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## On a Class of Nonlocal Problems with Applications to Mathematical Biology

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## On a Class of Nonlocal Problems with Applications to Mathematical Biology

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*To our friends Lynn Erbe and Herb Freedman*

**Abstract.** We consider the problem of finding positive solutions to a family of nonlocal partial differential equations, with mixed boundary conditions. It is a feature of these problems that maximum principle methods are not applicable, and we employ topological methods in the analysis. A key role is played by related eigenvalue estimates.

### 1 Introduction

Let  $B_\delta(x, y)$  denote a mollifier in  $R^N$ , i.e.,  $B_\delta(x, y) = B_\delta(|x - y|) \in C_0^\infty$ ,  $\int_{R^N} B_\delta(x, y) dy = 1$  for any  $x$ ,  $B_\delta(|x - y|) = 0$  if  $|x - y| \geq \delta$ ,  $B_\delta(|x - y|)$  bounded away from zero if  $|x - y| < \mu < \delta$ . Consider a smooth bounded domain  $\Omega \subset R^N$  and define for any  $u \in L^1(\Omega)$ ,

$$\bar{u}(x) = \int_{\Omega} B_\delta(x, y) u(y) dy.$$

In this paper we consider the problem of finding positive solutions to the nonlocal partial differential equation:

$$-\nabla[a\nabla u + \vec{b}u] = [h - g\bar{u}]u \quad (1)$$

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subject to the mixed boundary conditions:

$$\begin{cases} u = 0 & \text{on } \partial\Omega_D; \\ \text{natural boundary conditions on } \partial\Omega_N, \text{ i.e., in this case :} & \\ a \frac{\partial u}{\partial \vec{n}} + (\vec{b} \cdot \vec{n})u = 0 & \end{cases} \quad (2)$$

where  $\partial\Omega_D$ ,  $\partial\Omega_N$  are disjoint parts of  $\partial\Omega$ ,  $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ ,  $\partial\Omega_D$  closed. We shall consider explicitly only cases where  $\partial\Omega_D$  and  $\partial\Omega_N$  are not trivial but simpler versions of our proofs work in the other cases. We also require that regularity conditions be obtained at points of  $\partial\Omega$  (resp.  $\partial\Omega_D \cap \overline{\partial\Omega_N}$ ) by means of biLipschitz maps to hemispheres (resp. quarter spheres).

Furthermore, given any constant  $K$  we ask that there exists a neighbourhood  $\mathcal{N}$  of  $\partial\Omega$  such that

$$\int_{\mathcal{N}} a |\nabla \omega|^2 + (\vec{b} \cdot \nabla \omega) \omega > K \int_{\mathcal{N}} \omega^2 \quad (3)$$

for any nontrivial  $\omega \in H_0^{1,2}(\mathcal{N} \cap (\Omega \cup \partial\Omega_N))$ . The reader interested in detailed descriptions of such conditions may find some of them given in, e.g., [18], [19], [22], [24] and elsewhere. As for the functions  $a, h, g, \vec{b}$  in (1), we recall their possible physical meanings:  $a(x)$  represents the diffusion process;  $\vec{b}(x) = (b_1(x), \dots, b_N(x))$  is a possible drift;  $h(x)$  the growth condition and  $g(x)$  a factor to describe the magnitude of the "crowding effects". We ask that  $a, h, g$  be nonnegative in  $L^\infty(\Omega)$  and piecewise smooth with  $\alpha < a(x) < \beta$  for some constants  $0 < \alpha < \beta$ ,  $a$  smooth near  $\partial\Omega_N$  and that  $\vec{b}$  be  $C^1$ . We thus allow for some abrupt environmental changes in the physical problem being described as was earlier done by H. I. Freedman and his coworkers, [10], [11], for local problems. Unlike such earlier approaches, we do not consider the problem in separate "patches", with suitable continuity conditions at the patch boundaries. Instead we formulate the problem globally in weak form and recover the patch continuity conditions as a consequence. This is indicated at the end of the paper. Consequently, we feel that this approach has also some advantages for purely local problems, although we do not pursue this topic here.

Observe that we require more smoothness for  $\vec{b}$  than for the other coefficients. This has to do with our method of proof, but we conjecture that the results are true for  $\vec{b}$  piecewise smooth, say.

Nonlocal equations similar to (1) have been previously introduced to model biological processes by [13], [8, 9], [5]. In these papers, further significant references may be found.

Some differences between the present results and earlier work with which we are familiar are as follows: as mentioned above, we consider the case of mixed boundary conditions, with piecewise smooth  $L^\infty$  coefficients in which  $g$  is also allowed to vanish somewhere in  $\Omega$ . Furthermore, the kernel  $B_\delta(x, y)$  vanishes for  $|x - y| > \delta$ . We do not require that the left-hand side of (1) be coercive. We feel that these conditions are more realistic than many often used, and certainly more general. They do introduce new difficulties in the proofs.

We note that here  $\bar{u}$  is calculated using only values of  $y$  inside  $\Omega$ , i.e., the effects on  $u(x)$ , for  $x$  with  $\text{dist}(x, \partial\Omega) < \delta$ , of values  $u(y)$  with  $y \notin \Omega$  are disregarded except through the boundary conditions. If such interaction effects were significant, then  $\Omega$  could be enlarged. It may also be of interest to assume that  $u$  is given outside  $\Omega$ . This is actually what we do here, with the choice  $u(y) \equiv 0$  if  $y \notin \Omega$ , for most of the paper, but the small  $\delta$  case is treated differently.

We emphasize to the reader that order arguments (i.e., upper/lower solutions) are in general not applicable to this family of problems, as was illustrated by explicit example in [5], even if  $g(x) = \varepsilon \simeq 0$ . The problem we consider is thus different from those considered in [3], [9], [12], [15], where if  $g(x)$  was small enough then order procedures were indeed applicable, and from those considered in, e.g., [20], [21] where order arguments can always be employed. We thus proceed by Topological Methods (Degree Theory).

The paper is in final form and structured as follows: We shall first consider the case where  $g$  is bounded (essentially) away from zero. Next, we will allow  $g$  to be zero at some points of  $\Omega$ . The case of  $\delta$  small will then be considered. Some comments, related results and open problems will conclude the paper.

Finally, we point out that our proofs work without any changes in the case that the right-hand side of (1) is replaced by the more general expression  $f(x, u, T(u))u$  under suitable assumptions on  $f$  and the operator  $T$ . More comments on this will be made in the last section.

## 2 Existence for general $\delta$

Note first that equation (1) and the maximum principle show that if  $0 \leq u \in L^\infty$ , nontrivial, then  $u > 0$ . Since most of the coefficients are only in  $L^\infty$ , to see this note that we must still have  $u \in C^{\alpha_0}(\bar{\Omega})$  for some  $\alpha_0$  determined by the problem data but not  $u$ , [24]. If  $u(P) = 0$  at some  $P \in \Omega$ , then  $\|u\|_{L^1(S_r(P))} = 0$  for small  $r$  by the generalized Harnack's Inequality, [14, p. 194], where  $S_r(P)$  denotes the ball of radius  $r$ , center  $P$ . It follows that  $u \equiv 0$  in  $S_r(P)$  and thus  $u > 0$  in  $\Omega$  since  $\Omega$  is connected.

As may be expected, we shall formulate results in the sequel based on suitable eigenvalue estimates. Specifically here we consider the two parameter local problem:

$$\ell_0 \omega = -\nabla[a \nabla \omega + \vec{b} \omega] + \eta g \omega = \mu_1(\eta) h \omega \quad (4)$$

subject to boundary conditions (2), where  $\mu_1(\eta)$  denotes the eigenvalue with a positive eigenvector. Problem (4) is classical, it and related problems have been studied by many authors, see, e.g., [4], [6]. It is well known that  $\omega > 0$  exists, and  $\mu_1(\eta)$  is monotone increasing in  $\eta$ .

Our specific assumption is:

$$\mu_1(0) < 1 < \mu_1(K) \quad (5)$$

for large  $K$ , i.e., "eigenvalue crossing" occurs for this related problem. It is fairly easy to see that if this condition is dropped, then the nonlocal problem (1) - (2) may not have a positive solution.

We collect such preliminary facts as well as other results for the reader's convenience in Lemma 0, and give brief proofs.

**Lemma 1** For  $\eta \geq 0$ :

- (a)  $\mu_1(\eta)$  exists and  $\omega > 0$  in  $\Omega \cup \partial\Omega_N$ . Moreover,  $\mu_1(\eta)$  and  $\omega$  are unique.
- (b)  $\mu_1(\eta) \uparrow \eta$ . If  $g > \alpha_0 > 0$  then  $\mu_1(\eta) \rightarrow \infty$  as  $\eta \rightarrow \infty$

$$(c) \theta_1(\eta) \leq \mu_1(\eta) \leq \theta_2(\eta)$$

where  $\theta_1(\eta)$ ,  $\theta_2(\eta)$  are the first eigenvalues of the formally self-adjoint problems:

$$-\nabla \left[ a \nabla \omega_1 + \frac{\vec{b}}{2} \omega_1 \right] + \frac{\vec{b} \cdot \nabla \omega_1}{2} + \eta g \omega_1 = \theta_1(\eta) h \omega_1$$

and

$$-\nabla \left[ a \nabla \omega_2 + \frac{\vec{b}}{2} \omega_2 \right] + \frac{\vec{b} \cdot \nabla \omega_2}{2} + \left[ \frac{|\vec{b}|^2}{4a} + \eta g \right] \omega_2 = \theta_2(\eta) h \omega_2$$

respectively, with the associated conditions (2). Here the existence of  $\theta_1(\eta)$  is assumed.

(d) If problem (1) - (2) has a positive solution then condition (5) holds.

(e) Let  $0 \leq u \in C^\alpha$  satisfy

$$\begin{cases} -\nabla[a \nabla u + \vec{b}u] \leq hu \\ \text{Boundary conditions (2)} \end{cases}$$

and assume  $P \in \Omega$ . Let  $\delta^* > 0$  be given, and let  $S_{\delta^*}(P)$  denote the sphere centered at  $P$  of radius  $\delta^*$ . Then there exists  $K, \varepsilon_0 > 0$  independent of  $u, P$  such that

$$\sup_{S_{\varepsilon_0}(P) \cap \Omega} u(x) \leq K \left( \int_{S_{\delta^*}(P) \cap \Omega} u^2 \right)^{1/2}.$$

(f) Let  $z \in C^\alpha(\bar{\Omega})$  satisfy  $-\nabla[a \nabla z + \vec{b}z] - \lambda h z = f$  with boundary conditions (2) and  $0 \geq f \in L^\infty$ . There exists  $\lambda_0 > 0$  independent of  $f, z$  such that if  $0 < \lambda < \lambda_0$  and  $z \geq 0$  then  $z \equiv 0$ .

**Proof** (a) That  $\mu_1(\eta)$  exists and  $\omega > 0$  is an immediate consequence of the fact  $\ell_0^{-1}$  is a positive map for  $\eta \geq 0$  (by the maximum principle applied to the formal  $L^2$  adjoint, ([23], p. 97)) from  $C^{\alpha_0}$  to itself if the coefficients are smooth. See, e.g., ([16], p. 67), and note that  $(\ell_0)^{-1}(hz) \geq cz$  for some  $c > 0, 0 \leq z \in C_0^\infty$  and if  $\omega \geq 0$  then  $\omega > 0$  in  $\Omega$  by our earlier comments at the start of this section. To obtain the result in our case we pass to the limit over a normalized (in  $L^2$ ) sequence, employing the estimates of part (c) to show that the sequence of approximating eigenvalues is bounded. That  $\omega > 0$  in  $\partial\Omega_N$  follows from the extension process of part (e).

To see that there is only one such  $\mu_1(\eta)$ , note that by the Fredholm Alternative we also have

$$\ell_0^*(\omega^*) \triangleq \mu_1^*(\eta) h \omega^*$$

for some  $\mu_1^*(\eta), \omega^* > 0$  where  $\ell_0^*$  denotes the formal adjoint, i.e.,

$$\mu_1(\eta)(h\omega^*, \omega) = (\omega^*, \ell_0 \omega) = \mu_1^*(\eta)(h\omega^*, \omega).$$

Since  $(h\omega^*, \omega) > 0$ , this shows  $\mu_1(\eta)$  is unique. Finally to see that  $\omega > 0$  is also unique (up to scalar multiples), note that if there were two such eigenfunctions:  $\omega, v$  then  $\ell_0(\omega - \varepsilon v) - \mu_1(\eta)(\omega - \varepsilon v) = 0$ . Set  $E = \{\varepsilon \mid \omega - \varepsilon v \geq 0 \text{ in } \Omega\}$ . Clearly  $0 \in E$ ,  $E$  is closed and if  $\varepsilon \in E$  then  $(\omega - (\varepsilon + \delta)v)^-$  must have support near  $\partial\Omega$  for small  $\delta > 0$ , since given any connected compact set  $K \subset \Omega$  then, again by Harnack's Inequality, either  $\omega - \varepsilon v \equiv 0$  in  $K$  or it is bounded away from zero in  $K$ . But the form associated with  $\ell_0$  is coercive over such functions, by inequality (3), whence  $\varepsilon + \delta \in E$  for  $\delta$  small, i.e.,  $E$  is also open, and the result follows.

(b) Again by the Fredholm Alternative (see also [2] for related concepts) we have for any  $0 < u \in \text{Domain}(\ell_0)$ ,  $\ell_0 u - \mu_1(\eta)hu \in C^\alpha$ , and assuming at first smooth coefficients with  $h > 0$  that:

$$(\omega^*, \ell_0 u - \mu_1(\eta)hu) = 0,$$

i.e., for some  $x_0 \in \Omega$ ,

$$\ell_0 u(x_0) = \mu_1(\eta)h(x_0)u(x_0)$$

whence

$$\inf_{x \in \Omega} \left( \frac{\ell_0 u}{hu} \right) \leq \mu_1(\eta) \leq \sup_{x \in \Omega} \left( \frac{\ell_0 u}{hu} \right).$$

For example, if  $-\nabla[a\nabla u_i + \vec{b}u_i] + \eta_i g u_i = \mu(\eta_i)h u_i$  with  $i = 1, 2$  and  $\eta_2 > \eta_1$ , then

$$\begin{aligned} \mu_1(\eta_1) &\leq \sup_{x \in \Omega} \left[ \frac{-\nabla[a\nabla u_2 + \vec{b}u_2] + \eta_1 g u_2}{h u_2} \right] \\ &= \sup_{x \in \Omega} \left[ \frac{\mu_2(\eta_2)h u_2 + (\eta_1 - \eta_2)g u_2}{h u_2} \right] \leq \mu_2(\eta_2). \end{aligned}$$

Passing to a limit then gives the result for our case. That  $\mu_1(\eta) \rightarrow \infty$  as  $\eta \rightarrow \infty$  for  $g > \alpha_0 > 0$  follows from  $\mu_1(\eta) \geq \theta_1(\eta)$  (shown in part (c)) and the Courant Min-Max Principle.

(c) Observe  $\theta_1(\eta) \leq \mu_1(\eta)$  follows from:

$$\mu_1(\eta) = \frac{\int_{\Omega} a |\nabla \omega|^2 + (\vec{b} \cdot \nabla \omega) \omega + \eta g \omega^2}{\int_{\Omega} h \omega^2}$$

while  $\theta_1(\eta)$  is the infimum of the right-hand side over  $\omega \in H_0^{1,2}(\Omega \cup \partial\Omega_N)$ ,  $h\omega \neq 0$ . Next we obtain  $\theta_2(\eta) \geq \mu_1(\eta)$  by modifying the arguments in [1]. Specifically, let  $\varphi \in C_0^\infty(\Omega \cup \partial\Omega_N)$ , and observe the identity

$$\begin{aligned} \int_{\Omega} (\omega^*)^2 a \left| \nabla \left( \frac{\varphi}{\omega^*} \right) \right|^2 + \omega^* \varphi \vec{b} \cdot \nabla \left( \frac{\varphi}{\omega^*} \right) + \frac{|\vec{b}|^2}{4a} \varphi^2 \\ = \int_{\Omega} a |\nabla \varphi|^2 + (\vec{b} \cdot \nabla \varphi) \varphi + \left( \frac{|\vec{b}|^2}{4a} + \eta g \right) \varphi^2 - \mu_1(\eta) \int_{\Omega} h \varphi^2. \end{aligned}$$

As in [1], we observe that the left-hand side is nonnegative definite and obtain the result by taking the infimum over  $\varphi$ .

(d) Suppose (1) - (2) has a solution  $0 < u \in L^\infty$ . Then

$$-\nabla[a\nabla u + \vec{b}u] - hu = -g\bar{u}u \leq 0 \quad (\neq 0).$$

We thus have (by the arguments in (b)) that there exists  $0 < \theta < 1$  and  $\omega > 0$  such that

$$-\nabla[a\nabla \omega + \vec{b}\omega] - \theta h \omega = 0$$

and  $\omega$  satisfies condition (2). In fact,  $\mu_1(0) = \theta < 1$  since  $\theta = 1$  implies  $g\bar{u}u = 0$ . Furthermore, if we choose  $K \geq \bar{u}$  in  $\bar{\Omega}$  then

$$-\nabla[a\nabla u + \vec{b}u] + gKu \geq hu.$$

For example,  $\mu_1(K) \geq 1$ . We can only have equality if  $u$  is actually the eigenvector, i.e.,  $K = \bar{u}$ . But in this case,  $\mu_1(K+1) > 1$ .

(e) Note that for each point  $Q \in \overline{\partial\Omega}_N$  we have a neighbourhood  $\mathcal{N}$  such that  $\Omega \cap \mathcal{N}$  can be mapped either to a hemisphere or a quarter sphere if  $Q \in \overline{\partial\Omega}_N \cap \partial\Omega_D$  of radius  $r = r(Q)$  by biLipschitz maps  $L$ . Without loss of generality,  $\mathcal{N} \subset S_{\delta^*}(Q)$ . Set  $\mathcal{M} = L^{-1}[S_{\varepsilon r}(L(Q))]$ ,  $0 < \varepsilon < 1$ . First cover  $\overline{\partial\Omega}_N \cap \partial\Omega_D$  in this way and then the remainder of  $\partial\Omega_N$  so that for each  $\mathcal{N}$  we have either  $\mathcal{N} \cap \partial\Omega_D = \emptyset$  or  $\mathcal{N}$  is a neighbourhood of some  $Q \in \overline{\partial\Omega}_N \cap \partial\Omega_D$ . By compactness of  $\overline{\partial\Omega}_N$  we have

$$\overline{\partial\Omega}_N \subset \bigcup_1^m \mathcal{N}_i$$

and thus

$$\overline{\partial\Omega}_N \subset \bigcup_1^m \mathcal{M}_i$$

for some  $\varepsilon$ ,  $0 < \varepsilon < 1$ . Clearly we have  $\text{dist}(\partial(\bigcup_i^m \mathcal{M}_i), \overline{\partial\Omega}_N) \geq \mu > 0$  for some  $\mu$ .

Now if  $\text{dist}(P, \overline{\partial\Omega}_N) \geq \mu$  then the result is immediate by [14] for some  $\varepsilon_0 < \mu/2$ . On the other hand, if  $\text{dist}(P, \overline{\partial\Omega}_N) < \mu$  then  $P \in \mathcal{M}_i$  for some  $i$ . Let  $L : \mathcal{N}_i \cap \Omega \rightarrow H_i$  where  $H_i$  is a hemisphere of radius  $r_i$  (and thus  $L(\mathcal{M}_i \cap \Omega)$  is a hemisphere of radius  $\varepsilon r_i$ ) if  $\mathcal{N}_i \cap \Omega_D = \emptyset$ .  $H_i$ ,  $L(\mathcal{M}_i \cap \Omega)$  are quarter spheres otherwise. We extend  $u$  as an even function to the whole sphere of radius  $r_i$  (resp. to the upper hemisphere) if  $H_i$  is a hemisphere (resp. quarter sphere) and extend the coefficients as outlined in [23]. In the latter case where  $H_i$  is a quarter sphere, we extend  $u$  to the whole sphere by taking  $u \equiv 0$  on the bottom hemisphere. Note that this is different from what was done in [23] where the extension across the Dirichlet boundary was as an odd function, not as zero. Since  $P \in \mathcal{M}_i$ ,  $\varepsilon_i < 1$ , we now apply the interior results of [14] to the extended problem in each sphere and conclude the result for  $\varepsilon_0$  small enough. Since  $L^{-1}$  is also Lipschitz, the result holds for  $P$  and  $\Omega$ . Finally, since the number of  $\mathcal{N}_i$ ,  $\mathcal{M}_i$  is finite, we need only take the smallest  $\varepsilon_0$  and largest  $K$ .

(f) Again by part (a) we have the existence of an eigenvalue  $\xi$  and positive eigenvector  $\tau$  to the problem

$$\mathcal{L}^* \tau = \xi \tau$$

with boundary conditions (2), where

$$\mathcal{L}(\omega) = -\nabla[a\nabla\omega + \vec{b}\omega].$$

We then note:

$$0 \geq (\tau, f) = (\tau, \mathcal{L}z - \lambda hz) \geq (\xi - \lambda|h|_{L^\infty})(\tau, z)$$

and the conclusion if  $\lambda_0 < \xi/|h|_{L^\infty}$ . □

Observe that Lemma 1(c) makes it possible to estimate  $\mu_1$  by first estimating  $\theta_1, \theta_2$  via the Courant Min-Max principle. We shall use this later.

**Lemma 2** *Let  $g > \alpha_0 > 0$  a.e.  $\Omega$ , and let  $a_n, g_n, h_n$  be smooth pointwise approximations to  $a, g, h$  with  $\alpha < a_n < \beta$ ,  $\alpha_0 \leq g_n \uparrow g$ ,  $0 \leq h_n \uparrow h$ . Consider the regularized problem:*

$$-\nabla[a_n \nabla u_n + \vec{b}u_n] = [h_n - g_n \bar{u}_n]u_n \quad (1^*)$$

subject to conditions (2). If assumption (5) holds, then problem (1\*) - (2) has a solution  $0 < u_n \in C^\alpha(\bar{\Omega})$  with  $\|u_n\|_{C^\alpha} \leq K_1$ ,  $\|u_n\|_{C^\alpha} \geq K_0$ , for some positive constants  $\alpha, K_1, K_0$ , independent of  $u_n, n$ .

**Remark** We recall that upper/lower solutions procedures cannot be applied to (1\*) due to the presence of  $\bar{u}$ . Note also that by Lemma 1(b), for assumption (5) to hold it suffices to have  $\mu_1(0) < 1$ .

**Proof** We may assume that  $\vec{b} \cdot \vec{n} > 0$  on  $\partial\Omega_N$ , for if not we need only make the variable change  $u_n = e^\omega v_n$  with  $\nabla\omega \cdot \vec{n} \gg 0$  with  $\omega$  chosen suitably. This will also change the coefficients and  $B_\delta$  but it will not affect their relevant properties. In particular, we will still have  $\mu_1(0) < 1$ . Next, add a large positive constant  $C_0$  to both sides of (1\*) so that the left-hand side becomes coercive over  $H_0^{1,2}(\Omega \cup \partial\Omega_N)$ . We thus obtain the equation:

$$\mathcal{L}u_n \equiv -\nabla[a_n \nabla u_n + \vec{b}u_n] + C_0 u_n = [h_n + C_0 - g_n \bar{u}_n]u_n.$$

With which we associate family of problems:

$$\mathcal{L}u_n = \lambda[h_n + C_0 - g_n \bar{u}_n]u_n^+, \tag{1**}$$

for  $0 \leq \lambda \leq 1$ . A routine bootstrap argument shows that any nontrivial solution  $u_n \in L^\infty$  of (1\*\*) must be classical except at  $\lambda_D : \partial\Omega_D \cap \partial\bar{\Omega}_N$ , and furthermore  $u_n \geq 0$  by choice of  $C_0$ . The earlier arguments then show  $u_n > 0$ . Observe that there exists a  $\lambda_0 > 0$  independent of  $n$  such that if  $\lambda \leq \lambda_0$  then  $u_n \equiv 0$  by Lemma 1(e) and  $u_n$  is thus bounded in this case. While, if  $\lambda_0 < \lambda \leq 1$  then  $u_n$  cannot have a positive maximum on  $\partial\Omega$  since on  $\partial\Omega_N$  we would have  $\frac{\partial u}{\partial n} = -(\vec{b} \cdot \vec{n})u < 0$  at such a point. It follows that if  $P_n \in \Omega$  is a maximum of  $u_n$  then from the equation we have at  $P_n$ :

$$-\text{div}(\vec{b}) + C_0 \leq \lambda[h_n + C_0 - g_n \bar{u}_n]$$

and we conclude  $\bar{u}_n(P_n)$  is bounded independently of  $n$ . By the definition of  $\bar{u}$  and by the assumed smoothness of  $\partial\Omega$  we have  $\int_{S_p(P_n) \cap \Omega} u_n \leq C$  for some sphere  $S_p(P_n)$ . We need only apply Lemma 1(e) to obtain

$$|u_n(P_n)| \leq K \left[ \int_{S_p(P_n) \cap \Omega} u^2 \right]^{1/2}$$

and thus

$$\|u_n\|_{L^\infty} = |u_n(P_n)| \leq K \left[ \|u_n\|_{L^\infty} \int_{S_p(P_n) \cap \Omega} u \right]^{1/2}$$

i.e.,

$$\|u_n\|_{L^\infty} \leq K' \int_{S_p(P_n) \cap \Omega} u$$

for some  $K'$ .

We conclude  $\|u_n\|_{L^\infty(\Omega)} < K$  for some  $K$  independent of  $n, \lambda$ . But then,  $u_n$  satisfies a differential equation with bounded coefficients and we conclude by, e.g., [24],  $\|u_n\|_{C^\alpha(\bar{\Omega})} \leq K_1$  for some  $\alpha, K_1$  independent of  $u_n, \lambda$ . On the other hand, note that if the right-hand side of (1\*\*) is replaced by:

$$[h_n + C_0 - g_n \bar{u}_n]u_n^+ + t\theta$$

with  $t \geq 0$  and some smooth  $\theta \geq 0$ , by (5) we must have for any nontrivial solution  $u$  some point  $P \in \Omega$  such that

$$\bar{u}_n(P) \geq c > 0 \quad (6)$$

with  $c$  independent of  $t$ , and thus  $\|u_n\|_{C^\alpha(\bar{\Omega})} \geq K_0$  with  $K_0$  independent of  $n, t$ . Since  $C^\alpha$  is compactly embedded in  $C^{\alpha_0}$  for  $\alpha > \alpha_0$ , we then have by a routine but lengthy argument using Degree Theory, [25], that (1\*\*) – (2) has a positive solution in some annulus  $B_R - B_r$  where  $B_p$  denotes the sphere of radius  $p$  in  $C^\alpha$  for some  $\alpha > 0$ .

□

**Theorem 1** *Problem (1) – (2) has a positive solution  $u$ .*

**Proof** Note that  $\{u_n\}$  are bounded in  $H^{1,2} \cap L^\infty$  and thus we may assume  $u_n \rightarrow u$  weakly in  $H^{1,2}$  and strongly in  $L^\xi$  for any large  $\xi$ . By (6),  $\bar{u}_n(P_n) \geq c$  and since we also may assume  $P_n \rightarrow P$  and thus  $\bar{u}_n(P_n) \rightarrow \bar{u}(P)$  by the mollifier properties, we must have  $0 \leq u$  nontrivial. Finally note that  $\int_\Omega a \nabla u \cdot \nabla \varphi$  is an equivalent inner product to the usual one in  $H^{1,2} \cap \{v|v = 0 \text{ on } \partial\Omega_D\}$  and thus

$$\int_\Omega a_n \nabla u_n \cdot \nabla \varphi - \int_\Omega a \nabla u \cdot \nabla \varphi = \int_\Omega [a_n - a] \nabla u_n \cdot \nabla \varphi + \int_\Omega a \nabla (u_n - u) \cdot \nabla \varphi.$$

The first term converges to zero by Holder's Inequality since  $0 < \alpha < a_n \uparrow a$  in  $L^\xi$  for any large  $\xi$ , while the second term tends to zero by inner product equivalence. Since convergence of the other terms is immediate, we conclude that  $u$  is a positive (weak) solution of (1) – (2) in  $L^\infty$ , and thus in  $C^\alpha(\bar{\Omega})$  for some  $\alpha$  and classical where the coefficients are smooth, except on  $\partial\Omega_D \cap \partial\bar{\Omega}_N$ .

□

We now pass to the situation where  $g \geq 0$  may actually vanish on parts of  $\Omega$ . We present the results assuming all coefficients to be smooth, since arguments identical to Theorem 1 can then be used to extend the results to more general cases. Elementary examples show that now the existence of positive solutions depends on the relation between the various coefficients in (1) – (2). We shall assume the following condition:

Let  $K > 0$  be given and consider the eigenvalue problem:

$$-\nabla[a \nabla \omega + \vec{b} \omega] + gKI(S_{\delta'}(x))\omega = e(K)h\omega \quad (7)$$

with boundary conditions (2) and some  $\delta' < \delta$  where  $\delta$  is as given in the definition of  $\bar{v}$ . We recall that  $S_{\delta'}(x)$  is the ball centered at  $x$  of radius  $\delta'$ ,  $I(S_{\delta'}(x))(y) = 1$  if  $y \in S_{\delta'}(x)$ ,  $= 0$  otherwise. We then assume that there exists a  $K$  such that given any  $x \in \Omega$  at which  $g(x) = 0$ ,  $e(\lambda)$  satisfies:

$$e(0) < 1 < e(K). \quad (8)$$

Clearly condition (8) excludes problems with very small  $\delta$ , all other quantities being fixed. For these situations an analysis can however be made by perturbation arguments as we show in the next section.

**Theorem 2** *If condition (8) holds, problem (1) – (2) has a positive solution.*

**Proof** The arguments are similar to those of Lemma 1. We thus only make the following comments where some differences do arise. If (8) holds, then any nonnegative solution  $u \in L^\infty$  of

$$-\nabla[a\nabla u + \vec{b}u] = \lambda[h - g\bar{u}]u$$

and conditions (2) must vanish for  $\lambda \leq e(0)$  and is thus clearly bounded. Note that  $e(0)$  depends only on the coefficients.

To show that any solution must be bounded in  $L^\infty$ , it suffices by our earlier arguments to show there exist numbers  $\xi > 0$ ,  $K_1 > 0$  independent of  $u$  such that  $\int_{S_\xi(P) \cap \Omega} u \leq K$ , at any maximum point  $P$ .

Set  $Z = \{x | g(x) = 0\}$ ,  $Q = \{x | \text{dist}(x, Z) \geq \delta_0\}$  with  $\delta_0, \delta''$  such that  $0 < \delta_0 < \delta' < \delta' + \delta_0 < \delta'' < \delta$  and choose  $\theta > 0$  such that  $g(x) \geq \theta$  for  $x$  in  $Q$ . Now if  $P \in Q$  then we proceed exactly as before. If  $P \in \Omega - Q$  then there exists  $y \in Z$  such that  $\|P - y\| < \delta_0$ . Condition (8) implies the existence of some  $x \in S_{\delta'}(y)$  such that  $K \geq \bar{u}(x)$ . Since  $\|x - P\| < \delta' + \delta_0 < \delta'' < \delta$  we obtain the result with  $\xi = \delta'' - (\delta' + \delta_0)$ . The rest of the proof is identical to the earlier one. □

### 3 The small $\delta$ case

We consider here the case of  $\delta$  small with  $g$  still allowed to vanish somewhere in  $\Omega$ , and smooth coefficients. Our starting points are the local problem:

$$-\nabla[a\nabla u + \vec{b}u] = [h - gu]u \tag{9}$$

subject to conditions (2), and the fact that the degree is invariant under perturbation. We basically exploit the observation that  $\bar{u}(x)$  and  $u(x)$  are near each other if  $\delta$  is sufficiently small. The existence of a solution to (9) will then imply the existence of a solution to (1).

We modify the definition of  $\bar{u}$  as follows: If  $u \in C^\alpha(\bar{\Omega})$  then there exists  $\hat{u}$ ,  $C^\alpha$  extension of  $u$ , with  $\|\hat{u}\|_{C^\alpha(\mathbb{R}^n)} \leq K\|u\|_{C^\alpha(\bar{\Omega})}$ . There are clearly infinitely many possible extensions, but here we select one such and then define  $\bar{u}$  by

$$\bar{u}(x) = \int_{\Omega} B_\delta(x, y)\hat{u}(y)dy$$

in this section. Note that this is different from the earlier  $\bar{u}$ , unless Dirichlet conditions are chosen on the whole  $\partial\Omega$  and we set  $u^* \equiv 0$  in  $\mathbb{R}^n - \Omega$ .

Unlike the earlier results, problem (9) - (2) can be considered by upper/lower solution arguments, but it is advantageous for our purposes to use only half of these arguments. Specifically, consider once again the related linear eigenvalue problem (4) with conditions (2), and again assume that condition (5) holds if  $K$  is large enough. It follows that we can find a value of  $K$  and a  $\theta > 0$  such that for any constant  $z > 0$  the problem:

$$\begin{cases} -\nabla[a\nabla\omega_1 + \vec{b}\omega_1] + Kg\omega_1 = (1 + \theta)h\omega_1 \\ a \frac{\partial\omega_1}{\partial n} + (\vec{b} \cdot \vec{n})\omega_1 = 0 & \text{on } \partial\Omega_N \\ \omega_1 = z & \text{on } \partial\Omega_D \end{cases} \tag{10}$$

has a positive solution  $\omega_1$ . This follows from the observation that for  $C_0$  sufficiently large the problem

$$\begin{cases} -\nabla[a\nabla\omega_2 + \vec{b}\omega_2] + [\lambda_0 g - (1 + \theta)h + C_0]\omega_2 = 0 \\ \omega_2 = 1 & \text{on } \partial\Omega_D \\ a \frac{\partial\omega_2}{\partial n} + (\vec{b} \cdot \vec{n})\omega_2 = 0 & \text{on } \partial\Omega_N \end{cases}$$

has a solution  $\omega_2 \geq 0$  by coercivity of the left-hand side. Indeed  $\omega_2 > 0$  in  $\bar{\Omega}$  since it cannot have a minimum of zero in the interior by the Generalized Harnack's Inequality while on the boundary either  $\omega_2 = 1$  or without loss of generality  $\vec{b} \cdot \vec{n} > 0$ . On the other hand the equation:

$$\ell_2 z_0 \equiv -\nabla[a\nabla z_0 + \vec{b}z_0] + [\lambda_0 g - (1 + \theta)h]z_0 = 0$$

with the same boundary conditions has a solution  $z_0$  by our eigenvalue assumptions. But then we have  $\ell_2(z_0 - \omega_2) \geq 0$  and thus  $z_0 - \omega_2 \geq 0$  by the same arguments as in Lemma 0(a). We then need only note that  $\omega_1 = zz_0$ . By the maximum principle, if  $z$  is large enough we have for the solution  $\omega_1$  of (10):

$$\begin{cases} -\nabla[a\nabla\omega_1 + \vec{b}\omega_1] + g\omega_1^2 \geq h\omega_1 \\ a \frac{\partial\omega_1}{\partial n} + (\vec{b} \cdot \vec{n})\omega_1 = 0 & \text{on } \partial\Omega_N \\ \omega_1 > 0 & \text{on } \partial\Omega_D \end{cases}$$

i.e.,  $\omega_1$  is an upper solution to (9) - (2).

**Theorem 3** *If (5) holds then problem (1) - (2) has for  $\delta$  small enough a positive solution.*

**Proof** Note that (9) may be rewritten as

$$-\nabla[a\nabla u + \vec{b}u] + cu = f(x, u) \quad (11)$$

with a  $c > 0$  chosen large enough so that the left-hand side is coercive and  $f$  is monotone in  $u$  for  $u$  in the interval  $0 \leq u \leq \omega_1$ . Let  $\tilde{u} = \min(\omega_1, u)$  and replace  $f(x, u)$  in (11) by  $f(x, \tilde{u})$ . We now apply the same Degree Theory arguments as before, setting

$$\tilde{T} = [-\nabla[a\nabla \cdot + \vec{b}\cdot] + c]^{-1}(f(x, \tilde{u}^+)),$$

we again find that  $\text{Deg}[I - \tilde{T}, B_R - B_r, 0] = 1$  for some  $0 < r < R$  with  $B_p$  the ball of radius  $p$  in  $C^\alpha(\bar{\Omega})$ .

For example, there is a positive solution of (9) - (2) since  $\omega_1$  is an upper solution and furthermore this solution is unique since if  $u_1, u_2$  are two solutions then we may construct a supersolution  $\omega^*$ , by solving (10) with  $z \gg 0$ , such that  $\omega^* > \max(u_1, u_2)$ . By the usual local problem iteration procedure we construct a sequence  $\{\omega_n\}$  such that  $\omega_0 = \omega^*$  and  $\omega_n \geq \omega_{n+1} \geq \max(u_1, u_2)$ . Furthermore,  $\omega_n$  converges to a solution pointwise, so that we may conclude the existence of three solutions  $u_1, u_2, u_3$  with  $u_3 \geq u_1$ . But then we have from (9) that  $gu_3 \geq gu_1$  and by the same arguments as in the Proof of Lemma 1(b) it follows that  $gu_3 = gu_1$  somewhere in  $\Omega$  and thus it follows that  $u_3 \equiv u_1$  and hence  $u_2 \equiv u_1$ . If need be we replace  $\omega_1$  by  $\omega^*$  and thus conclude without loss of generality that the (unique)

positive solution  $u$  satisfies  $u < \omega_1$  in  $\bar{\Omega}$ . If we now choose a small ball in  $C^\alpha$  about the unique local problem solution just found, then the perturbed problem:

$$-\nabla[a\nabla u + \vec{b}u] + cu = f(x, \tilde{u}^+) + g\tilde{u}\tilde{u}^+ - g\tilde{u}\tilde{u}^+$$

must also have a solution in this ball for  $\delta$  small enough, since  $\|\tilde{u} - \tilde{u}\|_{L^\infty(\Omega)} \leq K\delta^\alpha$  by the (present) definition of  $\bar{u}$ . In summary, we have found a solution  $u^*$  of

$$-\nabla[a\nabla u^* + \vec{b}u^*] + cu^* = [h + c - g(\tilde{u}^*)](\tilde{u}^*)^+$$

near  $u$ . We conclude for  $\delta$  small enough, that  $u^* \leq \omega_1$ , i.e.,  $\tilde{u}^* = u^*$  and we have thus found a solution to the original nonlocal problem.  $\square$

#### 4 Examples

In this section we illustrate the above results with a pair of concrete examples.

**Example 1** Suppose that in Equation (1) we have  $a \equiv 1$ ,  $\vec{b} \equiv \vec{0}$ ,  $h = 1$  and  $\Omega = (0, L)$ . Finally assume Dirichlet conditions:  $u(0) = u(L) = 0$ . We find that  $e(0) = \frac{\pi^2}{L^2}$ . Suppose  $g$  is given by:

$$g = \begin{cases} 0 & \text{for } x \in (\frac{L}{2} - \varepsilon, \frac{L}{2} + \varepsilon) \\ g_0 \text{ (constant)} & \text{otherwise} \end{cases}$$

with  $0 < \frac{L}{2} - \varepsilon < \frac{L}{2} + \varepsilon < L$ . Now if  $\delta > 2\varepsilon$  as  $K \rightarrow \infty$  in (7),  $e(K)$  converges to a value at least equal to the smaller of the three eigenvalues obtained by considering the three segments:  $(0, x - \delta)$ ,  $(\frac{L}{2} - \varepsilon, \frac{L}{2} + \varepsilon)$ ,  $(x + \delta, L)$ . By symmetry and eigenvalue monotonicity, this will be least for  $\Omega^* = (0, \frac{L}{2} + \varepsilon - \delta)$ . If we choose  $\frac{L}{2} > \delta + \varepsilon \geq 3\varepsilon$  then the least eigenvalue will be no less than  $\frac{\pi^2}{(\frac{L}{2} + \varepsilon - \delta)^2}$ . A sufficient requirement for the existence of a positive solution to (1) - (2) thus becomes:

$$\pi^2 < L^2 < \frac{L^2 \pi^2}{(\frac{L}{2} + \varepsilon - \delta)^2}.$$

Observe that  $g_0$  does not appear in this condition. Its magnitude will have an effect on the size/shape of the solution  $u$  but not on its existence.

**Example 2** Assume now  $g > 0$  in  $\bar{\Omega}$ , that the coefficients are smooth and the boundary conditions are purely Dirichlet. To apply our results we need only show that the eigenvalue problem

$$-\nabla[a\nabla\omega + \vec{b}\omega] = \mu_1(0)h\omega$$

with conditions (2) has least eigenvalue  $\mu_1(0) < 1$ , since  $\mu_1(K) > 1$  for  $K$  large by Lemma 1(b).

We apply Lemma 1(c) by comparing  $\mu_1(0)$  with  $\theta_2(0)$  where this is given by:

$$-\nabla[a\nabla\omega_2] + \left[ -\frac{1}{2} \operatorname{div}(\vec{b}) + \frac{|\vec{b}|^2}{4a} \right] \omega_2 = \theta_2(0)h\omega_2.$$

That  $\mu_1(0) < 1$  is thus ensured by showing the existence of a  $\omega_3$  such that

$$\int_{\Omega} \beta |\nabla \omega_3|^2 + (\vec{b} \cdot \nabla \omega_3) \omega_3 + \frac{|\vec{b}|^2}{4\alpha} \omega_3^2 - h \omega_3^2 < 0,$$

where we recall  $0 < \alpha < a < \beta$ . This becomes:

$$\int_{\Omega} \beta |\nabla \omega_3|^2 + \left[ -\frac{1}{2} \operatorname{div}(\vec{b}) + \frac{|\vec{b}|^2}{4\alpha} - h \right] \omega_3^2 < 0.$$

It is easy to obtain sufficient conditions for this to be the case. Suppose, for example, that  $\Omega$  contains a cube  $C = \prod_{i=1}^N \{0 < x_i < L\}$ . The least eigenvalue for the Dirichlet Problem associated with the cube is  $(\frac{\pi}{L})^{2N}$  with associated eigenvector  $v = \prod_{i=1}^N \sin(\frac{\pi x_i}{L})$ . We pick  $\omega_3$  by setting  $v = 0$  in  $\Omega - C$  and thus obtain our result if:

$$\frac{\beta \pi^{2N}}{2^N L^N} + \int_C \left[ -\frac{1}{2} \operatorname{div}(\vec{b}) + \frac{|\vec{b}|^2}{4\alpha} - h \right] \prod_{i=1}^N \sin^2\left(\frac{\pi x_i}{L}\right) < 0.$$

### 5 Open questions and related problems

A natural question concerns the uniqueness of the solutions to (1) - (2) found in the preceding sections. In general, no uniqueness results are known to the authors, but in some specific cases, uniqueness can be established. We note, see also [1], that if  $B_\delta$  is constant, i.e., if we consider

$$-\nabla[a\nabla u + \vec{b}u] = \left[ h - g \int_{\Omega} u \right] u,$$

then the solution must be unique:  $u$  is the least eigenvector of (10) with  $e(\mu) = 1$  and  $\int_{\Omega} u = \mu$ . We conjecture that the solution is unique for (1) - (2), but observe that if we allow the kernel  $B_\delta$  to be smooth but otherwise arbitrary, then this is not the case. To see this, consider, e.g., the least eigenvalue Dirichlet problem for:

$$-\Delta u + g_1(1 + \varepsilon_1)u + g_2(1 + \varepsilon_2)u = \mu_1(\varepsilon_1, \varepsilon_2)u,$$

with  $g_1, g_2$  smooth nonnegative functions with disjoint supports. We may choose  $\varepsilon_1, \varepsilon_2$  such that  $\mu_1(\varepsilon_1, 0) = \mu_1(0, \varepsilon_2)$ . The choice of any two functions  $f_1, f_2$  such that:

$$\int_{\Omega} f_1 u_1 = (1 + \varepsilon_1), \quad \int_{\Omega} f_1 u_2 = 1, \quad \int_{\Omega} f_2 u_1 = 1, \quad \int_{\Omega} f_2 u_2 = (1 + \varepsilon_2)$$

yields a nonlocal problem with kernel  $g_1(x)f_1(y) + g_2(x)f_2(y)$ ,  $g = 1$ ,  $h = \mu_1(\varepsilon_1, 0)$  and two positive solutions:  $u_1, u_2$ .

Finally, as mentioned earlier,  $[h - g\bar{u}]u$  can be replaced by a more general  $f$  and  $\bar{u}$  can be replaced by a more general operator  $Tu$ . One case arises from a kernel  $B_\delta(|x - y|)$  which satisfies similar properties to the ones postulated before, except  $B_\delta(0) = \infty$ . One possible such situation comes from the system

$$\begin{cases} -\Delta u = [h - gv]u \\ -\Delta v = u \end{cases}$$

with, for simplicity, Dirichlet conditions. We then obviously have:

$$-\Delta u = [h - gT(u)]u$$

$$\text{where } T(u) = (-\Delta)^{-1}(u) = \int_{\Omega} G(x, y)u(y)dy$$

and  $G(x, y)$  represents the Green's Function for  $-\Delta$ . The earlier procedures are applicable as follows: Let  $u > 0$  be a solution of

$$-\Delta u = \lambda[h - gT(u)]u$$

with  $0 < \lambda_0 < \lambda < 1$ . Choose  $\Omega_\varepsilon = \{x | x \in \Omega, \text{dist}(x, \partial\Omega) < \varepsilon\}$  and let  $\omega > 0$  solve in  $\Omega_\varepsilon$ :

$$\begin{aligned} -\Delta\omega - h\omega &= 0 \\ \omega &= 1 \quad \text{on } (\partial\Omega_\varepsilon) \cap \Omega \\ \omega &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

This is possible if  $\partial\Omega$  is regular enough and  $\varepsilon$  is small. Since  $-\Delta u - hu \leq 0$  in  $\Omega_\varepsilon$  we then have:  $u \leq \omega \|u\|_{L^\infty(\bar{\Omega})}$  and thus the maximum of  $u$  must occur inside some compact set  $K \ll \Omega$ . But if  $P \in K$  then  $v \geq c \int_{B_{\delta/2}(P)} u$  for some constants  $c, \delta > 0$ . Indeed,  $v$  exceeds  $-\Delta z = u$  in  $B_\delta(P)$ ,  $z = 0$  on  $\partial B_\delta(P)$ , for some small  $\delta$  independent of  $u, P$ , i.e.,

$$\begin{aligned} v(P) &\geq c_1 \int_{|P-y|<\delta} \left[ \frac{1}{|P-y|^{n-2}} - \frac{1}{\delta^{n-2}} \right] u \\ &\geq c_2 \int_{B_{\delta/2}(P)} u. \end{aligned}$$

We can thus bound  $\|u\|_{L^\infty}$  as before.

Note also that if the coefficients are assumed regular near  $\partial\Omega$ , then the neighbourhood  $\mathcal{N}$  of the introduction need only be a neighbourhood of  $\partial\Omega_D \cap \bar{\partial\Omega}_N$ . If  $\vec{b}/a$  is a gradient field, then this condition can be dropped altogether.

As a final observation, we point out that if the coefficients in (1) are subject to a finite jump at some point  $x_0 \in \Gamma$ , where  $\Gamma$  is a smooth surface in  $\Omega$  then besides the  $C^\alpha$  continuity of  $u$  we may also conclude by [17, p. 222] that, the "flux matching" condition holds. For example, if  $a(x)$  has a jump at  $x_0$  then  $a \frac{\partial u}{\partial n} |_{x_0}$  is continuous, where  $n$  here denotes the normal to  $\Gamma$  in a chosen direction.

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