Remark 1: Observe that we can choose the function g in such a way that if $s(t_0, \overline{x}(t_0)) = 0$ for some $t_0 \in [0, T]$, then $s(t, \overline{x}(t)) = 0$ for any $t \in [0, T]$. In [1], we proposed $g(t, x, u) = (\partial/\partial t)s(t, x) + ((\partial/\partial x)s(t, x))f(t, x, u)$. With this selection, one has $0 = g(t, \overline{x}(t), \overline{u}(t)) = (d/dt)s(t, \overline{x}(t))$ for almost all $t \in [0, T]$. From the theory of variable structure systems, the solution $u^*(t, x)$ of the algebraic equation $g(t, x, u) = 0$ is called "equivalent control" for system (1) and (2).

Some comments on Assumptions H1)–H6) can be found in [1].

In what follows, we will select the functions s and g to address the control problem of tracking a prescribed bounded trajectory $x_d(t)$, $t \in [0, \infty)$. For simplicity, we henceforth limit our discussion to linear time-invariant systems. Extensions to some nonlinear problems can be obtained generalizing the results in [6]–[8]. Moreover, we will make the standard assumption that the input matrix is full column rank. Under these assumptions it is well known that it is possible to find a change of variables bringing the system to the following form:

$$\dot{x} = f(t, x, u) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} x + \begin{pmatrix} 0 \\ B_2 \end{pmatrix} u \tag{5}$$

where $A_{11} \in R^{(n-m)\times(n-m)}$, $A_{22} \in R^{m\times m}$, $B_2 \in R^{m\times m}$, $x = (x_1^T\ x_2^T)^T$, $x_1 \in R^{n-m}$, $x_2 \in R^m$, $u \in R^m$.

The following is a well-known result of linear system theory [5].

Lemma 1: If (5) is completely controllable, then the pair (A_{11}, A_{12}) is completely controllable.

B. Integral Control

Let $x_d(t)$ be a given bounded trajectory that the system is required to follow. Define the function $s = s(t, x)$ by $s(t, x) = H(x_d(t) - x - e^{Ct}(x_d(0) - x(0)))$, where $H \in R^{m\times n}$ and $C \in R^{n\times n}$ are real matrices to be selected, and define the function g as follows:

$$g(t, x, u) = \frac{\partial}{\partial t} s(t, x) + \frac{\partial}{\partial x}(s(t, x))f(t, x, u) + \Gamma s(t, x) \tag{6}$$

where $\Gamma \in R^{m\times m}$ is a matrix to be selected.

Now partition $H = (H_1\ \ H_2)$, where $H_1 \in R^{m\times(n-m)}$ and $H_2 \in R^{m\times m}$, and denote by $\mathrm{Re}\lambda_{\min}(A)$ the minimum of the real parts of all the eigenvalues of the matrix A. Then we can state the following.

Theorem 2: Let δ, β, γ, and η be given positive numbers. Assume that:

1) $\mathrm{Re}\ \lambda_{\min}(H_2B_2) \geq \beta$;
2) $\mathrm{Re}\ \lambda_{\min}(A_{12}H_2^{-1}H_1 - A_{11}) \geq \beta + \gamma$;
3) $\mathrm{Re}\ \lambda_{\min}(\Gamma) \geq \eta$;
4) $\mathrm{Re}\ \lambda_{\min}(-C) \geq \beta$.

Moreover, assume that the following matching condition on the nominal trajectory is satisfied:

$$\dot{x}_{d1} = A_{11}x_{d1} + A_{12}x_{d2}.$$

Then there exists $\epsilon_1 > 0$ with the following property. Let $\epsilon \in (0, \epsilon_1]$, $u_0 \in R^m$, and let $(x(t, \epsilon), u(t, \epsilon))$ be the solution to (1) and (2) [where f, g are as defined in (5) and (6)] which satisfies $(x(0, \epsilon), u(0, \epsilon)) = (x_0, u_0)$. Then

$$|x_d(t) - x(t, \epsilon)| \leq \delta + ae^{-\beta t} \tag{7}$$

$$u(t, \epsilon) = \frac{1}{\epsilon} H\left(x_d(t) - x(t, \epsilon) - e^{Ct}(x_{d0} - x_0)\right.$$
$$\left. + \Gamma \int_0^t \left(x_d(\tau) - x(\tau, \epsilon) - e^{C\tau}(x_{d0} - x_0)\right) d\tau\right) + u_0 \tag{8}$$

for all $t \in [0, \infty)$; here a is a positive constant depending on the data.

Proof: Assumption 1) guarantees that the algebraic equation

$$\begin{aligned} g(t, x, u) := (H_1 H_2)\Bigg[\dot{x}_d(t) - \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} x \\ - \begin{pmatrix} 0 \\ B_2 \end{pmatrix} u - Ce^{Ct}(x_{d0} - x_0) \\ + \Gamma\left(x_d(t) - x - e^{Ct}(x_{d0} - x_0)\right)\Bigg] = 0 \end{aligned} \tag{9}$$

has a unique solution $u^*(t, x)$ for any $(t, x) \in [0, \infty) \times R^n$ *(equivalent control).*

Furthermore, for any $(t, x) \in [0, \infty) \times R^n$ the equilibrium point $u^*(t, x)$ of (2) turns out to be globally exponentially stable, and Assumption 1) assures that the exponential stability is uniform in $[0, \infty) \times R^n$.

Now note that (9) on a trajectory $x = x(t)$, $t \in [0, \infty)$, is equivalent to $0 = (d/dt)s(t, x(t)) + \Gamma s(t, x(t))$.

Then the reduced-order system can be rewritten as

$$\dot{\overline{x}} = A\overline{x} + B\overline{u} \tag{10}$$

$$\dot{s} = -\Gamma s. \tag{11}$$

Moreover, since $s(0, x_0) = 0$, we have

$$s(t, \overline{x}(t)) = H\left(x_d(t) - \overline{x}(t) - e^{Ct}(x_{d0} - x_0)\right) = 0, \quad t \in [0, \infty) \tag{12}$$

and so, from (6), we obtain

$$\overline{u}(t) = (H_2B_2)^{-1}H\left(\dot{x}_d(t) - A\overline{x}(t) - Ce^{Ct}(x_{d0} - x_0)\right). \tag{13}$$

Since (11) is exponentially stable by 3), Assumption H6) is satisfied if we show that $\overline{x} = \overline{x}(t)$ is exponentially stable. In order to check this, we define the error variable $\overline{e}(t) = x_d(t) - \overline{x}(t)$ whose dynamic behavior is described by the differential equation $\dot{\overline{e}} = \dot{x}_d(t) - A(x_d(t) - \overline{e}) + Bu^*(t, x_d(t) - \overline{e})$. It is easy to show that $\overline{e} = 0$ is an exponentially stable equilibrium point of this equation. Indeed, partitioning $\overline{e}^T = (\overline{e}_1^T \overline{e}_2^T)$ in accordance with the state partition, (12) can be written as

$$\begin{aligned} \overline{e}_2 &= x_{d2}(t) - \overline{x}_2(t) \\ &= H_2^{-1}\left(H_1(x_{d1}(t) - \overline{x}_1(t)) + He^{Ct}(x_{d0} - x_0)\right). \end{aligned} \tag{14}$$

Then, by using (10), (13), (14), and the matching condition 5) we have

$$\dot{\overline{e}}_1 = (A_{11} - A_{12}H_2^{-1}H_1)\overline{e}_1 - A_{12}H_2^{-1}He^{Ct}(x_{d0} - x_0). \tag{15}$$

Then, by 2), all the assumptions of Theorem 1 are satisfied, hence

$$\begin{aligned} \lim_{\epsilon\to 0} x(t, \epsilon) &= \overline{x}(t), \quad \text{uniformly in } [0, \infty) \\ \lim_{\epsilon\to 0} u(t, \epsilon) &= \overline{u}(t), \quad \text{uniformly in } [t_1, \infty) \end{aligned} \tag{16}$$

whenever $t_1 > 0$. Thus, given $\delta > 0$ there exists $\epsilon_1 > 0$ such that for any $\epsilon \in (0, \epsilon_1]$ we have that $|x(t, \epsilon) - \overline{x}(t)| \leq \delta$, $\forall t \in [0, \infty)$.

Finally, by Assumptions 2) and 4) we get

$$\begin{aligned} |\overline{x}_1(t) - x_{d1}(t)| \\ \leq Le^{-\beta t}\left(|\overline{x}_1(0) - x_{d1}(0)| + \frac{\|A_{12}H_2^{-1}H_1\|\,|x_{d0} - x_0|}{\gamma}\right). \end{aligned} \tag{17}$$

Here we use the estimate $\|e^{Xt}\| \leq Le^{(-a+\gamma)t}$ where $\mathrm{Re}\lambda_{\min}(-X) \geq a$, $L = L(\gamma)$, and $\gamma > 0$ is sufficiently small [9, Proposition 3, p. 4].