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Existence and optimal control for periodic parabolic equations with nonlocal terms

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Motivated by models which have been proposed for some problems in mathematical biology and fisheries management and elsewhere, we consider a nonlinear periodic parabolic problem and an associated cost functional J. A key feature of our problem is the presence of a nonlocal term which—as we show by direct example—renders the standard monotonicity methods invalid. We therefore employ topological methods to deal both with existence of solutions and of minima of J over the control set. Some considerations are also presented on related systems and on the question of uniqueness.

1. Introduction

In this paper, we consider the nonlinear periodic parabolic problem

$$\begin{aligned} &u_t - \Delta u = f(x, t, \phi(u), u, m)u \quad \text{in } Q_T, \\ &u = 0 \ \text{ on } \partial\Omega \times [0, T], \qquad u|_{t=T} = u|_{t=0} \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

and associated cost functional $J(u, m)$ which is to be minimized over all pairs (u, m) with $0 \leqslant u$ solution of (1.1) corresponding to $m \in U \subset L^\infty(Q_T)$. Here Ω is a smooth bounded domain in R^n, $Q_T = \Omega \times (0, T)$, $m = m(x, t)$ represents a control function and $\phi(w)$ is a continuous bounded functional from $L^p(\Omega)$ to R for $p \geqslant 1$ with $\phi(w) \geqslant 0$ if $w \geqslant 0$, $\phi(0) = 0$, and $\phi(w) \geqslant C\|w\|_{L^1(\Omega)}$ for some $C > 0$ and $0 \leqslant w \in L^p(\Omega)$. The precise form of the cost functional J will be given in Section 5, as will the control set U. Finally, we assume that f is smooth in all its variables. We shall also impose the following positivity and growth properties on f.

(A) For any given $m(x, t) \in U$, there exists $M(x, t) \in L^\infty(Q_T)$ such that

$$f(x, t, \alpha, \beta, m(x, t)) \leqslant [M(x, t) - \eta\alpha]$$

for some positive constant η if α and β are nonnegative.

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(B) For any given $m(x,t) \in U$, we have

$$\frac{1}{T}\int_0^T f(x,t,\phi(u),u(x,t),m(x,t))\,\mathrm{d}t > \mu_1 \quad \text{for a.a. } x \in \Omega$$

if $0 \leqslant u(x,t) \in C^\alpha(Q_T)$ and $\|u\|_{L^\infty(Q_T)}$ is sufficiently small, where μ_1 denotes the least eigenvalue of $-\Delta$ in Ω with Dirichlet boundary conditions. This is the same as the least eigenvalue of the linear periodic parabolic problem given by the left hand side of (1.1).

Our considerations motivated by the model considered by Calsina, Perello, & Saldana (1994) and Calsina & Perello (1995) to describe biological evolution involving nonlocal effects. Indeed, in their case, $f = x - \int_0^1 u$. Unlike their results, we consider here periodic solutions and optimal control, with $\Omega \subset R^n$ rather than $\Omega \subset R^1$ and, furthermore, $m = m(x,t)$ instead of $m = m(x)$. We remark that the control aspects of some related nonlocal problems were recently considered by Lenhart Protopopescu (1994). We emphasize, however, that some of our interest in this problem comes from the observation that, unlike the situation in their problems and in some other apparently similar nonlocal problems (Pao 1992), and in many other local ones (e.g. Cantrell & Cosner 1989; Hess 1991), monotonicity methods fail here, as shown by the following example given here in detail for the reader's convenience.

THEOREM 1 Consider the problem

$$\begin{aligned} u_t - \Delta u &= \left[k(t) - \textstyle\int_\Omega u\right]u \quad \text{in } Q_T, \\ u &= 0 \quad \text{on } \partial\Omega \times [0,T], \end{aligned} \tag{1.2}$$

with $(1/T)\int_0^T k > \mu_1$, smooth of period T. Then (1.2) has at most one positive periodic solution u of period T. If $\Omega = (-L,L) \subset R^1$ and

$$\left(\frac{1}{T}\int_0^T k - \frac{\pi^2}{4L^2}\right)\frac{\pi}{2} > k_{\max},$$

then u does not lie in the order interval determined by the upper/lower solution pair $(k_{\max}/2L, 0)$.

Proof. Let φ_1 denote the eigenvector of $-\Delta$ for the least eigenvalue μ_1. Note that (1.2) implies

$$(u,\varphi_1)_t + \mu_1(u,\varphi_1) = k(t)(u,\varphi_1) - (u,1)(u,\varphi_1),$$

where $(\alpha,\beta) = \int_\Omega \alpha\beta$. Integrating with respect to time, we obtain

$$\exp\left[\int_0^T (\mu_1 - h(t))\right](u,\varphi_1)\,|_{t=T} = (u,\varphi_1)|_{t=0},$$

where $h(t) = k(t) - (u,1)$. It follows by periodicity that either $(u,\varphi_1)|_{t=0} = 0$, and thus $(u,\varphi_i)|_t = 0$ for all t, or

$$\exp\left[\int_0^T (\mu_1 - h(t))\right] = 1,$$

i.e.

$$\int_0^T \int_\Omega u = \int_0^T k - \mu_1 T. \tag{1.3}$$

Since $u, \varphi_1 > 0$, the former possibility cannot hold and (1.3) is valid.

Repeating the process with some other eigenfunction φ_i, where $i > 1$, we observe that (1.3) implies $(u, \varphi_i) = 0$, since μ_1 is simple. We conclude that $u = \theta_1\varphi_1$ for some $\theta_1 = \theta_1(t)$, and for convenience set $\int_\Omega \varphi_1 = 1$.

As we shall show in Theorem 3 below, (1.2) has a unique solution. Let us assume now that $\Omega = (-L, L) \subset R^1$, and we observe that φ_1, with $\int_{-L}^{L} \varphi_1 = 1$, can be explicitly calculated, implying $\varphi_1(0) = \pi/4L$. Moreover, $\mu_1 = \pi^2/4L^2$. On the other hand, estimate (1.3) gives

$$(\theta_1)_{\max} \geqslant \frac{1}{T}\int_0^T k - \mu_1$$

and thus, for some $t_0 \in [0, T]$,

$$u(t_0, 0) = \left(\frac{1}{T}\int_0^T k - \mu_1\right)\frac{\pi}{4L}.$$

Finally, observe that, if

$$\left(\frac{1}{T}\int_0^T k - \frac{\pi^2}{4L^2}\right)\frac{\pi}{2} > k_{\max},$$

then there are no solutions in the order interval. □

Note that these arguments also show that, if $\int_0^T k \leqslant \mu_1 T$, then problem (1.2) does not have a positive periodic solution. This shows that condition B cannot be removed. Furthermore, a routine scaling argument shows that the result of Theorem 1 also holds if $\int_\Omega u$ is replaced in (1.2) by $\varepsilon \int_\Omega u$, for any $\varepsilon > 0$. That is, upper and lower solution procedures fail regardless of how small the nonlocal perturbation happens to be.

The difficulty arises from the fact that, in these equations, $f(x, t, \alpha, \beta, \gamma)\beta + K\beta$ may not be monotone in β for any constant K. Consequently we approach the existence problem entirely by topological methods, reminiscent of those employed by Allegretto & Nistri (1994), Brezis & Turner (1977), and others for steady-state local problems. Observe that $M(x, t)$ in condition A is a fixed function, so that the well-known 'eigenvalue crossing' condition for existence is replaced by the presence of $\phi(u)$.

Our results on existence and uniqueness are extensions of those given by Hess (1991: §V.38), since we do not require (in Hess's notation) that $b > 0$, nor that q be increasing. Furthermore, unlike Hess, we also present a uniqueness result for some nonlocal problems. The paper is structured as follows: we first consider the question of the existence and uniqueness of positive periodic solutions for problem (1.1) and for related systems. We then formally introduce the cost functional $J(u, m)$ which is to be minimized over all pairs (u, m) which satisfy(1.1). The precise form of J is motivated by a similar functional given by Kolosov (1996) for an optimal-control problem concerning fisheries management.

Finally, by a *solution* to (1.1) we mean at least a weak solution in the usual sense, i.e. $u \in C([0, T], L^2(\Omega)) \cap L^2((0, T), H_0^{1,2}(\Omega))$, and we assume that quantities are extended by periodicity, so that u is a solution for all t.

Note added in revision

It has been brought to our attention by Prof. Norman Dancer (private communication) that condition B can be replaced by assuming that the principal eigenvalue of the linearization at 0 is positive. Similarly, for the system cases discussed below, one can assume that $(u, 0)$ or $(0, v)$ are both stable (or both unstable), plus a nondegeneracy condition. The stability condition can then be reduced to a condition on a single equation.

2. Existence

While, in condition A, we allow M and η to depend on the control m, for notational convenience we shall suppress in the rest of this section explicit reference in f to m, as well as to (x, t).

Set $u = G(f(\phi(v), v^+)v^+)$ with $v \in L^\infty(Q_T)$ iff u satisfies (1.1) with $f(\phi(u), u)u$ replaced by $f(\phi(v), v^+)v^+$. We observe the following result

LEMMA 1 (a) Let $v \in L^\infty(Q_T)$. Then $u = G(f(\phi(v), v^+)v^+)$ is a compact continuous map from $L^\infty(Q_T)$ to $L^\infty(Q_T)$. Furthermore u and ∇u belong to $C^\alpha(\bar{Q}_T)$ for some $\alpha > 0$, where ∇u denotes the gradient of u with respect to x.

(b) If $u \in L^\infty(Q_T)$ satisfies $u = G(f(\phi(u), u^+)u^+)$, then u nontrivial implies $u > 0$.

(c) Let $0 \leqslant \omega \in C^1(\Omega) \cap H_0^{1,2}(\Omega)$ solve weakly the steady-state problem $-\Delta\omega = g$ for some $g \in L^2(\Omega)$. Then $\int_\Omega g \geqslant 0$.

(d) Let u be a positive solution of (1). Then

$$\operatorname*{ess\,inf}_x \frac{1}{T}\int_0^T f(x, t, \phi(u), u, m) \leqslant \mu_1.$$

Proof. (a) By assumption, $f(\phi(v), v^+)v^+ \in L^\infty(Q_T)$ for any $v \in L^\infty(Q_T)$, and the periodicity condition gives

$$\|u(\cdot, 0)\|_{L^2(\Omega)} \leqslant K\|f(\phi(v), v^+)v^+\|_{L^\infty(Q_T)}.$$

We then immediately conclude that $\|u\|_{L^2(Q_T)}$ satisfies a similar bound, and we may apply classic local estimates (Ladyzenskaya, Solonnikov, & Uraltseva 1968) to bound u in $L^\infty[\Omega \times [T/2, 3T/2]]$, and thus for all t by periodicity, in terms of $\|f(\phi(v), v^+)v^+\|_{L^\infty(Q_T)}$. Classical arguments in the same reference also show that u and ∇u belong to $C^\alpha(\bar{Q}_T)$ for some $\alpha > 0$. The compactness and continuity follow from the smoothness of f and the compact embedding $C^\alpha \hookrightarrow L^\infty$.

(b) We pass to the limit in the Steklov averages $(u_h)^-$ in the standard way (Ladyzenskaya *et al.* 1968: p.85) and conclude

$$\left(-\frac{1}{2}\int_\Omega (u^-)^2\right)\Big|_0^T - \int_\Omega\int_0^T |\nabla u^-|^2 = \int_\Omega\int_0^T [f(\phi(u), u^+)]u^-u^+ = 0$$

by the smoothness of f. Periodicity then implies $\int_\Omega \int_0^T |\nabla u^-|^2 = 0$ and thus $u = u^+ \geqslant 0$. Finally, since $u \in C^\alpha(\bar{Q}_T)$ by Part (a), and since u is periodic and nontrivial, we may assume $u(x,\xi) \not\equiv 0$ for some $\xi \leqslant 0$. Let $0 \leqslant \psi(x) \in C_0^\infty(\Omega)$ be nontrivial with $\psi(x) < u(x,\xi)$, and, for a constant $M > 0$, let v solve the problem

$$v_t - \Delta v + Mv = 0 \quad \text{for } t > \xi \text{ and } x \in \Omega,$$

$$v(x,\xi) = \psi(x) \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega \times [0,T].$$

Since $f(\phi(u),u) \in L^\infty(\bar{Q}_T)$, we can choose the constant M large enough and conclude that $u \geqslant v > 0$ by the strong maximum principle (Smoller 1983) for $t > \xi$, and thus $u > 0$ for all t.

(c) We choose a test function φ_ε given by

$$\varphi_\varepsilon(x) = \begin{cases} \omega(x)/\varepsilon & \text{if } \omega(x) \leqslant \varepsilon, \\ 1 & \text{if } \omega(x) > \varepsilon, \end{cases}$$

and note that $\int_\Omega g\varphi_e \to \int_{\Omega\cap\{\omega>0\}} g$ as $\varepsilon \to 0$, while

$$\int_\Omega g\varphi_\varepsilon = \frac{1}{\varepsilon}\int_{\{\omega\leqslant\varepsilon\}} |\nabla\omega|^2 \geqslant 0.$$

Observe that, on $S \triangleq \{\omega = 0\}$, we have $g = 0$ a.e. since, by standard elliptic regularity theory, $g \in L^2(\Omega)$ implies $\omega \in H^2(\Omega)$ with $-\Delta\omega = g$ a.e. But, on S, $\omega = \nabla\omega = \Delta\omega = 0$ and thus $\int_\Omega g = \int_{\Omega\cap\{\omega>0\}} g \geqslant 0$.

(d) Let $\varphi \in C_0^\infty(\Omega)$, and note that a direct calculation using Picone's identity (Allegretto 1971) gives

$$\int_0^T \int_\Omega u^2 \left|\nabla\left(\frac{\varphi}{u}\right)\right|^2 = \int_0^T \int_\Omega |\nabla\varphi|^2 - \int_0^T \int_\Omega \varphi^2 |f(x,t,\phi(u),u,m)] + \int_0^T \int_\Omega \varphi^2 \frac{u_t}{u}.$$

We observe that the last integral on the right-hand side vanishes by the periodicity of u, and then let $\varphi \to \varphi_1$ in $H_0^1(\Omega)$ to obtain the result. □

LEMMA 2 Let $\lambda \in [0,1]$. Then there exists a number R, independent of λ, such that no solutions $u \in L^\infty(Q_T)$ of

$$u = \lambda G(f(\phi(u),u^+)u^+) \tag{2.1}$$

satisfy $\|u\|_{L^\infty(Q_T)} = R$.

Proof. If u satisfies (2.1), for some $\lambda \in [0,1]$, then $u \geqslant 0$, and u and ∇u belong to $C^\alpha(\bar{Q}_T)$ by Lemma 1. We claim that $\|u\|_{L^1(Q_T)}$ is bounded uniformly in λ. Indeed, choose $\varphi \in C_0^\infty(\Omega)$ and integrating both sides of (2.1) with f replaced by λf, in weak form, over Q_T gives

$$\int_\Omega \varphi u\Big|_0^T + \int_\Omega \nabla\varphi\nabla\left[\int_0^T u\right] = \lambda\int_\Omega\left[\int_0^T fu\right]\varphi.$$

The first term on the left-hand side vanishes by periodicity, and we apply Lemma 1(c) to

$\omega = \int_0^T u$ and conclude that $\int_{Q_T} fu \geqslant 0$. Consequently, by Assumption A and the fact that $\phi(w) \geqslant C\|w\|_{L^1(\Omega)}$ for some $C > 0$ with $0 \leqslant w \in L^p(\Omega)$, we have

$$\int_0^T \left(\int_\Omega u\right)^2 \leqslant C_1 \int_0^T \left[\int_\Omega u\right]$$

for some constant C_1. Cauchy's inequality then yields $\|u\|_{L^1(Q_T)} \leqslant C_2$. We next observe that the positivity of u and condition A imply that u satisfies the linear inequality

$$u_t - \Delta u \leqslant Mu. \tag{2.2}$$

We employ (2.2), periodicity, and a result of Di Benedetto (1993: Theorem 5.2, p.128) to conclude that

$$u(x,t) \leqslant C_3 \left[\int_0^{3T/2} \int_\Omega u\right]$$

for $t \in [T/2, 3T/2]$, with C_3 independent of u, and therefore $\|u\|_{L^\infty(Q_T)} \leqslant C_r$. ☐

LEMMA 3 Let $\theta = \theta(x) > 0$ be a fixed smooth function in Ω, and $\delta \geqslant 0$. Then all nontrivial solutions $u \in L^\infty(Q_T)$ of

$$u = G(f(\phi(u), u^+)u^+) + \delta G(\theta)$$

must satisfy $\|u\|_{L^\infty(Q_T)} \geqslant r > 0$ for some r independent of u and δ.

Proof. As before, any such nonnegative solution u is in C^α. If $\|u\|_{L^\infty(Q_T)}$ is small, then so is $\phi(u)$, and condition B and Lemma 1(d) are contradictory, and thus $\|u\|_{L^\infty(Q_T)} > r$ for some r independent of u and δ. ☐

Let deg denote the Leray–Schauder degree (Zeidler 1986; Smoller 1983).

LEMMA 4 There exist r and R such that
(a) $\deg(u - G(f(\phi(u), u^+)u^+), B_R, 0) = 1$,
(b) $\deg(u - G(f(\phi(u), u^+)u^+), B_r, 0) = 0$,
where B_ξ denotes the ball centered at the origin of radius ξ in $L^\infty(Q_T)$.

Proof. (a) By Lemma 2, the degree is well defined on B_R and furthermore $1 = \deg(u, B_R, 0) = \deg(u - G(f(\phi(u), u^+)u^+), B_R, 0)$.

(b) By Lemma 3, the degree is again well defined on B_r and the same for all $\delta > 0$. If it were different from zero, there would be a solution of the problem

$$u_t - \Delta u = f(\phi(u), u^+)u^+ + \delta\theta \quad \text{in } Q_T,$$

$$u = 0 \quad \text{on } \partial\Omega \times [0, T], \qquad u(\cdot, T) = u(\cdot, 0) \quad \text{in } \Omega,$$

with $u(\cdot, 0) \geqslant 0$ and $\|u\|_{L^\infty(Q_T)} < r$. In such a case, $f(\phi(u), u)u^+$ is bounded in $L^\infty(Q_T)$, and so is the solution $\tilde{u}$ of

$$\tilde{u}_t - \Delta\tilde{u} = f(\phi(u), u)u \quad \text{in } Q_T,$$

$$\tilde{u} = 0 \quad \text{on } \partial\Omega \times [0, T], \qquad \tilde{u}(\cdot, 0) = 0 \quad \text{in } \Omega.$$

On the other hand, if θ is again chosen to be φ_1, the positive eigenvector of $-\Delta$, then $v = t\varphi_1$ satisfies

$$v_t - \Delta v \leqslant [1 + T\mu_1]\varphi_1 \quad \text{in } Q_T,$$

$$v = 0 \quad \text{on } \partial\Omega \times [0, T], \quad v(\cdot, 0) = 0 \quad \text{in } \Omega.$$

Hence setting $\omega = u - \tilde{u} - \alpha v$ with $\alpha > 0$ gives

$$\omega_t - \Delta\omega \geqslant 0$$

if $\delta > (1 + T\mu_1)\alpha$, with $\omega(\cdot, 0) \geqslant 0$. We conclude that $u \geqslant \tilde{u} + \alpha v$, contradicting the assumption that there is a small-norm solution for all δ. ☐

THEOREM 2 If conditions A and B hold, problem (1.1) has a positive solution $u \in C^\alpha[\bar{Q}_T]$.

Proof. The proof of the theorem is immediate from the properties of the Leray–Schauder degree (Smoller 1983): Ch. 12), Lemmas 3 and 4, and Lemma 1(b). ☐

3. Systems

For some systems, the previous results hold with only minor changes in the assumptions if one is interested in the existence of positive solutions. To be definite and for presentational simplicity (the same proofs hold for somewhat more general situations) consider the system (Calsina *et al.* 1994; Calsina & Perello 1995)

$$\begin{aligned} \ell u \equiv u_t - \Delta u &= \left[m - \alpha \int_\Omega u - \beta \int_\Omega v\right] u^+ \equiv f_1(x, t, \phi(u), \phi(v), u), \\ \ell v \equiv v_t - \Delta v &= \left[M - \gamma \int_\Omega u - \delta \int_\Omega v\right] v^+ \equiv f_2(x, t, \phi(u), \phi(v), v), \end{aligned} \tag{3.1}$$

to which we seek positive periodic solutions, subject to the same boundary conditions as before, and with $m, M, \alpha, \beta, \gamma$, and δ smooth positive functions of (x, t). The earlier procedures show immediately the existence of solutions $(u, 0)$ and $(v, 0)$. If we seek the existence of co-existence solutions (u, v) with $u, v > 0$, then we proceed as follows: Put $G(\tau) = \ell^{-1}(\tau)$ subject to periodic and Dirichlet conditions. The properties of the degree and the earlier arguments show that

$$\deg\{(u, v) - (G(f_1(x, t, \phi(u), 0, u), G(f_2(x, t, 0, \phi(v), v)), A_{[\varepsilon', R]} \times A_{[\varepsilon', R]}, 0\} = 1, \tag{3.2}$$

where $A_{[\varepsilon', R]} = \{u | \varepsilon' < \|u\|_\infty < R\} \subset L^\infty(Q_T)$ for some $\varepsilon', R > 0$. To show existence, we need only show that there are no solutions (u, v) on $\partial[A_{[\varepsilon', R]} \times A_{[\varepsilon', R]}]$ for $f_1(x, t, \phi(u), 0, u)$ (resp. $f_2(x, t, 0, \phi(v), v)$) replaced by $f_1(x, t, \phi(u), \lambda\phi(v), u)$ (resp. $f_2(x, t, \lambda\phi(u), \phi(v), v)$) in (3.2) with $0 \leqslant \lambda \leqslant 1$. If $\|u\|_{L^\infty} = R$ or $\|v\|_{L^\infty} = R$ then no such solutions are possible by the earlier arguments. If $\|u\|_{L^\infty} = \varepsilon'$, then observing that (u, v) are actually classical solutions, we apply the maximum principle to the second equation and obtain $(\inf\delta)\|v\|_{L^1(\Omega)}(t_0) \leqslant \|M\|_{L^\infty(Q_T)}$ for some $t_0 \in [0, T]$. Furthermore, integrating the second equation in (3.1) and employing the positivity of v gives, for $t \in (t_0, t_0 + T)$,

$$(\|v\|_{L^1(\Omega)}(t))' \leqslant (\|M\|_{L^\infty})\|v\|_{L^1(\Omega)}(t). \tag{3.3}$$

Here we have employed the fact that $\int_\Omega(-\Delta v) = \int_{\partial\Omega}(-\partial v/\partial n) \geqslant 0$. Integrating (3.3) gives $\|v\|_{L^1(\Omega)}(t) \leqslant \mathrm{e}^{\|M\|_{L^\infty}T}\|v\|_{L^1(\Omega)}(t_0)$ for $t \in [0, T]$, and thus $\max_t[\|v\|_{L^1(\Omega)}(t)] \leqslant \mathrm{e}^{\|M\|_{L^\infty}T}\|M\|_{L^\infty}/\inf\delta$. If we substitute this estimate into the first equation and assume that $m - (\beta/\inf\delta)\mathrm{e}^{\|M\|_{L^\infty}T}\|M\|_{L^\infty} > \mu_1$ then—again by our earlier arguments—there is no solution (u, v) with $\|u\|_{L^\infty} = \varepsilon'$ for ε' sufficiently small.

There is an identical estimate for $v : M - (\gamma/\inf\delta)\mathrm{e}^{\|m\|_{L^\infty}T}\|m\|_{L^\infty} > \mu_1$, and if both hold, then there is a solution (u, v) in $A_{[\varepsilon',R]} \times A_{[\varepsilon',R]}$. We observe that, as a consequence of the special nature of (3.1), these estimates give explicit conditions on the coefficients. We remark that our conditions may be viewed as asking that the 'cross terms' β and γ be sufficiently small for a given M, m, α, and δ. An analogous result holds if α and δ are 'small' relative to M, m, β, and γ. To see this, consider first the system

$$\ell u = \left[m - \beta\int_\Omega v\right]u, \qquad \ell v = \left[M - \gamma\int_\Omega u\right]v.$$

We apply the earlier approach and conclude, in the same way, the existence of a nontrivial solution pair $0 \leqslant u, 0 \leqslant v$. That indeed both u and v must actually be positive is immediate from the assumption that $m > \mu_1$ and $M > \mu_1$ and the form of the equations. A routine homotopy argument then shows the existence of a positive solution (u, v) for system (3.1) if α and δ are small enough, although it appears difficult to obtain explicit estimates in this case. Some restrictions on these terms are unavoidable, as the following elementary example indicates: Suppose that $M, m, \alpha, \delta, \beta$, and γ are all positive constants. Let (u, v) represent a positive solution; then, if we let φ_1 again denote the positive eigenfunction for $-\Delta$, we obtain

$$\begin{aligned}(u, \varphi_1)_t + \mu_1(u, \varphi_1) &= \left[m - \alpha\int_\Omega u - \beta\int_\Omega v\right](u, \varphi_1),\\ (v, \varphi_1)_t + \mu_1(v, \varphi_1) &= \left[M - \gamma\int_\Omega u - \delta\int_\Omega v\right](v, \varphi_1).\end{aligned} \tag{3.4}$$

We divide the equations in (3.4) by (u, φ_1) and (v, φ_1) respectively and integrate with respect to T to get, by periodicity,

$$\begin{aligned}(m - \mu_1)T &= \alpha\int_{Q_T} u + \beta\int_{Q_T} v,\\ (M - \mu_1)T &= \gamma\int_{Q_T} u + \delta\int_{Q_T} v.\end{aligned} \tag{3.5}$$

Recalling that $u, v > 0$, we note that equations (3.5) show that some restrictions—more general than the sufficient conditions we obtained earlier—must be imposed on α, β, γ, and δ.

Indeed, explicitly solving (3.5) for $\int_{Q_T} u$ and $\int_{Q_T} v$ shows that at least one of these will be nonpositive if $\alpha\delta - \beta\gamma \neq 0$ and

$$[\delta(m-\mu_1) - \beta(M-\mu_1)][\alpha(M-\mu_1) - \gamma(m-\mu_1)] \leqslant 0.$$

This implies a necessary condition for the existence of positive periodic solutions which is in qualitative agreement with our sufficient criteria for existence. In particular, if one of β and γ is near zero and the other is large, then no positive solution pair can exist.

We conclude this section with some brief considerations on other systems. Suppose first that f_1 and f_2 are replaced by

$$\tilde{f_1} = \left[m - \alpha u - \beta \int_\Omega v - \alpha' \int_\Omega u - \beta' v\right] u^+,$$
$$\tilde{f_2} = \left[M - \gamma \int_\Omega u - \delta v - \gamma' u - \delta' \int_\Omega v\right] v^+, \tag{3.6}$$

with $\delta, \alpha > 0$, giving a 'competitive' system. In this case, an analogous explicit result holds, since the maximum principle gives $\|v\|_{L^\infty} \leqslant \|M\|_{L^\infty}/\inf \delta$. This can be directly employed in the first equation to obtain an estimate of β and β'. The estimates of γ and γ' are established identically. On the other hand, if $\delta = \alpha = 0$ but $\delta', \alpha' > 0$, then this procedure is in principle applicable, but we were unable to obtain explicit conditions on β, β', γ, and γ' due to the difficulty of calculating the embedding constant C in the estimate: $\|v\|_{L^\infty(Q_T)} \leqslant C\|v\|_{L^1(Q_T)}$, which would then be needed to determine the conditions on β and β'.

Similar remarks apply to cooperative systems. Consider, as an explicit example, the case where

$$f_1 = \left(m - \alpha \int_\Omega u + \beta \int_\Omega v\right) u, \qquad f_2 = \left(M + \gamma \int_\Omega u - \delta \int_\Omega v\right) v. \tag{3.7}$$

Integrating over Q_T the equations (3.7) after multiplying by u and v respectively gives the energy estimates

$$\frac{1}{2}\left(\log \|u\|^2_{L^2(\Omega)}\right)_t \leqslant \bar{m} - \mu_1 - \underline{\alpha} \int_\Omega u + \bar{\beta} \int_\Omega v,$$
$$\frac{1}{2}\left(\log \|v\|^2_{L^2(\Omega)}\right)_t \leqslant \bar{M} - \mu_1 + \bar{\gamma} \int_\Omega u - \underline{\delta} \int_\Omega v,$$

where $\bar{m} \geqslant m \geqslant \underline{m}$ and the same notation is employed for the other coefficients. If we next integrate over $[0, T]$, we obtain an immediate explicit estimate for $\|v\|_{L^1(Q_T)}$ and $\|u\|_{L^1(Q_T)}$ if $\bar{\beta}$ and $\bar{\gamma}$ are sufficiently small. Integrating equations (3.7) directly over $[t_1, t_2]$ then gives an explicit estimate for $\max \|u\|_{L^1(\Omega)}$ and $\max \|v\|_{L^1(\Omega)}$ and thus for $\|u\|_{L^\infty(Q_T)}$ and $\|v\|_{L^\infty(Q_T)}$ as before. Note that repeating the arguments which led to equations (3.5) shows once again that requirements on β and γ cannot be completely removed.

4. Uniqueness

In this section we consider the question of uniqueness of positive periodic solutions. Observe that, as mentioned above, this topic is not discussed in §V.38 of Hess (1991).

THEOREM 3 Assume that f in (1.1) has the form

$$f(x,t,\phi(u),u,m) = M(x,t) - h(t,\phi(u))$$

for some smooth functions M and h. Suppose further that:

(i) $\phi(\alpha u) = \alpha\phi(u)$ for any constant $\alpha > 0$, $\phi(u)$ differentiable with respect to t;
(ii) if $z_1(t) < z_2(t)$ in $(0,T)$, then $\int_0^T h(t,z_1(t))\,dt \neq \int_0^T h(t,z_2(t))\,dt$.

Then (1.1) has at most one positive periodic solution.

Proof. It is convenient to introduce the functionals:

$$F_1(u,t) \triangleq \exp\int_0^t h(\xi,\phi(u)(\xi))d\xi$$

and

$$F_2(u,t) \triangleq \exp\left[-\frac{t}{T}\int_0^T h(\xi,\phi(u)(\xi))d\xi\right].$$

Motivated by some of the arguments of Calsina & Perello (1995), if u denotes a solution of (1.1), we set

$$w = u[F_1(u,t)\cdot F_2(u,t)]. \tag{4.1}$$

A direct calculation then shows that $w > 0$ satisfies the linear problem

$$\begin{aligned} &w_t - \Delta w = [M(x,t) + \ln F_2(u,1)]w, \\ &w = 0 \quad \text{on } \partial\Omega, \qquad w(T) = w(0). \end{aligned} \tag{4.2}$$

We conclude from the positivity of w that $w = \alpha z$ for some constant α, where z denotes the positive eigenvector corresponding to the least eigenvalue of the linear periodic problem (4.2) (Hess 1991). Note that $\ln F_2(u,1)$ is a constant, and thus by the uniqueness of the eigenvalue with a positive eigenvector, if v denotes any other positive solution of (1.1), it follows that

$$\gamma \triangleq \ln F_2(u,1) = \ln F_2(v,1) \tag{4.3}$$

and $r = \beta z$ for some constant β, where

$$r = vF_1(v,t)\cdot F_2(v,t).$$

Without loss of generality, we may assume $\beta \geqslant \alpha$. By condition (i) of the theorem statement, we have

$$\phi(u)F_1(u,t) = \alpha\phi(z)\exp(t\gamma) \triangleq \alpha f_1.$$

Differentiating both sides with respect to t gives

$$[\phi(u)]'F_1(u,t) + \phi(u)h(t,\phi(u)(t))F_1(u,t) = \alpha f_1'$$

and, dividing by $F_1(u,t)$,

$$[\phi(u)]' + \phi(u)\left[h(t,\phi(u)(t)) - \frac{f_1'}{f_1}\right] = 0. \tag{4.4}$$

Observe that $\phi(v)$ also satisfies (4.4). By the intermediate-value theorem and the assumed smoothness of h, we observe that $\theta(t) \triangleq \phi(u) - \phi(v)$ satisfies

$$\theta' + Z(t)\theta = 0$$

for some function $Z(t)$. Furthermore, $\theta(0) = \phi(u)(0) - \phi(v)(0) = (\alpha - \beta)\phi(z)(0)$. We conclude that

$$\phi(u) - \phi(v) = \mathrm{e}^{-\int_0^t Z(\xi)\mathrm{d}\xi}(\alpha - \beta)\phi(z)(0).$$

It follows that either $\alpha = \beta$ and consequently $\phi(u) \equiv \phi(v)$, or else $\alpha < \beta$ and $\phi(u) < \phi(v)$. In the first instance, $w = r$ and $F_1(v,t) = F_1(u,t)$. We conclude by (4.3) that $u \equiv v$. In the second case, by assumption (ii) of the theorem statement, we have $\ln F_2(u,1) \neq \ln F_2(v,1)$, but this is not possible since it contradicts (4.3). ☐

COROLLARY 1 Let $f = M(x,t) - \sum_{i=1}^{p} h_i(t)(\int_\Omega u)^{n_i}$ for smooth functions M and h_i and constants $n_i \geqslant 1$ $(i = 1, \ldots, p)$. Then (1.1) has precisely one solution.

5. Optimization problems

In this section we associate with problem (1.1) a cost functional $J(u,m)$ to minimize on all the pairs (u,m) which satisfy (1.1). For simplicity, we explicitly consider the case where $f(x,t,\phi(u),u,m) = m - \int_\Omega u$, which is the physically significant prototype.

The general form of the considered functional is

$$J(u,m) = \int_0^T \int_\Omega F(x,t,u(x,t),m(x,t))\,\mathrm{d}x\,\mathrm{d}t, \tag{5.1}$$

where $F : Q_T \times R^1 \times R^1 \to R^1$ is smooth in all its variables.

For a concrete example we note that if (1.1) represents the model of the evolution of a biological species involving nonlocal effects, $u(x,t)$ is the density of population at x at time $t \geqslant 0$, and $m(x,t)$ is the controlled rate of growth of the population without nonlocal effects, then the function F may have the following form:

$$F(x,t,u(x,t),m(x,t)) = c(x,t,m(x,t)) - g(x,t,u(x,t),m(x,t)), \tag{5.2}$$

where the function $c : Q_T \times R^1 \to R^1$ represents the cost of controlling the rate of growth to the value $m(x,t)$, and where the function $g : Q_T \times R^1 \times R^1 \to R^1$ represents the benefits due to the presence of a population of density u and rate of growth m. Clearly, the meaning of c and g can be also interchanged.

As mentioned in Section 1 a similar functional has been considered by Kolosov (1996) for an optimal-control problem concerning fisheries management. Kolosov represents the biological model of a fish population by an ordinary differential equation, since the spreading of the population over the region Ω is not considered there. In this situation, the control function is $m(t) = r - qv(t)$, and $c(t, m(t))$ is represented as $cv(t)$, where $c > 0$, r is the rate of natural growth of population, q is the catchability coefficient, and $v(t)$ is the rate of fish caught at time $t \geqslant 0$ which is the control parameter. Finally, $g(t, u(t), m(t)) = pqu(t)m(t)$, where p denotes the price of fish per unit of mass and q takes into account the environmental condition for the growth of the population.

Keeping in mind the physical significance of the control parameter m in the above applications, we consider here the following control set:

$$U = \left\{0 \leqslant m \in L^\infty(Q_T) : \mu_1 + \varepsilon \leqslant m(x,t) \leqslant r_2 \text{ for some } \varepsilon, r_2 > 0 \text{ and a.e. in } Q_T\right\}.$$

In Section 2 we proved in particular that, under assumptions A and B, for any $m \in U$ there is a positive solution of (1.1). Furthermore there exist $r, R > 0$ such that $r < \|u\|_{L^\infty(Q_T)} < R$ for any positive solution u of (1.1) corresponding to the controls $m \in U$. We will show in this section that $J(u, m)$ attains a minimum on the solution set

$$S = \{(u, m) : u \text{ is a positive solution of (1.1) corresponding to } m \in U\}.$$

First we consider some preliminary definitions and results. For any fixed $m \in U$, let $\psi_m : Q_T \times R^1 \times R^1 \to R^2$ be the map given by

$$\psi_m(x,t,\alpha,\beta) = \{(f(x,t,\alpha,\beta,m(x,t))\beta, F(x,t,\beta,m(x,t))\}$$

for a.a. $(x,t) \in Q_T$ and any $\alpha, \beta \in R^1 \times R^1$.

Consider the multivalued map $\psi : Q_T \times R^1 \times R^1 \multimap R^1 \times R^1$ defined as follows:

$$\psi(x,t,\alpha,\beta) = \{(f(x,t,\alpha,\beta,\gamma)\beta, F(x,t,\beta,\gamma)) : \gamma \in [\mu_1 + \varepsilon, r_2]\}.$$

Since F is smooth and $\gamma \in [\mu_1 + \varepsilon, r_2]$, it is immediate to see that ψ has compact convex values in $R^1 \times R^1$. Furthermore, it is upper-semicontinuous.

Define now the multivalued Nemytskii operator generated by ψ, namely $\Psi : L^\infty(Q_T) \multimap L^\infty(Q_T) \times L^\infty(Q_T)$, as follows:

$$\begin{aligned}\Psi(v) = \{&(a,b) \in L^\infty(Q_T) \times L^\infty(Q_T) :\\ &(a(x,t), b(x,t)) \in \psi(x,t,\phi(v), v(x,t)) \text{ for a.a. } (x,t) \in Q_T\}.\end{aligned}$$

We have the following result (Lasry & Robert 1990).

LEMMA 5 The multivalued map Ψ has closed convex values; it maps bounded sets into bounded sets; and its composition with a linear compact map is upper-semicontinuous. Moreover, $\Psi(v) = \{\Psi_m(v) : m \in U\}$ for any $v \in L^\infty(Q_T)$, where

$$\Psi_m(v)(x,t) = \psi_m(x,t,\phi(v), v(x,t)) \text{ for a.a. } (x,t) \in Q_T.$$

We are now in the position of proving the following.

THEOREM 3 Under conditions A and B, the cost functional J attains its minimum in S.

Proof. For any $m \in U$ and any $u \in L^\infty(Q_T)$, we set

$$(u, w) = \mathcal{G}(\psi_m(\phi(v), v^+))$$

if and only if u satisfies (1.1) with $f(x,t,\phi(u),u,m)u$ replaced by $f(x,t,\phi(v),v^+,m)v^+$ and w is given by

$$\dot{w}(t) = \int_\Omega F(x,t,v^+(x,t),m(x,t))\,dx \quad \text{a.e. in } Q_T,$$
$$w(0) = 0.$$

Let $\{u_n, m_n\}_{n\in\mathbb{N}} \subset S$ be a minimizing sequence. By Lemma 1(b) we have that

$$(u_n, w_n) = \mathcal{G}(\psi_{m_n}(\phi(u_n), u_n)) \quad \text{for any } n \in \mathbb{N},$$

or equivalently, by using the Nemytskii operator generated by ψ_{m_n},

$$(u_n, w_n) = \mathcal{G}(\Psi_{m_n}(u_n)).$$

Therefore $(u_n, w_n) \in \mathcal{G}(\Psi(u_n))$ for any $n \in \mathbb{N}$. By Lemmas 1(a) and 5, the multivalued map $u \multimap \mathcal{G}(\Psi(u))$ is compact and upper-semicontinuous with convex closed values. Thus, by the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $L^\infty(Q_T)$, passing to a subsequence if necessary, we have $u_n \to u_0$ and $w_n \to w_0$ in $L^\infty(Q_T)$. Furthermore, $(u_0, w_0) \in \mathcal{G}(\Psi(u_0))$ since $\mathcal{G} \circ \Psi$ has closed graph.

Finally, by the property $\Psi(u_0) = \bigcup_{m\in U} \Psi_m(u_0)$, there exists $m_0 \in U$ such that

$$(u_0, w_0) = \mathcal{G}(\Psi_{m_0}(u_0)),$$

or equivalently $(u_0, m_0) \in S$ and $w_0(T) = J(u_0, m_0)$ is the minimum of J on S. □

REMARK 1 If f and F are linear in the control m, then the multivalued map $S : U \multimap L^\infty(Q_T) \times L^\infty(Q_T)$ defined by

$$S(m) = \{(u, w) \in L^\infty(Q_T) \times L^\infty(Q_T) : (u, w) = \mathcal{G}(\Psi_m(u))\}$$

has a closed graph in the w-$L^2(Q_T) \times L^\infty(Q_T) \times L^\infty(Q_T)$ topology. Here w-$L^2(Q_T)$ denotes the weak topology in $L^2(Q_T)$. As a consequence, since a minimizing sequence $\{(u_n, m_n)\}_{n\in\mathbb{N}}$ converges to $\{(u_0, m_0)\}$, with $m_0 \in U$ in the $L^\infty(Q_T)\times$ w-$L^2(Q_T)$ topology, we have that $(u_0, w_0) \in S(m_0)$.

6. Finite-dimensional approximations

In this section we show how minimizing sequences for J can be obtained by finite-dimensional arguments. Consider a smooth Schauder basis $\{m_i\}_{i\in\mathbb{N}}$ for $L^2(Q_T)$. For any $p \in \mathbb{N}$ we consider the finite-dimensional control set

$$U_p = U \cap \operatorname{span}\{m_1, \ldots, m_p\}.$$

Define the solution set S_p relative to U_p as

$$S_p = \{(u, m) \in L^\infty(Q_T) \times U : m \in U_p \text{ and } (u, m) \in S\}.$$

Under our conditions, $S_p \neq \emptyset$ for any $p \in \mathbb{N}$, and we have

$$\overline{\bigcup_{p=1}^{\infty} S_p} \subseteq S, \tag{6.1}$$

where the closure is in the $L^\infty(Q_T) \times L^2(Q_T)$ topology. Furthermore,

$$U = \overline{\bigcup_{p=1}^{\infty} U_p}, \tag{6.2}$$

where the closure is in the $L^2(Q_T)$ topology.

We have the following result.

LEMMA 6 If, for any $m \in U$, problem (1.1) has a unique positive solution u, then

$$\overline{\bigcup_{p=1}^{\infty} S_p} = S.$$

Proof. By virtue of (6.1), it suffices to prove that $\overline{\bigcup_{p=1}^{\infty} S_p} \supseteq S$. For this, let $(u_0, m_0) \in S$, and observe that by (6.2) there exists a sequence of controls $\{m_i\}_{i\in\mathbb{N}}$ in $\bigcup_{p=1}^{\infty} U_p$ such that $m_i \to m_0$ in $L^2(Q_T)$. For any $i \in \mathbb{N}$, let u_i be the positive solution of (1.1) corresponding to the control m_i. On the other hand, the sequence $\{u_i\}_{i\in\mathbb{N}}$ is compact in $L^\infty(Q_T)$. Hence, passing to a subsequence if necessary, we have that $\{(u_i, m_i)\}_{i\in\mathbb{N}}$ in $\bigcup_{p=1}^{\infty} S_p$ converges in the $L^\infty(Q_T) \times L^2(Q_T)$ topology to (u_0, m_0). This concludes the proof. ☐

Since S_p is compact in $L^\infty(Q_T) \times L^2(Q_T)$ for any $p \in \mathbb{N}$, and since J is continuous in this space, there exists a pair $(u_p, m_p) \in S_p$ such that $J(u_p, m_p) = J_p$, where $J_p = \inf_{(u,m)\in S_p} J(u, m)$. We can state the following.

THEOREM 4 Under the assumptions of Lemma 6, the sequence $\{(u_p, m_p)\}_{p\in\mathbb{N}}$ is minimizing.

Proof. The sequence $\{J_p\}_{p\in\mathbb{N}}$ is nonincreasing; hence $\hat{J}_0 = \lim_{p\to\infty} J_p = \inf_{p\in\mathbb{N}} J_p$. We have to prove that $\hat{J}_0 = \min_{(u,m)\in S} J(u, m)$.

Assume that $\hat{J}_0 > J_0$ and let $\varepsilon = \hat{J}_0 - J_0 > 0$. By Lemma 6, there exists $p \in \mathbb{N}$ and a pair $(\hat{u}_p, \hat{m}_p) \in S_p$ such that

$$J_0 \leqslant J(\hat{u}_p, \hat{m}_p) < J_0 + \varepsilon. \tag{6.3}$$

On the other hand,

$$J(\hat{u}_p, \hat{m}_p) \geqslant J(u_p, m_p) \geqslant \hat{J}_0,$$

and so

$$J(\hat{u}_p, \hat{m}_p) - J_0 \geqslant \varepsilon,$$

contradicting (6.3). ☐

To conclude, we consider heuristically the minimization of a cost functional associated with a system. For a competitive system of two equations of the form (1.1) with the nonlinear terms given by (3.6), we consider a cost functional of the form

$$\begin{aligned} J(u, v, m, M, \Lambda) = {} & \int_0^T \int_\Omega [c_1(x, t, m(x, t)) - (1 - \Lambda) g_1(x, t, u(x, t), m(x, t))]\, dx\, dt \\ & + \int_0^T \int_\Omega [c_2(x, t, M(x, t)) - \Lambda g_2(x, t, v(x, t), M(x, t))]\, dx\, dt, \end{aligned}$$

where the functions m, M, c_i, g_i $(i = 1, 2)$ have the same meaning as the corresponding quantities at the beginning of this section. The parameter $\Lambda \in [0, 1]$ can be viewed as a control, for instance, of the benefit of sharing the fishing between two different species of density u and v respectively. Therefore the functional J can represent the payoff or the cost of fishing two species which share their limited sources in presence of competitive local effects.

Following the approach presented above, it is easy to show that there exists the minimum of $J(u, v, m, M, \Lambda)$ on $S \times [0, 1]$, where

$$S = \{(u, v, m, M) \in L^\infty(Q_T) \times L^\infty(Q_T) \times L^\infty(Q_T) \times L^\infty(Q_T) :$$

$$(u, v, m, M) \text{ satisfies the system}\}.$$

The case of cooperative systems (3.7) can be handled in the same way.

Finally we mention that it is also possible to consider the competitive system as a two-player game with conflicting objectives. In this case the optimal-control problem consists in showing the existence of a saddle point of a suitable objective functional (Lenhart & Protopopescu 1994).

7. Conclusions

In this paper, we have considered a nonlinear periodic parabolic equation and an associated cost functional J. We have obtained existence and uniqueness results for a positive solution u and conditions which ensure that J attains a minimum over (u, m) with m from a suitable control set U. A scheme under which the minimizing sequence can be constructed by means of finite-dimensional approximations was also presented. This depended critically on the existence of a unique solution to the equation. That this can be obtained under more general conditions than those given in Section 4 is conjectured, but remains an open question. Some results of the same nature were also presented for related systems.

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