Differential Inclusions and Optimal Control
Lecture Notes in Nonlinear Analysis
Volume 2, 1998, pages 9–18

CONTROL AND OPTIMIZATION OF NONLOCAL STEADY-STATE PROBLEMS

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1. Introduction

In a previous paper [1] the authors considered the problem of finding positive solutions to the nonlocal elliptic partial differential equation

$$\text{(I)} \qquad -\nabla[a\nabla u + \vec{b}\,u] = [h - g\,\overline{u}]u \qquad \text{in } \Omega$$

with associated mixed boundary conditions

$$\text{(II)} \qquad u = 0 \quad \text{on } \partial\Omega_D \qquad \text{and} \qquad a\frac{\partial u}{\partial n} + (\vec{b}\cdot\vec{n})u = 0 \quad \text{on } \partial\Omega_N$$

where $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, $\partial\Omega_D \cap \partial\Omega_N = \emptyset$, $\partial\Omega_D$ closed and we require that regularity conditions hold at points of $N = \partial\Omega_D \cap \overline{\partial\Omega_N}$ (see [7], [8], [9], [11]). Ω is a smooth domain in $\mathbb{R}^n$. Here the nonlocal term $\overline{u}$ is given by

$$\overline{u}(x) = \int_\Omega B_\delta(x,y)u(y)\,dy$$

where $B_\delta(x,y) = B_\delta(|x-y|) \in C_0^\infty$ is a mollifier in $\mathbb{R}^n$, i.e., $\int_{\mathbf{R}^n} B_\delta(x,y)\,dy = 1$ for any x. $B_\delta(|x-y|) = 0$ if $|x-y| \geq \delta$, $B_\delta(|x-y|)$ bounded away from zero if $|x-y| < \mu < \delta$.

1991 *Mathematics Subject Classification.* 49J20, 93C10, 93C20.

Key words and phrases. Nonnegative solutions, nonlinear elliptic equation, nonlocal term, optimal control.

Research supported in part by NSERC (Canada) and CNR (Italy)

The solutions of (I)-(II) represent the steady-states of an evolution equation similar to those previously introduced by [4], [5] and [2], (see also the extensive references therein).

The aim of this paper is to consider a related control problem to (I)-(II). Namely, motivated also by the results in [11], we consider here the problem

$$-\nabla[a\nabla u + \vec{b}\,u] = [h - g\,\overline{u}]u - f\sqrt{\varepsilon^2 + u^2}I(u > 0) \tag{1}$$

subject to mixed boundary conditions

$$u = u_d \geq 0 \quad \text{on } \partial\Omega_D \qquad \text{and} \qquad a\frac{\partial u}{\partial n} + (\vec{b}\cdot\vec{n})u = 0 \quad \text{on } \partial\Omega_N, \tag{2}$$

where $u \geq 0$ is the density of population of a species, $I(u > 0)$ is the characteristic function of the set $\{x \mid u(x) > 0\}$, $\varepsilon > 0$ and the term $f\sqrt{\varepsilon^2 + u^2}I(u > 0)$ is the so-called "harvesting" term. The function $f(x)$ represents the harvesting intensity, which is the control parameter, $a(x)$ the diffusion process, $\vec{b}(x) = (b_1(x)\dots b_n(x))$ a possible drift, $h(x)$ the intrinsic rate of growth of the species and $g(x)$ the crowding effect. The assumptions on all the above functions will be presented later in Section 1.

The parameter $\varepsilon > 0$ introduces a discontinuity in the harvesting term which makes it possible to have "dead zones": nonempty extinction regions $(u = 0) \subset \Omega$. This is fundamentally different from the problem considered in [1].

Following [10] we associate with (1)-(2) a cost functional to be maximized on the set of all the pairs (u, f) with $u \geq 0$ solution to (1)-(2) corresponding to f from a certain class of functions. For given $\alpha > 0$ the form of the cost functional is

$$J_\alpha(u, f) = \int_\Omega q_1 I(u \geq \alpha) + q_2 f u - q_3 f, \tag{3}$$

where $q_i(x)$, $i = 1, 2, 3$, are nonnegative weights to be chosen. The meaning of maximizing J_α is to maximize the region of Ω in which the density is bounded away from zero taking into account the economic benefit of the harvesting $q_2 f u$ and its cost $-q_3 f$.

We point out that (3) is slightly different from the functional considered in [10], where the cost functional contained the term $I(u > 0)$ instead of the term $I(u \geq \alpha)$, $\alpha > 0$. This modification is necessary since our approach does not allow us to prove the existence of the maximum even in very particular cases of the cost functional containing $I(u > 0)$.

The paper is organized as follows. In Section 1 we first prove in Theorem 1 the existence of nonnegative solutions of (1)-(2) with $u_d \equiv 0$ corresponding to a given control function $f \in L^\infty(\Omega)$, $0 \leq f < M$ a.e. in Ω, for some constant M. The positivity of such solutions in dependence of $\varepsilon > 0$ is also studied. Then in Theorem 2 the case of Dirichlet boundary conditions on both $\partial\Omega_D$ (with $u_d \geq 0$, $u_d \not\equiv 0$), and $\partial\Omega_N$ is considered. It is shown how the extinction region depends on the parameter $\varepsilon > 0$ for a given f.

In Section 2 we assume as the set of admissible controls the closure in the $L^2(\Omega)$-topology of the class of functions considered in Section 1. The existence of nonnegative solutions of (1)-(2) corresponding to all the admissible controls is proved by means of the results of Section 1.

Finally, the properties of the solution map S which associates to f the set of nonnegative solutions of (1)-(2) are investigated. The more relevant property of S, which allows us to solve the associated optimization problem is the closure of its graph in the w-$L^2(\Omega) \times L^\infty(\Omega)$ topology, where w-$L^2(\Omega)$ denotes the weak topology in $L^2(\Omega)$.

We would like to point out that the approach presented here in Section 1 for the existence results is quite different from that of [10]. In particular the presence in (1) of the nonlocal term $\overline{u}$ does not allow us to use any method based on order arguments as was shown earlier in [2]. Moreover, we consider here the case of mixed boundary conditions and the state equation is of more general form.

The optimization problem is also solved in a completely different way. In fact, in [10] the solution map is shown to be single-valued monotone and differentiable with respect to the control f. These properties permit the differentiation of the cost functional and the use of the steepest ascent algorithm in searching for a local maximizer of the cost functional. In our case the solution map is, in general, a multivalued one without any other relevant property than the closure of its graph in a suitable topology. In fact, the closure of the graph of S in the w-$L^2(\Omega) \times L^\infty(\Omega)$ topology will permit us to solve the proposed optimization problem.

2. Existence

We consider here the nonlocal problem

$$-\nabla[a\nabla u + \vec{b}\,u] = [h - g\,\overline{u}]u - f\sqrt{\varepsilon^2 + u^2}\,I(u > 0) \tag{1}$$

subject to mixed boundary conditions

$$u = u_d \geq 0 \quad \text{on } \partial\Omega_D \qquad \text{and} \qquad a\,\frac{\partial u}{\partial n} + (\vec{b}\cdot\vec{n})u = 0 \quad \text{on } \partial\Omega_N. \tag{2}$$

We search for nonnegative solutions of (1)-(2). As mentioned in the Introduction, (1)-(2) is a possible biological model for the steady-states of a species whose density of population is $u \geq 0$ in presence of a nonlocal term $\overline{u}(x) = \int_{\mathbb{R}^n} B_\delta(|x - y|)u(y)\,dy$ and of a discontinuous ($\varepsilon > 0$) harvesting term of which the function f is the control parameter.

We assume the following conditions:

(A) the function a is a piecewise smooth function, smooth near $\partial\Omega_N$, satisfying $0 < a_1 \leq a(x) \leq a_2$ for a.a. $x \in \Omega$; $f, g, h \in L^\infty(\Omega)$ with $g(x)$, $h(x) > c > 0$, $f \geq 0$ for a.a. $x \in \Omega$ and some constant c, u_d and $\vec{b} = (b_1, \ldots, b_n)$ are smooth functions in $\overline{\Omega}$.

(B) if $u_d \equiv 0$ the function f satisfies

$$\operatorname{ess}\inf_{x\in\Omega}[h(x) - f(x)] > \mu_1,$$

where μ_1 is the least eigenvalue of $-\nabla[a\nabla u + \vec{b}\,u]$ subject to (2).

To better understand the meaning of assumption (B) suppose $u > 0$, rewrite (1) in the form

$$-\nabla[a\nabla u + \vec{b}\,u] = [h - f - g\,\overline{u}]u - \frac{\varepsilon^2}{\sqrt{\varepsilon^2 + u^2} + u} f$$

and consider the second term on the right hand side as a perturbation for ε small. For $\varepsilon = 0$ and Dirichlet boundary conditions in [2] it has been shown that while (B) may be modified, it cannot be removed.

For convenience we introduce the linear operator $G : L^\infty(\overline{\Omega}) \to C^\alpha(\overline{\Omega})$ defined as follows

$$u = Gw, \qquad w \in L^\infty(\Omega)$$

if and only if u satisfies (1)-(2) with the right hand side replaced by w, i.e. $G = (-\nabla[a\nabla\,\cdot\, + \vec{b}\,\cdot\,])^{-1}$. By well known regularity results, see [11], it follows that G is a continuous and compact operator for α small.

We consider first the case when $u_d \equiv 0$, i.e., homogeneous boundary conditions. We have

Theorem 1. *Under assumptions* (A)-(B) *and* $u_d \equiv 0$ *there exist* $\varepsilon_0, \varepsilon_1 > 0$ *such that*

(a) *if* $0 \le \varepsilon < \varepsilon_0$ *then* (1)-(2) *has a nonnegative nontrivial solution;*
(b) *if* $\varepsilon > \varepsilon_1$ *then* (1)-(2) *has only the solution* $u \equiv 0$ *in* $L^\infty(\Omega)$.

Proof. (a) We draw on the results of [1] and [2], and thus only present a short proof for the reader's convenience emphasizing the few differences.

Without loss of generality, we assume that the left hand side of (1) is coercive, otherwise we add a linear term to both sides. Let $0 \le \lambda \le 1$ and consider first the equation

$$-\nabla[a\nabla u + \vec{b}\,u] = \lambda[h - f - g\,\overline{u}]u^+$$

subject to boundary conditions (2). If λ is small enough, the only solution is $u \equiv 0$. Indeed, solutions are nonnegative and satisfy

$$-\nabla[a\nabla u + \vec{b}\,u] - \lambda[h - f]u = -\lambda g\,\overline{u}\,u \le 0,$$

whence $u \equiv 0$ since the least eigenvalue of the operator on the left hand side is positive for λ small. Suppose thus $0 < \lambda_0 \le \lambda \le 1$ and that $0 \le u \in C^\alpha$, for some α, is a solution. Let a_n, h_n, f_n, g_n denote smooth approximations to a, h, f, g respectively. Without loss of generality, we assume that they satisfy the same L^∞ bounds as the original coefficients, and that they converge in L^p for any large p. We may also assume that $\vec{b}\cdot\vec{n} > 0$ on $\partial\Omega_N$ for if not, we replace u by

$v = e^{-w}u$, with $\nabla w \cdot \vec{n} \gg 0$ on $\partial\Omega_N$, chosen suitably in what follows. Consider now the linear eigenvalue problem

$$-\nabla[a_n \nabla u_n + \vec{b}\, u_n] - \lambda[h_n - f_n - g_n\, \overline{u}\,]u_n = k_n u_n$$

subject to conditions (2). Observe that k_n exists, and it is bounded above and below by constants independent of n. If we normalize the eigenvectors by $\|u_n\|_{L^\infty} = 1$, it follows that u_n is bounded in $H^{1,2} \cap C^\alpha$, see [11], for some $\alpha > 0$, and we may assume $0 \leq u_n \to \omega$ weakly in $H^{1,2}$ and strongly in C^{α_0} for some $\alpha_0 < \alpha$. Since $\omega \geq 0$, and thus $\omega > 0$ by the maximum principle, we conclude that $k_n \to 0$ and $\omega \equiv u/\|u\|_{L^\infty}$. Finally, suppose u_n assumes its max in $\overline{\Omega}$ at P_n. In view of the boundary conditions, $P_n \notin \partial\Omega$ as there either $u_n = 0$ or $a(\partial u_n/\partial n) = -(\vec{b} \cdot \vec{n})u_n < 0$. It follows that $P_n \in \Omega$, and from the equation we get

$$\overline{u}\,(P_n) \leq \frac{\lambda[h_n - f_n] + \operatorname{div}(\vec{b}\,) + k_n}{\lambda g_n} \leq K$$

for a constant K independent of n, λ. Next suppose $P_n \to P$, then $\overline{u}\,(P) = \lim \overline{u}(P_n)$ and for any $Q \in \Omega$, $u(Q) \leq u(P)$ by equicontinuity of the u_n. By a reflection process, see [11], and the generalized Harnack inequality [6], we conclude first that u is bounded in L^∞ and then in C^α for some α. Since $h(x) - f(x) > \mu_1$ for a.a. $x \in \Omega$, we also have that $\|u\|_{C^\alpha}$ cannot be too small and thus, as in [1] and [2], $\operatorname{Deg}(I - T, B_R - B_r, 0) = 1$ where B_ρ is the ball of radius ρ centered at zero in $C^\alpha(\overline{\Omega})$, and $T_\omega = G([h - f - g\overline{\omega}]\omega^+)$ subject to (2). Let $0 \leq I_m(u) \leq 1$ denote a smooth approximation to $I(u)$, $I_m(\xi) = 0$ if $\xi \leq 0$, and consider the perturbation $Z(\omega) = G(\varepsilon^2 f I_m(\omega)/(\sqrt{\varepsilon^2 + \omega^2} + \omega))$ on $B_R - B_r$. If we choose $\varepsilon > 0$ small enough, independent of m, then $\operatorname{Deg}(I - T - Z, B_R - B_r, 0) = 1$ and thus there exists a solution $0 \leq \omega_m$ of

$$-\nabla[a\nabla\omega_m + \vec{b}\,\omega_m] = [h - f + g\,\overline{\omega}_m]\omega_m - \frac{\varepsilon^2 f I_m(\omega_m)}{\sqrt{\varepsilon^2 + \omega_m^2} + \omega_m}.$$

We again have $\omega_m \to \omega$ weakly in $H^{1,2}$, strongly in C^α with $\omega \geq 0$, nontrivial. Since $\varepsilon^2 f I_m(\omega_m)/(\sqrt{\varepsilon^2 + \omega_m^2} + \omega_m)$ is bounded, we may take it to be weakly convergent in L^2 to a function z. If $\omega(x) > 0$, then $z = \varepsilon^2 f/(\sqrt{\varepsilon^2 + \omega^2} + \omega)$, while on the set $\{\omega = 0\}$ we have $\nabla\omega = 0$ almost everywhere. We assumed the coefficients are piecewise smooth and thus, on $\Omega - \Gamma$ we have $\omega \in H^{2,2}$, where Γ is a set of measure zero. We conclude that ω satisfies $-\nabla[a\nabla\omega + \vec{b}\omega] = [h - f + g\overline{\omega}]\omega + z$, a.e. on $\Omega - \Gamma$ whence $z = 0$ a.e. on the set $\{\omega = 0\}$.

(b) Again assume without loss of generality that $\vec{b} \cdot \vec{n} > 0$ on $\partial\Omega_N$. Suppose that $u \geq 0$, $u \not\equiv 0$, is a solution and select a constant $c_0 > 0$ such that the linear problem

$$-\nabla[a\nabla\omega + \vec{b}\,\omega] = [h - f - g\,\overline{u}\,]\omega + c_0\omega$$

subject to (2) does not have eigenvalue zero. Let once again a_n, h_n, f_n, g_n be smooth approximations and consider the problem

$$-\nabla[a_n\nabla\omega_n + \vec{b}\,\omega_n] - [h_n - f_n + c_0 - g_n\overline{u}]\omega_n = \left\{\frac{-\varepsilon^2 f}{\sqrt{\varepsilon^2 + u^2} + u} I(u > 0) - c_0 u\right\}_n,$$

where $\{\cdot\}_n$ denotes a mollifier. This problem has a solution $\omega_n \in C^\alpha$, and we may assume as before that $\omega_n \to u$ in C^{α_0} and weakly in $H^{1,2}$. Since $u \ge 0$, $u \not\equiv 0$, then $\max \omega_n > 0$ for n large, and $\overline{u}(P_n) < K$ where $\max \omega_n = \omega_n(P_n)$. We conclude as in (a), that $\overline{u}(P) \le K_0$ if $u(P) = \max u$ and consequently that $u \le K_1$ in Ω, by the generalized Harnack inequality, with K_1 independent of u, ε. We then have

$$-\nabla[a\nabla u + \vec{b}\,u] \le \left(hK_1 - \frac{\varepsilon^2 f}{\sqrt{\varepsilon^2 + K_1^2} + K_1}\right) \operatorname{sign}(u).$$

For $\varepsilon > 0$ sufficiently large, the right hand side is nonpositive in Ω and thus $u \le 0$ in Ω, i.e. $u \equiv 0$. □

We now pass to the case $u_d \ge 0$, $u_d \not\equiv 0$. In this case $u \equiv 0$ as a solution is impossible, we do not require condition (B), and we can show the existence of a nonnegative solution of (1)-(2), for any $\varepsilon > 0$, by using the earlier proof to show that $\operatorname{Deg}(I - T, B_R, 0) = 1$, where $T : C^\alpha(\Omega) \to C^\alpha(\Omega)$ is the compact operator composition of G with the Nemytskii operator generated by the right hand side of (1) regularized, and then passing to a limit. Briefly but specifically, suppose a, h, g, f are smooth. We consider

$$-\nabla[a\nabla u + \vec{b}\,u] = \lambda\left([h - f - g\,\overline{u}]u^+ - \frac{\varepsilon^2 f}{\sqrt{\varepsilon^2 + u^2} + u} I_m(u > 0)\right),$$

subject to $u = \lambda u_d$ on $\partial\Omega_D$ and the same natural boundary conditions on $\partial\Omega_N$, for $0 \le \lambda \le 1$. Again if λ is small, then $-\nabla[a\nabla u + \vec{b}\,u] - \lambda[h - f]u \le 0$ and $u = \lambda u_d$ on $\partial\Omega_D$ shows that u is bounded in terms of u_d. Otherwise, either $\|u\|_{L^\infty} = \lambda\|u_d\|_{L^\infty}$ or we again have $\overline{u}(P) \le K$ for a $P \in \overline{\Omega}$ at which u assumes its maximum. If P is away from $\partial\Omega_D \cap \overline{\partial\Omega_N}$, we proceed exactly as before to bound u in L^∞ and then in C^α for some α. Otherwise we map a neighborhood of P to a quarter sphere, and reflect the coefficients as before and u as an even function to the whole of the upper hemisphere. Since u_d extends as a C^α function, we use results in [6] to bound u in L^∞ and then in C^α. Observe that the L^∞ bound is independent of ε, m and the C^α bound is independent of m.

We now prove that for $\varepsilon > 0$ sufficiently large there exist extinction zones for the solutions of (1) found above subject to the Dirichlet boundary conditions $u = u_d \ge (\not\equiv)\ 0$ on $\partial\Omega_D$. We have the following result.

Theorem 2. *Assume* (A). *Let $\delta > 0$ be given. Then there exists $\varepsilon_0 > 0$ such that if $\varepsilon > \varepsilon_0$ then $\mu(u > 0) \le \delta$.*

We need first the following.

Lemma 1. *Let $\{A_n\}$ be a sequence of measurable sets contained in Ω with $\mu(A_n) > \delta > 0$. Then there exists a subsequence and a function τ, $0 \le \tau \le 1$ such that $I(A_n) \to \tau$ in $L^1(\Omega)$ and $\int_\Omega \tau \ge \delta$.*

Proof. Let $f_n = I(A_n)$ for any $n \in \mathbb{N}$, $\{f_n\}$ is a sequence of bounded measurable functions and so, passing to a subsequence if necessary, $f_n \to \tau$ weakly in

$L^2(\Omega)$. Therefore by the Banach-Saks's Theorem $(1/n)\sum_{i=1}^n f_i \to \tau$ in $L^2(\Omega)$. But

$$\left\| \frac{1}{n}\sum_{i=1}^n f_i \right\|_{L^1(\Omega)} \geq \delta$$

and thus

$$\left\| \frac{1}{n}\sum_{i=1}^n f_i \right\|_{L^2(\Omega)} \geq \gamma > 0$$

for some $\gamma > 0$, which implies that $\tau \not\equiv 0$. Moreover, by passing again to a subsequence if necessary, $\{(1/n)\sum_{i=1}^n f_i\}$ converges a.e. in Ω to τ, thus τ takes only values in $[0,1]$ and $\|\tau\|_{L^1} \geq \delta$, whence $\mu(\tau \neq 0) \geq \delta$. □

We prove now Theorem 2.

Proof of Theorem 2. Suppose not, since $g\bar{\omega}\omega \geq 0$ we have

$$-\nabla[a\nabla\omega + \vec{b}\,\omega] - [h-f]\omega \leq -\frac{f\varepsilon^2}{\sqrt{\varepsilon^2+\omega^2}+\omega} I(\omega > 0)$$

and as mentioned above, $0 \leq \omega \leq k$ for some k independent of ε. I.e.,

$$-\nabla[a\nabla\omega + \vec{b}\,\omega] \leq [h-f]k - \frac{f\varepsilon^2}{\sqrt{\varepsilon^2+k^2}+k} I(\omega > 0).$$

Let $\varepsilon_n \to \infty$ and suppose $I(\omega_n > 0) > \delta$. Since $\omega_n \leq k$, we may assume $\omega_n \to \omega \geq 0$ weakly in $H^{1,2}$ and strongly in L^2. Let z_n solve

$$-\nabla[a\nabla z_n + \vec{b}\,z_n] = fI(\omega_n > 0)$$

with $z_n = 0$ on $\partial\Omega_D$, while r solves

$$-\nabla[a\nabla r + \vec{b}\,r] = [h-f]k$$

and $r = u_d$ on $\partial\Omega_D$. Since $I(\omega_n > 0) \to \tau$ in L^1 (and thus L^p for large p) then $z_n \to z$ in $C^\alpha(\overline{\Omega})$ for some $\alpha > 0$ with

$$-\nabla[a\nabla z + \vec{b}\,z] = f\tau.$$

Furthermore, since $f\tau \geq 0$ is nontrivial, $z > 0$ in Ω. Finally, for any given $M > 0$, $Mz_n + \omega_n \leq r$ if ε is large enough, i.e., $Mz + \omega \leq r$ whence $\omega < 0$ somewhere in Ω if M is large enough, contradicting $\omega \geq 0$. □

Remark 1. Observe that if f vanishes somewhere in Ω, then the previous arguments will show the result if $f\tau \not\equiv 0$, i.e., if $\mu\{f(x) = 0\} < \delta$.

3. Control and Optimization

As the set of admissible controls for problem (1)-(2) we consider

$$V = \{f \in L^\infty(\Omega) \mid 0 \le f(x) \le M \text{ for a.a. } x \in \Omega\},$$

where $M = \operatorname{ess\,inf}_{x\in\Omega}[h(x) - \mu_1]$ if $u_d \equiv 0$. Otherwise $M > 0$ is chosen for convenience. Obviously

$$V = \overline{\{f \in L^\infty(\Omega) \mid 0 \le f(x) < M \text{ for a.a. } x \in \Omega\}},$$

where the closure is in the $L^2(\Omega)$-topology. Since V is convex and closed it is weakly closed in the w-$L^2(\Omega)$ topology and vice versa. In Section 1 for $\varepsilon > 0$ as determined in Theorems 1 and 2, we proved that the set

$$S(f) = \{u \in L^\infty \mid u \ge 0 \text{ is a solution of (1)-(2)}\}$$

is a nonempty (compact) set for any $f \in L^\infty(\Omega)$ with $0 \le f(x) < M$ for a.a. $x \in \Omega$.

We can prove the following.

Theorem 3.

(a) *$S(f) \neq \emptyset$ for any $f \in V$;*
(b) *$S : V \to L^\infty(\Omega)$ has closed graph in the w-$L^2(\Omega) \times L^\infty(\Omega)$-topology;*
(c) *$S(V)$ is a compact set in $L^\infty(\Omega)$.*

Proof. (a) This is established in Section 1 for $0 \le f < M$. Otherwise, let $f_m \in V$ with $0 \le f_m < M$ such that $f_m \to f$ in $L^2(\Omega)$ weakly and suppose u_m solves (1), (2) with I replaced by I_m. We still have u_m uniformly bounded, and by a limit argument conclude the existence of a function $u \in S(f)$. Observe that $u \equiv 0$ is possible if $u_d \equiv 0$.

(b) Let $\{(u_n, f_n)\} \subset L^\infty(\Omega) \times V$, $u_n \in S(f_n)$ such that $u_n \to u$ in $L^\infty(\Omega)$, $u \ge 0$ and $f_n \to f \in V$ in $L^2(\Omega)$ weakly. The same arguments as in (a) show that $u \in S(f)$.

(c) Let $\{u_n\} \subset S(V)$ be any sequence and let $\{f_n\} \subset V$ be such that $u_n \in S(f_n)$. By passing to a subsequence if necessary we have $u_n \to u$ in $L^\infty(\Omega)$ and $f_n \to f$ in $L^2(\Omega)$ weakly and so $u \in S(f)$.

Remark 2. The previous results guarantee that the multivalued map S is upper-semicontinuous in the w-$L^2(\Omega) \times L^\infty(\Omega)$-topology (see [3], Corollary 1, p. 42).

Finally, we can solve our optimization problem. Consider the cost functional

$$J_\alpha(u, f) = \int_\Omega q_1 I(u \ge \alpha) + q_2 f u - q_3 f, \qquad \alpha > 0.$$

We assume that the nonnegative weights $q_i \in L^2(\Omega)$, $i = 1, 2, 3$. We can prove the following.

Theorem 4. *For any* $\alpha > 0$, *the cost functional* $J_\alpha(u, f)$ *attains a maximum on the set* $\mathcal{S} = \{(u, f) \in L^\infty(\Omega) \times V \mid u \in S(f),\ f \in V\}$.

Proof. Let $\{(u_n, f_n)\} \subseteq \mathcal{S}$ be a maximizing sequence. We have $u_n \to u_0$ in $L^\infty(\Omega)$ and $f_n \to f_0$ in $L^2(\Omega)$ weakly. Consider

$$\begin{aligned}\lim_{n\to\infty} \int_\Omega q_1 I(u_n \geq \alpha) + q_2 f_n u_n - q_3 f_n \\ &\leq \limsup_{n\to\infty} \int_\Omega q_1 I(u_n \geq \alpha) + \limsup_{n\to\infty} \int_\Omega q_2 f_n u_n - q_3 f_n \\ &\leq \int_\Omega \limsup_{n\to\infty} q_1 I(u_n \geq \alpha) + \limsup_{n\to\infty} \int_\Omega q_2 f_n u_n - q_3 f_n .\end{aligned}$$

On the other hand,

$$\limsup_{n\to\infty} I(u_n \geq \alpha) \leq I(u_0 \geq \alpha)$$

a.e. in Ω. Rewriting the second term as

$$\int_\Omega [q_2 u_n - q_3] f_n = \int_\Omega [q_2 u_0 - q_3] f_n + \int_\Omega q_2 [u_n - u_0] f_n ,$$

by our assumptions on q_2, q_3 we get

$$\lim_{n\to\infty} J_\alpha(u_n, f_n) \leq J_\alpha(u_0, f_0)$$

with $u_0 \in S(f_0)$ by Theorem 3.

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