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STABILITY AND INSTABILITY OF PERIODIC SOLUTIONS OF A DAMPED WAVE EQUATION IN A THIN DOMAIN

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1. Introduction

In the previous papers [5] and [6] the authors showed the existence of periodic solutions

$$w^\varepsilon = \begin{pmatrix} u^\varepsilon \\ v^\varepsilon \end{pmatrix}$$

with respect to the time t of a damped wave system in the non-autonomous and autonomous cases respectively.

The considered system in the non-autonomous case is of the form

$$(1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = v \\ \dfrac{\partial v}{\partial t} = \Delta_X u + \dfrac{\partial^2 u}{\partial Y^2} - \beta v - \alpha u + g(t, X, Y, u) \end{cases}$$

with Neumann boundary condition:

$$(2) \qquad \frac{\partial u}{\partial \nu_\varepsilon} = 0 \qquad \text{on } \partial Q_\varepsilon,$$

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where α and β are positive constants, g is an appropriate smooth function T-periodic with respect to time t, and (X, Y) is a generic point of the "thin domain" $Q_\varepsilon = \Omega \times (0, \varepsilon) \subset \mathbb{R}^{N+1}$.

The method employed in [5] and [6] consists in assuming that the "reduced" problem at $\varepsilon = 0$ in the domain Ω admits an isolated T_0-periodic solution

$$w^0 = \begin{pmatrix} u^0 \\ v^0 \end{pmatrix},$$

$T_0 = T$ in the non-autonomous case, and then in searching for conditions under which this solution extends to one for the problem (1)-(2) in Q_ε. The main tool is the topological degree for nonlinear compact operators.

It must be observed that in the autonomous case, i.e. when g is independent of t in (1), the assumed T_0-periodic solution of the reduced problem is not isolated. To overcome this difficulty we normalize the unknown period $T > 0$, in general different from T_0, of the sought-after periodic solution of (1)-(2) by introducing T as a parameter in system (1) by means of the substitution $t \to (T_0/T)t$.

Furthermore, in this case additional assumptions are required on the linearized reduced system. Under these assumptions it is possible to prove the existence of a continuous functional $T = T(w)$, $w = \begin{pmatrix} u \\ v \end{pmatrix}$, such that $T(w^0) = T_0$ and such that w^0 is an isolated fixed point with topological index different from zero of the $T(\,\cdot\,)$-parametrized reduced problem.

The aim of this paper is to derive the stability properties of the periodic solution w^ε defined in Q_ε, for small $\varepsilon > 0$, from those of the T_0-periodic solution w^0 in Ω.

Specifically in the non-autonomous case, we will prove that if the latter is stable or unstable then for $\varepsilon > 0$ sufficiently small the former is also stable or unstable. This result will be obtained by considering the first order approximation L^ε of the Poincaré map V^ε associated to (1)-(2) at w^ε. It is well known (see for instance [2] and [3]) that if all the $\lambda \in \sigma(L^\varepsilon)$, the spectrum of L^ε, satisfy the inequality $|\lambda| \leq q < 1$ then w^ε is stable. On the contrary, if there exists $\lambda \in \sigma(L^\varepsilon)$ such that $|\lambda| > 1$ then w^ε is unstable. In our case these situations can be treated by using the fact that L^ε is a condensing operator with constant $k < 1$ (see [7] and [8]) and the properties of its spectrum. In fact, it turns out (see [1]) that if $\lambda \in \sigma(L^\varepsilon)$ satisfies $|\lambda| > k + d$, whenever $d > 0$, then it is an eigenvalue of finite multiplicity.

The same will be done for the autonomous case to obtain orbital (in)stability of the T-periodic solution w^ε defined in Q_ε from that of the T_0-periodic solution w^0 defined in Ω. The additional problem in this case is that we must prove the simplicity of the eigenvalue 1 of the linearization of (1) around w^ε for $\varepsilon > 0$

sufficiently small, from that at $\varepsilon = 0$. Obviously, the assumptions for orbital (in)stability will concern all the other eigenvalues of the spectrum of the linearization.

The paper is organized as follows. In Section 2 we provide assumptions, definitions and preliminary results to be used in the sequel. In Section 3 we treat the non-autonomous case and in Section 4 the autonomous one.

2. Assumptions, definitions and preliminary results

We first consider the case when g depends on t and we assume the following conditions on $g : [0,T] \times \Omega \times [0,\varepsilon_0) \times \mathbb{R} \to \mathbb{R}$:

— g is of class C^1 jointly in the variables t, X, Y and u and it is T-periodic in $t : g(t+T, X, Y, u) \equiv g(t, X, Y, u)$. Moreover, g satisfies the following estimates:

$$|g_X(t,X,Y,u)| \leq a(1+|u|^{\theta+1}),$$
$$|g_Y(t,X,Y,u)| \leq a(1+|u|^{\theta+1}),$$
$$|g_u(t,X,Y,u)| \leq a(1+|u|^{\theta}),$$

for all values of its arguments t, X, Y, u. Here $a > 0$ is a suitable constant and $\theta \in [0,\infty)$ if $N = 1$, $\theta \in [0, 2/(N-1))$ if $N > 1$.

Observe that the growth rate θ is strictly less than the critical value $2/(N-1)$. This is because the validity of the Sobolev compact embedding result is crucial in our approach.

Following [4], for fixed $\varepsilon > 0$ we introduce new variables $X = x$, $Y = \varepsilon y$. System (1) becomes

$$\text{(3)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = v, \\ \dfrac{\partial v}{\partial t} = \Delta_x u + \dfrac{1}{\varepsilon^2}\dfrac{\partial^2 u}{\partial y^2} - \beta v - \alpha u + g(t, x, \varepsilon y, u) \end{cases}$$

and boundary condition (2) takes the form

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial Q,$$

where $Q = \Omega \times (0,1)$ and ν denotes the outward unit normal vector to Q. We suppose that Ω is a C^2-smooth domain.

For the reader convenience, we now give the most relevant definitions which permit to rewrite (1)-(2) as a fixed point problem in a suitable space. More details can be found in [5].

Following [5] (which in turn follows Hale-Raugel [4]), we introduce the following Banach spaces when $\varepsilon > 0$. Let X_ε^1 be the space $H^1(Q)$ with the norm

$$\left(\|u\|_{1Q}^2 + \frac{1}{\varepsilon^2}\left\|\frac{\partial u}{\partial y}\right\|_{0Q}^2\right)^{1/2}.$$

Here and below, $\|\cdot\|_{0Q}$ denotes the norm in $L^2(Q)$ and $\|\cdot\|_{1Q}$ that in $H^1(Q)$. Let $U_\varepsilon(t)$ be the semigroup generated by the system of linear equations

$$\begin{cases} \dfrac{\partial u}{\partial t} = v, \\ \dfrac{\partial v}{\partial t} = \Delta_x u + \dfrac{1}{\varepsilon^2}\dfrac{\partial^2 u}{\partial y^2} - \beta v - \alpha u, \end{cases}$$

with boundary condition (2). It is known (see [5]) that $U_\varepsilon(t)$ is a C_0-semigroup in the space

$$Y_\varepsilon^1 = X_\varepsilon^1 \times L^2(Q) \ni (u, v) = w.$$

One has the exponential estimate

$$\|U_\varepsilon(t)\|_{Y_\varepsilon^1 \to Y_\varepsilon^1} \le ce^{-\gamma t}, \qquad (t \ge 0),$$

where $c, \gamma > 0$. By introducing for $\varepsilon > 0$ the linear operator

$$A_\varepsilon = \begin{pmatrix} 0 & I \\ \Delta_x + \dfrac{1}{\varepsilon^2}\dfrac{\partial^2}{\partial y^2} - \alpha & -\beta \end{pmatrix}$$

with Neumann boundary condition, we can write

$$U_\varepsilon(t) = e^{A_\varepsilon t}, \qquad t \ge 0.$$

In the sequel by a solution of any differential equation we mean a solution of the corresponding integral equation obtained by the variation-of-constants formula.

Now let $C_T(Y_\varepsilon^1)$ be the space of all continuous, T-periodic functions $w = \begin{pmatrix} u \\ v \end{pmatrix}$ from $\mathbb{R}$ into Y_ε^1 with the usual norm

$$\|w\| = \sup_{t \in [0,T]} \|w(t)\|_{Y_\varepsilon^1}.$$

Define the following maps on $C_T(Y_\varepsilon^1)$:

$$f_\varepsilon(w)(t)(x, y) = \begin{pmatrix} 0 \\ g(t, x, \varepsilon y, u(t, x, y)) \end{pmatrix}$$

and

$$J_\varepsilon w(t) = U_\varepsilon(t)(I - U_\varepsilon(T))^{-1} \int_0^T U_\varepsilon(T - s)w(s)\, ds + \int_0^t U_\varepsilon(t - s)w(s)\, ds.$$

Then define

$$F_\varepsilon(w) = J_\varepsilon f_\varepsilon(w).$$

Using the Sobolev embedding theorem together with the theory of nonlinear Nemytskii operators, it is easy to show that F_ε maps $C_T(Y_\varepsilon^1)$ into itself and is completely continuous, i.e. it is continuous and it maps bounded sets into relatively compact sets. We give now the following

Definition 1. A fixed point of the completely continuous operator $F_\varepsilon : C_T(Y_\varepsilon^1) \to C_T(Y_\varepsilon^1)$ is a T-periodic solution of (1)-(2).

It is known that a fixed point of F_ε is always a T-periodic distributional solution of (1)-(2).

Next we pose the limit problem at $\varepsilon = 0$. Let $U_0(t)$ $(t \geq 0)$ be the semigroup generated by the linear system

$$\begin{cases} \dfrac{\partial u}{\partial t} = v, \\ \dfrac{\partial v}{\partial t} = \Delta_x u - \beta v - \alpha u, \end{cases}$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega.$$

Observe that $U_0(t) = e^{A_0 t}$, $t \geq 0$, where

$$A_0 = \begin{pmatrix} 0 & I \\ \Delta_x - \alpha & -\beta \end{pmatrix}$$

with Neumann boundary condition.

Let $w_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ be an element of $H^1(\Omega) \times L^2(\Omega)$. Then $U_0(t)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ is in $H^1(\Omega) \times L^2(\Omega)$ and one has the estimate

$$\|U_0(t)\|_{H^1(\Omega)\times L^2(\Omega)\to H^1(\Omega)\times L^2(\Omega)} \leq ce^{-\gamma t},$$

where $c, \gamma > 0$. Defining $i : \Omega \to Q$ by $i(x) = (x, 0)$, we obtain an inclusion $\mathcal{J} : H^1(\Omega) \times L^2(\Omega) \to Y_\varepsilon^1$ with $\mathcal{J}(u,v)(x,y) = (u(x), v(x))$. The map $\mathcal{J}$ is an isometry for all $0 < \varepsilon < \varepsilon_0$, and we identify $U_0(t)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ with the element $\mathcal{J}U_0(t)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ of Y_ε^1.

Define an operator F_0 on $C_T(H^1(\Omega) \times L^2(\Omega))$ as follows:

$$F_0(w) = J_0 f_0(w),$$

where J_0 has the same form as J_ε with $U_\varepsilon(t)$ replaced by $U_0(t)$ and

$$f_0(w)(t)(x) = \begin{pmatrix} 0 \\ g(t,x,0,u(t,x)) \end{pmatrix}.$$

Then $F_0 : C_T(H^1(\Omega) \times L^2(\Omega)) \to C_T(H^1(\Omega) \times L^2(\Omega))$ and it is completely continuous.

We identify the T-periodic solutions of the system

$$\begin{cases} \dfrac{\partial u}{\partial t} = v, \\ \dfrac{\partial v}{\partial t} = \Delta_x u - \beta v - \alpha u + g(t,x,0,u) \end{cases} \tag{4}$$

together with the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \tag{5}$$

with the fixed points of the operator F_0. The main result proved in [5] is the following existence result, here $\operatorname{ind}(\,\cdot\,,\,\cdot\,)$ indicates the topological index.

Theorem A. *If the problem* (4)-(5) *admits an isolated T-periodic solution* $w^0 = \begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \in C_T(H^1(\Omega) \times L^2(\Omega))$ *with* $\operatorname{ind}(F_0, w^0) \neq 0$, *then for sufficiently small $\varepsilon > 0$ the problem* (4)-(5) *admits a T-periodic solution* $w^\varepsilon = \begin{pmatrix} u^\varepsilon \\ v^\varepsilon \end{pmatrix} \in C_T(Y^1_\varepsilon)$ *and*

$$\|w^\varepsilon - \mathcal{J} w^0\|_{C_T(Y^1_\varepsilon)} \to 0 \qquad \text{as } \varepsilon \to 0.$$

The proof of Theorem A is mainly based on the following result, which we repeat here for the reader's convenience since it will be used in the next sections.

Lemma A. *Suppose that there exist $r > 0$, $\varepsilon_n \to 0$ and*

$$\begin{pmatrix} u^* \\ v^* \end{pmatrix} \in C_T(H^1(\Omega) \times L^2(\Omega))$$

such that the problem (1), (2) *admits T-periodic solutions*

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} \in C_T(Y^1_{\varepsilon_n})$$

with $\left\| \begin{pmatrix} u_n \\ v_n \end{pmatrix} - \mathcal{J}\begin{pmatrix} u^* \\ v^* \end{pmatrix} \right\|_{C_T(Y^1_{\varepsilon_n})} = r$. *Then there exist a T-periodic solution* $\begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix}$ *of* (4)-(5) *and a subsequence* $\left\{ \begin{pmatrix} u_{k_n} \\ v_{k_n} \end{pmatrix} \right\}$ *of* $\left\{ \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right\}$ *such that*

$$\left\| \begin{pmatrix} u_{k_n} \\ v_{k_n} \end{pmatrix} - \mathcal{J}\begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} \right\|_{C_T(Y^1_{\varepsilon_n})} \to 0 \quad \text{as } n \to \infty$$

with

$$\left\| \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} - \begin{pmatrix} u^* \\ v^* \end{pmatrix} \right\|_{C_T(H^1(\Omega)\times L^2(\Omega))} = r.$$

Finally, for the autonomous case

$$\tag{6} \begin{cases} \dfrac{\partial u}{\partial t} = v, \\ \dfrac{\partial v}{\partial t} = \Delta_x u + \dfrac{1}{\varepsilon^2}\dfrac{\partial^2 u}{\partial y^2} - \beta v - \alpha u + g(x, \varepsilon y, u) \end{cases}$$

together with Neumann boundary conditions

$$\tag{7} \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial Q,$$

the substitution $t \to (T_0/T)t$ produces the dependence on T of all the operators introduced before. Specifically, as it can be easily verified we have

$$F_\varepsilon(T, w) = J_\varepsilon(T) f_\varepsilon(T, w),$$

where

$$f_\varepsilon(T, w)(t)(x, y) = \begin{pmatrix} 0 \\ (T/T_0) g(x, \varepsilon y, u(t, x, y)) \end{pmatrix},$$

$$J_\varepsilon(T)w(t) = U_\varepsilon(T, t)[I - U_\varepsilon(T, T_0)]^{-1} \int_0^T U_\varepsilon(T, T_0 - s)w(s)\, ds + \int_0^t U_\varepsilon(T, t - s)w(s)\, ds,$$

and $U_\varepsilon(T, t)$ is the semigroup genarated by

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{T}{T_0} v, \\ \dfrac{\partial v}{\partial t} = \dfrac{T}{T_0}\left[\Delta_x u + \dfrac{1}{\varepsilon^2}\dfrac{\partial^2 u}{\partial y^2} - \beta v - \alpha u\right]. \end{cases}$$

Similar formulas hold for $\varepsilon = 0$. Obviously, a fixed point of $F_\varepsilon(T, \cdot) : C_{T_0}(Y_\varepsilon^1) \to C_{T_0}(Y_\varepsilon^1)$ for some $T > 0$ is a T-periodic solution of (6)-(7).

The following existence result was proved in [6].

Theorem B. *Suppose that the system*

$$\begin{cases} \dfrac{\partial u}{\partial t} = v, \\ \dfrac{\partial v}{\partial t} = \Delta_x u - \beta v - \alpha u - g(x, 0, u) \end{cases}$$

together with

$$\frac{\partial u}{\partial \nu} = 0 \qquad \textit{on } \partial\Omega$$

has a T_0*-periodic solution* $w^0 = \begin{pmatrix} u^0 \\ v^0 \end{pmatrix}$ *in the classical sense such that the linearized system*

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &= \psi, \\ \frac{\partial \psi}{\partial t} &= \Delta_x \varphi - \beta\psi - \alpha\varphi + g_u(x, 0, u^0(t,x))\varphi \end{aligned}$$

has no T_0*-periodic solutions which are linearly independent of* $(\partial w^0/\partial t)$*. Furthermore, we suppose that it does not possess any solution of the form* $\widetilde{w}(t,x) + (t/T_0)w^0(t,x)$*, where* $\widetilde{w} = \begin{pmatrix} \widetilde{\varphi} \\ \widetilde{\psi} \end{pmatrix}$ *is* T_0*-periodic with respect to* t*. Then for sufficiently small* $\varepsilon > 0$ *problem* (6)-(7) *admits a* T_ε*-periodic solution* $w^\varepsilon = \begin{pmatrix} u^\varepsilon \\ v^\varepsilon \end{pmatrix}$ *with* $T_\varepsilon \to T_0$ *and*

$$\|\widehat{w}^\varepsilon - \mathcal{J} w^0\|_{C_{T_0}(Y_\varepsilon^1)} \to 0 \qquad \text{as } \varepsilon \to 0,$$

where $\widehat{w}^\varepsilon(t) = w^\varepsilon((T_\varepsilon/T_0)t)$.

In the next two sections, for $\varepsilon > 0$ sufficiently small, we investigate the stability of the periodic solution w^ε of (1)-(2) (resp. (6)-(7)).

3. The non-autonomous case

For $\varepsilon > 0$ define the Poincaré map $V^\varepsilon : Y_\varepsilon^1 \to Y_\varepsilon^1$ associated to (1)-(2) as follows:

$$V^\varepsilon(p) = U_\varepsilon(T)p + \int_0^T U_\varepsilon(T-s) f_\varepsilon(w)(s)\, ds,$$

where $w(t) \in Y_\varepsilon^1$, $t \in [0,T]$, is a solution of (1)-(2) with $w(0) = p$ and T is the fixed period of the nonlinearity g.

As a direct consequence of Theorem A we have that for $\varepsilon > 0$ sufficiently small the Poincaré map has a fixed point $w^\varepsilon(0) \in Y_\varepsilon^1$ which represents the initial conditions of the T-periodic solution w^ε of (1)-(2).

Consider the linearization $L^\varepsilon : Y_\varepsilon^1 \to Y_\varepsilon^1$ of V^ε around w^ε:

$$L^\varepsilon q = U_\varepsilon(T)q + \int_0^T U_\varepsilon(T-s) f_\varepsilon'(w^\varepsilon(s))\psi(s)\, ds,$$

where $\psi(t) \in Y_\varepsilon^1$, $t \in [0,T]$, is the solution of the linearization of (1) around w^ε such that $\psi(0) = q$.

For $\varepsilon = 0$ we also define $V^0 : H^1(\Omega) \times L^2(\Omega) \to H^1(\Omega) \times L^2(\Omega)$ and the corresponding linearization L^0 around w^0. The linear map $L^\varepsilon : Y^1_\varepsilon \to Y^1_\varepsilon$ is k-condensing with respect to the measure of noncompactness of Kuratovskii generated by a suitable equivalent norm in the space Y^1_ε (see [8]). Moreover, in [7] this condensivity property was proved for some special measures of noncompactness defined by means of the Hausdorff measure of noncompactness.

It follows that (see [1]) for any $d > 0$ the points $\lambda \in \sigma(L^\varepsilon)$ for which $|\lambda| > k+d$ are eigenvalues of finite multiplicity.

In the sequel we will study the stability of w^ε by means of the properties of the spectrum of $L^0 : H^1(\Omega) \times L^1(\Omega) \to H^1(\Omega) \times L^2(\Omega)$ and the corresponding stability results [1]. In this section we assume the conditions of Theorem A.

We will first prove the following result.

Theorem 1. *Assume that for any $\lambda \in \sigma(L^0)$ we have that $|\lambda| < 1$. Then for sufficiently small $\varepsilon > 0$ the T-periodic solution w^ε of (1)-(2) is stable.*

Proof. Assume the contrary, then by [1] there exist two sequences $\varepsilon_n \to 0$ and $\lambda_n \in \sigma(L^{\varepsilon_n})$ such that $|\lambda_n| \geq 1$. For any $n \in \mathbb{N}$ let $q_n \in Y^1_{\varepsilon_n}$, $\|q_n\|_{Y^1_{\varepsilon_n}} = 1$ and $L^{\varepsilon_n} q_n = \lambda_n q_n$.

Put $e^{\mu_n T} = \lambda_n$, $n \in \mathbb{N}$, and let $\varphi_n(t) = e^{-\mu_n t}\psi_n(t)$, where $\psi_n(t)$ is the solution of the linearization of (1) around w^ε which we denote by

$$\dot{\psi} = A_{\varepsilon_n}\psi + B_{\varepsilon_n}(t)\psi$$

with $\psi_n(0) = q_n$. It turns out that φ_n is a T-periodic solution of the linear equation

$$\dot{\varphi} = (A_{\varepsilon_n} - \mu_n)\varphi + B_{\varepsilon_n}(t)\varphi. \tag{8}$$

In fact, we have $\varphi_n(T) = e^{-\mu_n T}\psi_n(T) = e^{-\mu_n T}\lambda_n q_n = q_n$. Thus $\varphi_n(T) = \varphi_n(0) = q_n$, moreover

$$\varphi_n(t) = e^{(A_{\varepsilon_n} - \mu_n)t} q_n + \int_0^t e^{(A_{\varepsilon_n} - \mu_n)(t-s)} B_{\varepsilon_n}(s)\varphi_n(s)\, ds,$$

and

$$\begin{aligned}\varphi_n(t+T) &= e^{(A_{\varepsilon_n}-\mu_n)(t+T)}q_n + \int_0^T e^{(A_{\varepsilon_n}-\mu_n)(t+T-s)}B_{\varepsilon_n}(s)\varphi_n(s)\,ds \\ &\quad + \int_T^{t+T} e^{(A_{\varepsilon_n}-\mu_n)(t+T-s)}B_{\varepsilon_n}(s)\varphi_n(s)\,ds \\ &= e^{(A_{\varepsilon_n}-\mu_n)t}\left[e^{(A_{\varepsilon_n}-\mu_n)T}q_n + \int_0^T e^{(A_{\varepsilon_n}-\mu_n)(T-s)}B_{\varepsilon_n}(s)\varphi_n(s)\,ds\right] \\ &\quad + \int_0^t e^{(A_{\varepsilon_n}-\mu_n)(t-\xi)}B_{\varepsilon_n}(\xi)\varphi_n(\xi+T)\,d\xi \\ &= e^{(A_{\varepsilon_n}-\mu_n)t}q_n + \int_0^t e^{(A_{\varepsilon_n}-\mu_n)(t-\xi)}B_{\varepsilon_n}(\xi)\varphi_n(\xi+T)\,d\xi.\end{aligned}$$

By the uniqueness of solutions of the Cauchy problem for the linearized equation we obtain that φ_n is a T-periodic solution of (8).

By Lemma A and Theorem A we get that $\varphi_n(t) \to \varphi_0(t)$ as $n \to \infty$, where $\varphi_0(t) \in H^1(\Omega) \times L^2(\Omega)$, $t \in [0,T]$, is a continuous T-periodic solution of

$$\dot{\varphi} = (A_0 - \mu_0)\varphi + B_0(t)\varphi,$$

the linearized system of (1) around w^0, with $\lambda_0 = \lim_{n\to\infty} \lambda_n$, $\|\varphi_0(0)\| = 1$ and $e^{-\mu_0 T} = \lambda_0$. This contradicts the fact that $|\lambda| < 1$ for any $\lambda \in \sigma(L^0)$.

We state now the instability result. We first give the following lemma.

Lemma 1. *Let* $q_0 \in H^1(\Omega) \times L^2(\Omega)$. *Then*

$$(\lambda I - L^\varepsilon)^{-1}\mathcal{J}q_0 \to \mathcal{J}(\lambda I - L^0)^{-1}q_0 \qquad \text{as } \varepsilon \to 0$$

uniformly with respect to $\lambda \in C$, *where* C *is a circle such that* $C \cap \sigma(L^0) = \emptyset$.

Proof. Note that as it is proved in Theorem 2 below, for sufficiently small $\varepsilon > 0$, $C \cap \sigma(L^\varepsilon) = \emptyset$ and so $(\lambda I - L^\varepsilon)^{-1}$ is well defined.

We argue by contradiction, then there exist $\delta_0 > 0$, $\varepsilon_n \to 0$ and $\lambda_n \to \lambda_0 \in C$, $\lambda_n \in C$, such that

$$\|(\lambda_n I - L^{\varepsilon_n})^{-1}\mathcal{J}q_0 - \mathcal{J}(\lambda_n I - L^0)^{-1}q_0\|_{Y^1_{\varepsilon_n}} \geq \delta_0. \tag{9}$$

Let $p_n = (\lambda_n I - L^{\varepsilon_n})^{-1}\mathcal{J}q_0$. We claim that the sequence $\{p_n\}$ is bounded, i.e. there exists $M > 0$ such that $\|p_n\|_{Y^1_{\varepsilon_n}} \leq M$ for any $n \in \mathbb{N}$.

In fact, assume $\|p_n\|_{Y^1_{\varepsilon_n}} \to \infty$ as $n \to \infty$ and recall that

$$\lambda_n p_n = L^{\varepsilon_n} p_n + \mathcal{J} q_0.$$

Let

$$\xi_n(t) = e^{A_{\varepsilon_n} t} p_n + \int\limits_0^t e^{A_{\varepsilon_n}(t-s)} B_{\varepsilon_n}(s) \xi_n(s)\, ds,$$

hence $L^{\varepsilon_n} p_n = \xi_n(T)$. Put $z_n(t) = (\xi_n(t)/\|p_n\|_{Y^1_{\varepsilon_n}})$, thus

$$z_n(t) = e^{A_{\varepsilon_n} t} z_n(0) + \int\limits_0^t e^{A_{\varepsilon_n}(t-s)} B_{\varepsilon_n}(s) z_n(s)\, ds$$

and

$$\lambda_n z_n(0) = z_n(T) + \frac{\mathcal{J} q_0}{\|p_n\|_{Y^1_{\varepsilon_n}}}.$$

Let $\lambda_n = e^{\mu_n T}$ and define

$$\eta_n(t) = e^{-\mu_n t} z_n(t).$$

Then

$$\eta_n(t) = e^{(A_{\varepsilon_n} - \mu_n)t} \eta_n(0) + \int\limits_0^t e^{(A_{\varepsilon_n} - \mu_n)(t-s)} B_{\varepsilon_n}(s) \eta_n(s)\, ds \tag{10}$$

where

$$\eta_n(0) = \eta_n(T) + e^{-\mu_n T} \frac{\mathcal{J} q_0}{\|p_n\|_{Y^1_{\varepsilon_n}}}, \qquad \|\eta_n(0)\|_{Y^1_{\varepsilon_n}} = 1.$$

Therefore, calculating $\eta_n(0)$ from the previous two relations and substituting in (10) we obtain

$$\begin{aligned}
\eta_n(t) = {} & e^{(A_{\varepsilon_n} - \mu_n)t} (I - e^{(A_{\varepsilon_n} - \mu_n)T})^{-1} \int\limits_0^T e^{(A_{\varepsilon_n} - \mu_n)(T-s)} B_{\varepsilon_n}(s) \eta_n(s)\, ds \\
& + e^{(A_{\varepsilon_n} - \mu_n)t} (I - e^{(A_{\varepsilon_n} - \mu_n)t})^{-1} e^{-\mu_n T} \frac{\mathcal{J} q_0}{\|p_n\|_{Y^1_{\varepsilon_n}}} \\
& + \int\limits_0^t e^{(A_{\varepsilon_n} - \mu_n)(t-s)} B_{\varepsilon_n}(s) \eta_n(s)\, ds.
\end{aligned}$$

Letting $n \to \infty$, since $\|p_n\|_{Y^1_{\varepsilon_n}} \to \infty$, Lemma A and Theorem A yield $\eta_n(t) \to \eta(t)$ where η is a T-periodic solution of

$$\dot{\varphi} = (A_0 - \mu_0)\varphi + B_0(t)\varphi$$

with $\mu_0 \in C$, which contradicts the fact that $C \cap \sigma(L^0) = \emptyset$.

Finally, by using the boundedness of $\{p_n\}$ and the same arguments as before we can prove that

$$\lambda_n p_n = L^{\varepsilon_n} p_n + \mathcal{J} q_0 \to \lambda_0 p_0 = L^0 p_0 + q_0, \qquad \lambda_0 \in C,$$

which is a contradiction with (9).

We are now in the position to proving the following result.

Theorem 2. *Assume that there exists $\lambda_0 \in \sigma(L^0)$ such that $|\lambda_0| > 1$, then for sufficiently small $\varepsilon > 0$ the T-periodic solution w^ε of (1)-(2) is unstable.*

Proof. Since λ_0 is an eigenvalue of finite multiplicity (see [1]) there is a closed disc D centered at λ_0 which does not contain points of $\sigma(L^0)$ different from λ_0 and $\operatorname{dist}(\partial D, 0) > 1$.

Furthermore, for $\varepsilon > 0$ sufficiently small there are no points of $\sigma(L^\varepsilon)$ lying on $C = \partial D$. In fact, assume there exist sequences $\varepsilon_n \to 0$ and $\lambda_n \in \sigma(L^{\varepsilon_n})$ with $\lambda_n \in C$. Then

$$\lambda_n q_n = L^{\varepsilon_n} q_n \qquad \text{for some } q_n \in Y^1_{\varepsilon_n}$$

and passing to the limit as $n \to \infty$ by Theorem A we obtain

$$\widehat{\lambda}_0 q_0 = L^0 q_0$$

with $\widehat{\lambda}_0 \in C$, $q_0 \in H^1(\Omega) \times L^2(\Omega)$ which is a contradiction.

Therefore for $\varepsilon > 0$ sufficiently small the Riesz's projector

$$P_\varepsilon q = -\frac{1}{2\pi i} \int\limits_C (\lambda I - L^\varepsilon)^{-1} q \, d\lambda$$

is well defined.

Let $q_0 \in H^1(\Omega) \times L^2(\Omega)$ such that $\lambda_0 q_0 = L^0 q_0$, $\|q_0\| = 1$. By Lemma A we get

$$P_\varepsilon q_0 \to P_0 q_0 \neq 0 \quad \text{as } \varepsilon \to 0,$$

and from this a contradiction if we assume the existence of a sequence $\varepsilon_n \to 0$ with the property that from $\lambda_n \in \sigma(L^{\varepsilon_n})$ it follows $\lambda_n \notin D$. Indeed in this case $P_{\varepsilon_n} \equiv 0$ for any $n \in \mathbb{N}$.

4. The autonomous case

Following the lines of the previous section we first define for $\varepsilon > 0$ and $T > 0$ the Poincaré map $V^\varepsilon_T : Y^1_\varepsilon \to Y^1_\varepsilon$ associated to (6)-(7) as follows

$$V^\varepsilon_T(p) = U_\varepsilon(T, T_0)p + \int\limits_0^{T_0} U_\varepsilon(T, T_0 - s) f_\varepsilon(T, w)(s) \, ds,$$

where $w(t) \in Y^1_\varepsilon$, $t \in [0, T_0]$, is a solution of (6)-(7) with $w(0) = p$ and the operators are those defined in Section 2.

We then consider the linearization $L^\varepsilon_T : Y^1_\varepsilon \to Y^1_\varepsilon$ of V^ε_T around $\widehat{w}^\varepsilon(t) = w^\varepsilon((T_\varepsilon/T_0)t)$, where w^ε is a T_ε-periodic solution of (6)-(7), whose existence for sufficiently small $\varepsilon > 0$ is guaranteed by Theorem B, namely

$$L^\varepsilon_T q = U_\varepsilon(T, T_0)q + \int\limits_0^{T_0} U_\varepsilon(T, T_0 - s) f'_\varepsilon(T, \widehat{w}^\varepsilon(s))\psi(s)\, ds,$$

where $\psi(t) \in Y^1_\varepsilon$, $t \in [0, T_0]$, is the solution of the linearization of (6) around $\widehat{w}^\varepsilon$, such that $\psi(0) = q$.

For $\varepsilon = 0$ we also define $V^0_T : H^1(\Omega) \times L^2(\Omega) \to H^1(\Omega) \times L^2(\Omega)$ and the corresponding linearization L^0_T around w^0. Consider the linear operator L^ε_T for T close to T_0. One can show that it has the condensivity properties of the operator L^ε of Section 3. Also its spectrum has the properties indicated for the spectrum of L^ε.

In the sequel we assume the conditions of Theorem B. Therefore, in particular, we assume the eigenvalue $1 \in \sigma(L^0_{T_0})$ is simple. In order to investigate orbital (in)stability of w^ε we need the following

Lemma 2. *If $\varepsilon > 0$ is sufficiently small the eigenvalue $1 \in \sigma(L^\varepsilon_{T_\varepsilon})$ is simple.*

Proof. We argue by contradiction, thus we assume that there exist sequences $\varepsilon_n \to 0$ and $q_n, q'_n \in Y^1_{\varepsilon_n}$ such that q_n, q'_n are linearly independent eigenvectors of $L^{\varepsilon_n}_{T_{\varepsilon_n}}$ with $\|q_n\|_{Y^1_{\varepsilon_n}} = \|q'_n\|_{Y^1_{\varepsilon_n}} = 1$ corresponding to $1 \in \sigma(L^{\varepsilon_n}_{T_{\varepsilon_n}})$.

For any $n \in \mathbb{N}$ we define a projection in $Y^1_{\varepsilon_n}$ as

$$P_n q = q - \langle q, q_n \rangle_n q_n,$$

where $\langle \cdot\, , \cdot \rangle_n$ denotes the scalar product in $Y^1_{\varepsilon_n} = X^1_{\varepsilon_n} \times L^2(Q)$ which generates the norm as it is defined in Section 2. Consider now

$$P_n q'_n = q'_n - \alpha_n q_n,$$

where $\alpha_n = \langle q'_n, q_n \rangle$. That is

$$q'_n = \alpha_n q_n + P_n q'_n.$$

From this we obtain

$$L^{\varepsilon_n}_{T_{\varepsilon_n}} q'_n = \alpha_n L^{\varepsilon_n}_{T_{\varepsilon_n}} q_n + P_n q'_n$$

or equivalently

$$P_n q'_n = L^{\varepsilon_n}_{T_{\varepsilon_n}} P_n q'_n.$$

Clearly $P_n q'_n \neq 0$, otherwise q_n, q'_n are linearly dependent. Therefore the previous equation can be rewritten as

$$\frac{P_n q'_n}{\|P_n q'_n\|_{Y^1_{\varepsilon_n}}} = L^{\varepsilon_n}_{T_{\varepsilon_n}} \frac{P_n q'_n}{\|P_n q'_n\|_{Y^1_{\varepsilon_n}}}.$$

Observe that $P_n q'_n$ is orthogonal to q_n. Now, passing to the limit as $n \to \infty$, by Lemma A (which still holds in the autonomous case) and Theorem B we obtain

$$\widehat{q}_0 = L^0_{T_0} \widehat{q}_0, \qquad \|\widehat{q}_0\|_{H^1(\Omega)\times L^2(\Omega)} = 1.$$

Furthermore,

$$\widehat{q}_0 = \frac{P_0 q_0}{\|P_0 q_0\|_{H^1(\Omega)\times L^2(\Omega)}},$$

where $q_0 \in H^1(\Omega) \times L^2(\Omega)$ is the normalized eigenvector of $L^0_{T_0}$ corresponding to the eigenvalue 1 and

$$P_0 q = q - \langle q, q_0 \rangle_0 \, q_0$$

is the projection in $H^1(\Omega) \times L^2(\Omega)$ defined by the usual norm $(\langle \cdot\,, \cdot \rangle_0)^{1/2}$ in this space which is the limit of the considered norm in $Y^1_{\varepsilon_n}$ as $n \to \infty$.

Since $\widehat{q}_0$ is orthogonal to q_0 we have a contradiction. Moreover by a similar procedure we can show that there is no adjoint vectors to q_n and so $1 \in \sigma(L^{\varepsilon_n}_{T_{\varepsilon_n}})$ is simple. This completes the proof.

To conclude it is sufficient to observe that we can repeat the same arguments employed in the non-autonomous case to establish the analogous of Theorems 1 and 2 for the orbital stability and instability respectively.

REFERENCES

[1] R. R. AKHMEROV, M. I. KAMENSKI, A. S. POTOPOV, A. E. RODKINA AND B. N. SADOVSKI, *Measures of noncompactness and condensing operators*, Birkhäuser Verlag, 1992.

[2] JU. L. DALECKII AND M.G. KREIN, *Stability of solutions of differential equations in Banach spaces*, Amer. Math. Soc. Trans. Math. Monographs, vol. 43, Providence Rhode Island, 1974.

[3] D. DANERS AND P. KOCH MEDINA, *Abstract evolution equations, periodic problems and applications*, Pitman Research Notes in Mathematical Series, vol. 279, Longman Scientific & Technical, 1992.

[4] J. HALE AND G. RAUGEL, *A damped hyperbolic equation on thin domains*, Trans. Amer. Math. Soc. **329** (1992), 185-219.

[5] R. JOHNSON, M.I. KAMENSKI AND P. NISTRI, *On periodic solutions of a damped wave equation in a thin domain using degree theoretic methods*, J. Differential Equations **40** (1997), 186-208.

[6] ———, *Existence of periodic solutions of an autonomous damped wave equation in thin domains*, J. Dynam. Differential Equations (to appear).

[7] M. KAMENSKI, *On the study of stability of periodic solutions for a new class of quasilinear equations in Banach spaces*, Dokl. Akad. Nauk (1997), 13-16. (Russian)

[8] O. LOPES AND S. CERON, *α-contractions and attractors for dissipative semilinear hyperbolic equations and systems*, Ann. Mat. Pura Appl. **160** (1991), 193-206.

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