u_0, with $\|u_0\|>1$, such that $\lambda_1=g^2(\|u_0\|)=0$ and $-u_0''(x)=0$ for almost all $x\in(0,1)$, $u_0(0)=u_0(1)=0$. Hence $u_0\equiv0$, contradicting the fact that $\|u_0\|>1$.

THEOREM 1.8. *Suppose that f is asymptotically super-linear. Then there exists* $\bar\lambda$ *such that for each* $\lambda>\bar\lambda$ *there exists exactly one non-trivial solution* u_λ *with* $\|u_\lambda\|\to1$ *as* $\lambda\to+\infty$.

Proof. By the Lemma 1.4 we have that $\lim_{\rho\to1+} g'(\rho)=-\infty$. Since $\lim_{\rho\to1+} g(\rho)=+\infty$, the result follows immediately from this, the continuity of g on $(1,\infty)$ and Lemma 1.6.

THEOREM 1.9. *Suppose that* $\lim_{p\to+\infty} f(p)/p=A(0<A\leqq+\infty)$. *Then* $\lim_{\rho\to+\infty} g(\rho)=A^{-\frac{1}{2}}\pi$. *If* $A=+\infty$ *then for each* $\lambda\in(0,\infty)$ *there exists at least one non-trivial solution of* (1.0).

Proof. The assertion is a direct consequence of Lemmas 1.5 and 1.6 and the continuity of g on $(1,\infty)$.

§2. SOME PARTICULAR NON-LINEARITIES

In view of what has been proved in Section 1 S has the structure illustrated in Figure 1 if we can first establish the following fact

(i) if $g'(\rho)=0$ and $\rho>1$ then $g''(\rho)>0$.

(We will see that the conditions which insure (i) imply the sub-linearity of f.)

Since (i) involves the second derivative of the function g, we first need to state the following:

LEMMA 2.1. *Let*

$$r(\rho)=-\tfrac{1}{2}F(\rho)^{-\frac{3}{2}}+\frac{F(\rho)^{-\frac{1}{2}}}{K(1)}-\int_1^\rho \{F(\rho)-F(\omega)\}^{-\frac{1}{2}}\frac{K'(\omega)}{K^2(\omega)}\,d\omega.$$

Then the function r is continuously differentiable on $(1,\infty)$ *and*

$$\frac{r'(\rho)}{K(\rho)}=\frac{3}{4}F(\rho)^{-\frac{5}{2}}-\frac{1}{2}\frac{F(\rho)^{-\frac{3}{2}}}{K(1)}-F(\rho)^{-\frac{1}{2}}\frac{K'(1)}{K^3(1)}-\int_1^\rho \{F(\rho)-F(\omega)\}^{-\frac{1}{2}}\left[\frac{K'(\omega)}{K^3(\omega)}\right]' d\omega$$

Proof.

$$\int_1^\rho \{F(\rho)-F(\omega)\}^{-\frac{1}{2}}\frac{K'(\omega)}{K^2(\omega)}\,d\omega=-2\int_1^\rho [\{F(\rho)-F(\omega)\}^{\frac{1}{2}}]'\frac{K'(\omega)}{K^3(\omega)}\,d\omega$$

$$=2F(\rho)^{\frac{1}{2}}\frac{K'(1)}{K^3(1)}+2\int_1^\rho \{F(\rho)-F(\omega)\}^{\frac{1}{2}}\left[\frac{K'(\omega)}{K^3(\omega)}\right]' d\omega.$$

By differentiation we obtain the assertion.

Now we are in a position to prove the main result of this section.

THEOREM 2.2. *Suppose that the function* $K:[1,\infty)\to(0,\infty)$ *satisfies the following conditions*

(a) $$K'(p)\leqq 0 \quad \textit{for all} \quad p\geqq 1$$

(b) $$\left[\frac{K'(p)}{K^3(p)}\right]'\leqq 0 \quad \textit{for all} \quad p\geqq 1.$$

Then there exists $\lambda_1>0$ *such that for each* $0\leqq\lambda\leqq\lambda_1$, *there is only the trivial solution for the problem* (1.0), *for each* $\lambda>\lambda_1$, *there are exactly two non-trivial solutions of* (1.0) *and for* $\lambda=\lambda_1$ there is exactly one non-trivial solution.

Proof. By Lemma 1.4 we have that $g'(\rho)=\sqrt{2}K(\rho)r(\rho)$ and by Lemma 2.1 we can consider

$$g''(\rho)=\sqrt{2}\{K(\rho)r'(\rho)+K'(\rho)r(\rho)\} \quad \text{on} \quad (1,\infty).$$

Since $K(\rho)>0$ for all $\rho\geqq 1$, if $g'(\rho)=0$ with $\rho>1$, then $r(\rho)=0$ and so $g''(\rho)$ has the same sign as $r'(\rho)$. Therefore we need only prove that if $r(\rho)=0$, with $\rho>1$, then $r'(\rho)>0$.

If $r(\rho)=0$, we get

$$\frac{F(\rho)^{-\frac{1}{2}}}{K(1)}=\frac{1}{2}F(\rho)^{-\frac{3}{2}}+\int_1^\rho \{F(\rho)-F(\omega)\}^{-\frac{1}{2}}\frac{K'(\omega)}{K^2(\omega)}\,d\omega.$$

It follows that

$$-\frac{1}{2}\frac{F(\rho)^{-\frac{3}{2}}}{K(1)}=-\frac{1}{4}F(\rho)^{-\frac{5}{2}}-\frac{F(\rho)^{-1}}{2}\int_1^\rho \{F(\rho)-F(\omega)\}^{-\frac{1}{2}}\frac{K'(\omega)}{K^2(\omega)}\,d\omega.$$

Replacing in the expression of $r'(\rho)$ we get

$$\frac{r'(\rho)}{K(\rho)}=\frac{1}{2}F(\rho)^{-\frac{5}{2}}-\frac{F(\rho)^{-1}}{2}\int_1^\rho \{F(\rho)-F(\omega)\}^{-\frac{1}{2}}\frac{K'(\omega)}{K^2(\omega)}\,d\omega$$

$$-F(\rho)^{-\frac{1}{2}}\frac{K'(1)}{K^3(1)}-\int_1^\rho \{F(\rho)-F(\omega)\}^{-\frac{1}{2}}\left[\frac{K'(\omega)}{K^3(\omega)}\right]' d\omega.$$

This completes the proof.

Note that the conditions of Theorem 2.2 imply that K on $(1,\infty)$ is a decreasing function. (Hence f is sub-linear.) In the case of non-linearities involving increasing concave functions, we can obtain for S the same results of Theorem 2.2 finding for $g'(\rho)$ an alternative representation.

For this, for $\rho>1$, we have

$$\frac{g(\rho)}{\sqrt{2}}=\left\{\int_1^\rho K(s)\,ds\right\}^{-\frac{1}{2}}+\int_1^\rho\left\{\int_\omega^\rho K(s)\,ds\right\}^{-\frac{1}{2}}d\omega=\rho^{-\frac{1}{2}}\left\{\int_{\rho^{-1}}^1 K(\rho z)\,dz\right\}^{-\frac{1}{2}}$$

$$+\rho^{\frac{1}{2}}\int_{\rho^{-1}}^1\left\{\int_t^1 K(\rho z)\,dz\right\}^{-\frac{1}{2}}dt=\rho^{\frac{1}{2}}\left[\rho^{-1}R(\rho,\rho^{-1})^{-\frac{1}{2}}+\int_{\rho^{-1}}^1 R(\rho,t)^{-\frac{1}{2}}\,dt\right]$$

where $R(\rho,t)=\int_t^1 K(\rho z)\,dz$.

LEMMA 2.3. *Let* $T(\rho)=\rho^{-1}R(\rho,\rho^{-1})^{-\frac{1}{2}}+\int_{\rho^{-1}}^1 R(\rho,t)^{-\frac{1}{2}}\,dt$. *Then T is twice continuously differentiable on* $(1,\infty)$. *Furthermore*

$$T'(\rho)=-\tfrac{1}{2}\rho^{-1}R(\rho,\rho^{-1})^{-\frac{3}{2}}\left\{\rho^{-2}K(1)+\int_{\rho^{-1}}^1 zK'(\rho z)\,dz\right\}$$

$$-\tfrac{1}{2}\int_{\rho^{-1}}^1 R(\rho,t)^{-\frac{3}{2}}\left\{\int_t^1 zK'(\rho z)\,dz\right\}dt$$

and

$$T''(\rho)=\tfrac{3}{4}\rho^{-1}R(\rho,\rho^{-1})^{-\frac{5}{2}}\left\{\rho^{-2}K(1)+\int_{\rho^{-1}}^1 zK'(\rho z)\,dz\right\}^2$$

$$+\tfrac{3}{4}\int_{\rho^{-1}}^1 R(\rho,t)^{-\frac{5}{2}}\left\{\int_t^1 zK'(\rho z)\,dz\right\}^2 dt$$

$$+\rho^{-4}R(\rho,\rho^{-1})^{-\frac{3}{2}}\{\tfrac{3}{2}K(1)-\tfrac{1}{2}K'(1)\}$$

$$-\tfrac{1}{2}\rho^{-1}R(\rho,\rho^{-1})^{-\frac{3}{2}}\int_{\rho^{-1}}^1 z^2K''(\rho z)\,dz$$

$$-\tfrac{1}{2}\int_{\rho^{-1}}^1 R(\rho,t)^{-\frac{3}{2}}\left\{\int_t^1 z^2K''(\rho z)\,dz\right\}dt.$$

Proof. For each $\rho>1$ there exists a positive constant $c(\rho)$ such that $K(p)\geqq c(\rho)$ for all $1\leqq p\leqq\rho$. Hence

$$R(\rho,t)=\int_t^1 K(\rho z)\,dz\geqq c(\rho)(1-t)\quad t\in(\rho^{-1},1).$$

Since $K(\rho z)$ is a twice continuously differentiable function on $(\rho^{-1},1)$ for all $\rho>1$, there exists a positive constant $d=d(\rho)$ such that

$$R(\rho,t)^{-\frac{3}{2}}\left|\int_t^1 zK'(\rho z)\,dz\right|\leqq d(\rho)(1-t)^{-\frac{1}{2}}\quad\text{for all}\quad t\in(\rho^{-1},1).$$

Therefore the formula for $T(\rho)$ can be differentiated term by term on $(1,\infty)$.

Similarly we can easily see that T' is differentiable on $(1, \infty)$. The proof of the lemma can be completed by routine calculation.

Let us now prove the following:

THEOREM 2.4. *Suppose that the function* $K\colon [1, \infty) \to (0, \infty)$ *satisfies the following conditions*

(a') $$\tfrac{3}{2}K(1) > K'(1)$$

(b') $$[p^{\frac{3}{2}}K'(p)]' \leqq 0 \quad \textit{for all} \quad p \geqq 1.$$

Then there exists $\lambda_1 > 0$ *such that for each* $0 \leqq \lambda < \lambda_1$ *there is only the trivial solution* $(0, \lambda)$ *for the problem* (1.0), *for each* $\lambda > \lambda_1$ *there are exactly two non-trivial solutions of* (1.0) *and for* $\lambda = \lambda_1$ *there is exactly one non-trivial solution.*

Proof. Since $g(\rho) = \sqrt{2}\rho^{\frac{1}{2}}T(\rho)$ we have that g is twice continuously differentiable on $(1, \infty)$ and

$$g'(\rho) = \sqrt{2}\left(\tfrac{1}{2}\rho^{-\frac{1}{2}}T(\rho) + \rho^{\frac{1}{2}}T'(\rho)\right).$$

If $g'(\rho) = 0$, then

$$g''(\rho) = \sqrt{2}\rho^{-\frac{1}{2}}\left(\tfrac{3}{2}T'(\rho) + \rho T''(\rho)\right).$$

We must now prove that $\frac{3}{2}T'(\rho) + \rho T''(\rho) > 0$. For this, we have that

$$\begin{aligned}
\tfrac{3}{2}T'(\rho) + \rho T''(\rho) &= \tfrac{1}{2}\rho^{-3}R(\rho, \rho^{-1})^{-\frac{3}{2}}\{\tfrac{3}{2}K(1) - K'(1)\} \\
&\quad - \tfrac{1}{2}\rho^{-1}R(\rho, \rho^{-1})^{-\frac{3}{2}}\left\{\int_{\rho^{-1}}^{1} [\tfrac{3}{2}zK'(\rho z) + \rho z^2K''(\rho z)]\,dz\right\} \\
&\quad - \tfrac{1}{2}\int_{\rho^{-1}}^{1} R(\rho, t)^{-\frac{3}{2}}\left\{\int_{t}^{1} [\tfrac{3}{2}zK'(\rho z) + \rho z^2K''(\rho z)]\,dz\right\} dt \\
&\quad + \tfrac{3}{4}R(\rho, \rho^{-1})^{-\frac{5}{2}}\left\{\rho^{-2}K(1) + \int_{\rho^{-1}}^{1} zK'(\rho z)\,dz\right\}^2 \\
&\quad + \tfrac{3}{4}\rho\int_{\rho^{-1}}^{1} R(\rho, t)^{-\frac{5}{2}}\left\{\int_{t}^{1} zK'(\rho z)\,dz\right\}^2 dt.
\end{aligned}$$

By our conditions on K, the proof is completed.

Note that the conditions (a') and (b') of the above Theorem imply that if $K'(p) \geqq 0$, for all $p \geqq 1$, then $K''(p) \leqq 0$. Moreover, in virtue of (a') and (b'), K is asymptotically sub-linear.

Lastly we prove the following:

THEOREM 2.5. *Suppose that the function* $K\colon [1, \infty) \to (0, \infty)$ *satisfies the following conditions*

(a'') $$\lim_{p \to +\infty} \frac{K(p)}{p} = A \quad \textit{where} \quad 0 < A \leqq +\infty$$

(b'') $$pK'(p) - K(p) \geqq 0 \quad \textit{for all} \quad p \geqq 1.$$

Then for each $\lambda \in (A^{-1}\pi^2, \infty)$ *there exists exactly one non-trivial solution of* (1.0). *If* $A < +\infty$ *then for each* $\lambda \in [0, A^{-1}\pi^2]$ *there is the only trivial solution* $u \equiv 0$.

Proof. By Theorem 1.9 the condition (a″) implies that $\lim_{\rho \to +\infty} g(\rho) = A^{-\frac{1}{2}}\pi$. Since $\lim_{\rho \to 1+} g(\rho) = +\infty$, the Theorem is proved if we show that $g'(\rho) < 0$ for all $\rho > 1$. For this, we consider the formula

$$g'(\rho) = 2^{-\frac{1}{2}}\rho^{-\frac{1}{2}}(T(\rho) + 2\rho T'(\rho)).$$

By Lemma 2.3 we get

$$\begin{aligned} T(\rho) + 2\rho T'(\rho) = \rho^{-1}R(\rho, \rho^{-1})^{-\frac{1}{2}} + \int_{\rho^{-1}}^{1} R(\rho, t)^{-\frac{1}{2}}\, dt \\ - R(\rho, \rho^{-1})^{-\frac{3}{2}}\left\{\rho^{-2}K(1) + \int_{\rho^{-1}}^{1} zK'(\rho z)\, dz\right\} \\ - \rho \int_{\rho^{-1}}^{1} R(\rho, t)^{-\frac{3}{2}}\left\{\int_{t}^{1} zK'(\rho z)\, dz\right\} dt. \end{aligned}$$

Using (b″) we obtain

$$\rho^{-1}R(\rho, \rho^{-1})^{-\frac{1}{2}} \leqq R(\rho, \rho^{-1})^{-\frac{3}{2}} \int_{\rho^{-1}}^{1} zK'(\rho z)\, dz$$

and

$$\int_{\rho^{-1}}^{1} R(\rho, t)^{-\frac{1}{2}}\, dt \leqq \rho \int_{\rho^{-1}}^{1} R(\rho, t)^{-\frac{3}{2}}\left\{\int_{t}^{1} zK'(\rho z)\, dz\right\} dt.$$

Hence $g'(\rho) < 0$, for all $\rho > 1$. This completes the proof.

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