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Existence of Periodic Solutions of an Autonomous Damped Wave Equation in Thin Domains[1]

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For a nonlinear autonomous damped wave equation in a thin domain we provide conditions ensuring the existence of periodic solutions in time. Our approach uses both methods developed by Hale and Raugel and methods based on the topological degree theory together with some results on the functionalization of parameter.

KEY WORDS: Autonomous hyperbolic nonlinear equation; periodic solutions; topological degree.

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1. INTRODUCTION

In this paper we prove an existence result for periodic solutions with respect to the time t of an autonomous damped wave equation in a thin domain.

The considered equation has the form

$$\frac{\partial^2 u}{\partial t^2} = \Delta_X u + \frac{\partial^2 u}{\partial Y^2} - \beta \frac{\partial u}{\partial t} - \alpha u + g(X, Y, u) \tag{1}$$

where α and β are positive constants and g is a suitable smooth function. Here (X, Y) is a generic point of the thin domain $Q_\varepsilon = \Omega \times (0, \varepsilon) \subset \mathbf{R}^{N+1}$,

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where Ω is a C^2-smooth bounded domain in $\mathbf{R}^N$ and $\varepsilon \in (0, \varepsilon_0)$ is a small parameter.

Associated to Eq. (1) we consider the Neumann boundary condition

$$\frac{\partial u}{\partial \nu_\varepsilon} = 0 \qquad \text{on } \partial Q_\varepsilon \tag{2}$$

We assume that the reduced problem at $\varepsilon = 0$ has a T_0-periodic solution z for some $T_0 > 0$. This solution is not isolated since the equation is autonomous. This creates a difficulty for the application of topological methods. To overcome it, we first normalize the unknown period of the sought-after periodic solution of (1)–(2) by replacing t by $(T_0/T)\, t$, $T > 0$, in Eq. (1). The resulting equation will depend on the parameter T and we will look for a solution pair (T, u) of this equation, with u T_0-periodic, which will represent a T-periodic solution of the original equation (1). Then we make suitable assumptions on the reduced linearized equation around $z = z(t, x)$. That is we assume that the eigenspace corresponding to the eigenvalue 1 of the resulting linear operator is one-dimensional, i.e., it is the $span\{z_t\}$ with $z_t = \partial z/\partial t$, and that the linearized equation does not have a solution of the form $y(t, x) + (t/T_0)\, z_t(t, x)$ where y is T_0-periodic.

Under these assumptions, following the lines of [3] it is possible to prove the existence of a continuous functional $T = T(w)$, $w = (u, u_t)$, such that $T(\hat{w}_0) = T_0$ where $\hat{w}_0 = (z, z_t)$ where $\hat{w}_0$ is an isolated fixed point, with topological index different from zero, of an operator parametrized by $T(\cdot)$ associated to the reduced problem.

Finally, by using suitable homotopies, it is shown that for sufficiently small $\varepsilon > 0$ problem (1)–(2) admits a T_ε-periodic solution u^ε such that $T_\varepsilon \to T$ and $u^\varepsilon \to z$ as $\varepsilon \to 0$ in a suitably defined space depending on $\varepsilon > 0$ (see [4]).

Analogous results have also been obtained in the non-autonomous case by the authors [7, 8]. In this case the period is assigned as the period of the time-dependent nonlinearity and no assumption on the linearized reduced problem is required.

This paper combines the topological methods mentioned above with the methods developed by Hale and Raugel to study the properties of the attractor A_ε defined by problem (1)–(2) under various boundary conditions ([1, 2, 4–6, 11]).

The paper is organized as follows. In Section 2 we introduce notations, definitions and some preliminary results which will be used in the sequel. In Section 3 we treat the linearized reduced problem. Finally, in Section 4 we formulate and prove the main existence result of periodic solutions of (1)–(2).

2. NOTATION, DEFINITIONS AND PRELIMINARY RESULTS

Following [4], for fixed $\varepsilon > 0$ we consider the change of variables $X = x$, $Y = \varepsilon y$. Equation (1) becomes

$$\frac{\partial^2 u}{\partial t^2} = \Delta_x u + \frac{1}{\varepsilon^2}\frac{\partial^2 u}{\partial y^2} - \beta\frac{\partial u}{\partial t} - \alpha u + g(x, \varepsilon y, u) \tag{3}$$

and the Neumann boundary condition takes the form

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial Q \tag{4}$$

where $Q = \Omega \times (0, 1)$ and ν denotes the outward unit normal vector to Q. We suppose that Ω is a C^2-smooth domain. We look for periodic solutions of (3)–(4) of some period $T > 0$. Since the period is unknown we normalize it to a given fixed period $T_0 > 0$ by replacing t by $(T_0/T)\,t$ resulting in the following problem for $\tilde{u}(t, x, y) = u((T/T_0)\,t, x, y)$:

$$\frac{T_0^2}{T^2}\frac{\partial^2 \tilde{u}}{\partial t^2} = \Delta_x \tilde{u} + \frac{1}{\varepsilon^2}\frac{\partial^2 \tilde{u}}{\partial y^2} - \frac{T_0}{T}\beta\frac{\partial \tilde{u}}{\partial t} - \alpha\tilde{u} + g(x, \varepsilon y, \tilde{u}) \tag{5}$$

where $t \in [0, T_0]$, with associated Neumann boundary condition

$$\frac{\partial \tilde{u}}{\partial \nu} = 0 \qquad \text{on } \partial Q \tag{6}$$

It is clear that, if for some $T > 0$, $\tilde{u}$ is a T_0-periodic solution of (5)–(6) then u is a T-periodic solution of (3)–(4).

For $\varepsilon > 0$ we introduce the following Banach spaces (see [4]). Let X_ε^1 be the space $H^1(Q)$ with the norm

$$\left(\|u\|_{1Q}^2 + \frac{1}{\varepsilon^2}\left|\frac{\partial u}{\partial y}\right|_{0Q}^2\right)^{1/2}$$

Here and below, $\|\cdot\|_{0Q}$ denotes the norm in $L^2(Q)$, and $\|\cdot\|_{1Q}$ that in $H^1(Q)$. Let $U_\varepsilon(T, t)$ be the semigroup generated by the system of linear equations

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{T}{T_0} v \\ \frac{\partial v}{\partial t} &= \frac{T}{T_0}\left[\Delta_x u + \frac{1}{\varepsilon^2}\frac{\partial^2 u}{\partial y^2} - \beta v - \alpha u\right] \end{aligned} \tag{7}$$

with the boundary condition (6). It is known, [4] and [6], that $U_\varepsilon(T, t)$, for any $T > 0$, is a C_0-semigroup in the space

$$Y_\varepsilon^1 \triangleq X_\varepsilon^1 \times L^2(Q) \ni (u, v)$$

In the somewhat more general problem considered in [4], this space is defined in another way which yields, however, the space Y_ε^1 in the case we are considering. For $T > 0$ in a bounded set, one has the exponential estimate

$$\| U_\varepsilon(T, t) \|_{Y_\varepsilon^1 \to Y_\varepsilon^1} \leqslant c e^{-\gamma t}, \qquad (t \geqslant 0)$$

where c and γ are positive constants, (see [4]).

If $u \in L^2(Q)$, define its projection by

$$(Pu)(x) = \int_0^1 u(x, y)\, dy, \qquad \text{then define}$$

$$\mathbf{P} = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}$$

Then P maps $L^2(Q)$ to $L^2(\Omega)$, $\mathbf{P}$ maps Y_ε^1 to $H^1(\Omega) \times L^2(\Omega)$ and

$$\|Pu\|_{j, \Omega} \leqslant \|u\|_{j, Q}, \qquad u \in H^j(Q), \qquad j = 0, 1 \tag{8}$$

This projection is important in relating problem (3)–(4) to the limiting problem in Ω obtained by letting $\varepsilon \to 0$. Now we discuss the function g. We assume that $g\colon \Omega \times [0, \varepsilon_0) \times \mathbf{R} \to \mathbf{R}$ is of class C^1 jointly in the variables x, Y and u, and that its derivatives satisfy the following estimates

$$|g_x(x, Y, u)| \leqslant a(1 + |u|^{\theta + 1}) \tag{9}$$

$$|g_Y(x, Y, u)| \leqslant a(1 + |u|^{\theta + 1}) \tag{10}$$

$$|g_u(x, Y, u)| \leqslant a(1 + |u|^{\theta}) \tag{11}$$

Here a is a positive constant, and θ is determined as follows: $\theta \in [0, \infty)$ if $N = 1$, and $\theta \in [0, 2/N - 1)$ for values $N \geqslant 2$ (recall that $\dim Q = N + 1$).

Let $C_{T_0}(Y_\varepsilon^1)$ be the space of all continuous, T_0-periodic functions $w = \binom{u}{v}$ from $\mathbf{R}$ into Y_ε^1 with the usual norm

$$\|w\| = \sup_{0 \leqslant t \leqslant T_0} \|w(t)\|_{Y_\varepsilon^1}$$

Define the following maps on $C_{T_0}(Y^1_\varepsilon)$

$$f_\varepsilon(T, w)(t)(x, y) = \begin{pmatrix} 0 \\ (T/T_0)\, g(x, \varepsilon y, u(t, x, y)) \end{pmatrix}$$

$$f_\varepsilon(w)(t)(x, y) = \begin{pmatrix} 0 \\ g(x, \varepsilon y, u(t, x, y)) \end{pmatrix}$$

and

$$J_\varepsilon(T)\, w(t) = U_\varepsilon(T, t)[I - U_\varepsilon(T, T_0)]^{-1} \int_0^{T_0} U_\varepsilon(T, T_0 - s)\, w(s)\, ds + \int_0^t U_\varepsilon(T, t-s)\, w(s)\, ds$$

Then define

$$F_\varepsilon(T, w) = J_\varepsilon(T)\, f_\varepsilon(T, w) \tag{12}$$

Using the estimate (8) with $j = 1$, one sees that the right-hand side of (12) is well-defined. Using the Sobolev embedding theory together with the theory of nonlinear Nemytskii operators [9], one can prove that, for any $T > 0$, $F_\varepsilon(T, \cdot)$ maps $C_{T_0}(Y^1_\varepsilon)$ into itself and is completely continuous, i.e., it is continuous and it maps bounded sets into relatively compact sets.

We identify the set of fixed points of $F_\varepsilon(T, \cdot)$, for some $T > 0$, as the class of T-periodic solutions which we will study. The question of the exact relation between the set of fixed points of F_ε and the set of T-periodic distributional solutions of (3)–(4) has been studied in [9, 10]. It is known that a fixed point of $F_\varepsilon(T, \cdot)$ is always a T-periodic distributional solution of (3)–(4).

Next we pose the limit problem at $\varepsilon = 0$. Let $U_0(T, t)$, $t \geq 0$, be the semigroup generated by the equations

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{T}{T_0} v \\ \frac{\partial v}{\partial t} &= \frac{T}{T_0} [\Delta_x u - \beta v - \alpha u] \end{aligned} \tag{13}$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial Q \tag{14}$$

Let $\binom{u_0}{v_0}$ be an element of $H^1(\Omega)\times L^2(\Omega)$. Then, for any $T>0$, $U_0(T,t)\binom{u_0}{v_0}$ is in $H^1(\Omega)\times L^2(\Omega)$, and, for T in a bounded set, one has the estimate

$$\|U_0(T,t)\|_{H^1(\Omega)\times L^2(\Omega)\to H^1(\Omega)\times L^2(\Omega)}\leqslant ce^{-\gamma t}$$

where c and γ are positive constants.

Writing $i\colon\Omega\to Q$ defined by $i(x)=(x,0)$, we obtain an inclusion $\mathscr{J}\colon H^1(\Omega)\times L^2(\Omega)\to Y^1_\varepsilon$ with $\mathscr{J}(u,v)(x,y)=(u(x),v(x))$. The map $\mathscr{J}$ is an isometry for all $0<\varepsilon<\varepsilon_0$, and we identify $U_0(T,t)\binom{u_0}{v_0}$ with the element $\mathscr{J}U_0(T,t)\binom{u_0}{v_0}$ of Y^1_ε.

Define, for $T>0$, an operator $F_0(T,\cdot)$ on $C_{T_0}(H^1(\Omega)\times L^2(\Omega))$ as follows:

$$F_0(T,w)=J_0(T)\,f_0(T,w)$$

where $J_0(T)$ has the same form as $J_\varepsilon(T)$ with $U_\varepsilon(T,t)$ replaced by $U_0(T,t)$ and

$$f_0(T,w)(t)(x)=\begin{pmatrix}0\\(T/T_0)\;g(x,0,u(t,x))\end{pmatrix}$$

$$f_0(w)(t)(x)=\begin{pmatrix}0\\g(x,0,u(t,x))\end{pmatrix}$$

Then $F_0(T,\cdot)$ maps $C_{T_0}(H^1(\Omega)\times L^2(\Omega))$ into itself and is completely continuous. We identify the T-periodic solutions of the problem

$$\frac{\partial^2u}{\partial t^2}=\Delta_xu-\beta\frac{\partial u}{\partial t}-\alpha u+g(x,0,u)\qquad(15)$$

$$\frac{\partial u}{\partial v}=0\qquad\text{on }\partial\Omega\qquad(16)$$

with the fixed points of the operator $F_0(T,\cdot)$.

In the sequel the double arrow will denote uniform convergence with respect to all relevant variables, and ind$(\cdot,\cdot)$ will denote the topological fixed point index.

3. THE REDUCED LINEARIZED EQUATION

Let $z=z(t,x)$ be a T_0-periodic solution of (15)–(16). Consider the linearized equation of (15)–(16) around $z=z(t,x)$, it can be written in the form of an integral equation as follows

$$w=(F_0)'_w(T_0,\hat{w}_0)\,w\qquad(17)$$

where $(F_0)'_w(T_0,\hat{w}_0)=J(T_0)(f_0)'_w(T_0,\hat{w}_0)$ and $\hat{w}_0=(z,z_t)$.

Let $(F_0)'_T(T_0, \hat{w}_0) = J'_0(T_0) f_0(T_0, \hat{w}_0) + J_0(T_0)(f_0)'_T(T_0, \hat{w}_0)$, we can state the following result.

Lemma 1. *Assume that there exists $T_0 > 0$ such that $F_0(T_0, \hat{w}_0) = \hat{w}_0$ for some $\hat{w}_0 = (z, z_t) \in C_{T_0}(H^1(\Omega) \times L^2(\Omega))$. Moreover assume that*

(a) $F_0(T, \hat{w}_0 + h) - F_0(T, \hat{w}_0) = (F_0)'_w(T, \hat{w}_0)\, h + \omega_1(T, h)$;
$F_0(T_0 + s, \hat{w}_0) = (F_0)'_T(T_0, \hat{w}_0)\, s + \omega_2(s)$ *where*

$$\frac{\|\omega_1(T, h)\|}{\|h\|} \rightrightarrows 0 \quad \text{as} \quad \|h\| \to 0, \qquad \frac{\|\omega_2(s)\|}{s} \to 0 \quad \text{as} \quad s \to 0$$

(We use a double arrow to indicate uniform convergence, in this case with respect to T.)

(b) the subspace spanned by the eigenvectors corresponding to the eigenvalue $1 \in \sigma((F_0)'_w(T_0, \hat{w}_0))$ is one-dimensional.

(c) the equation

$$(F_0)'_T(T_0, \hat{w}_0) = w - (F_0)'_w(T_0, \hat{w}_0)\, w \tag{18}$$

has no solution $w \in C_{T_0}(H^1(\Omega) \times L^2(\Omega))$.

Then there exists a continuous functional $T = T(w)$ such that $T(\hat{w}_0) = T_0$ and furthermore, the fixed point $\hat{w}_0$ of the operator $\Gamma_0(w) \triangleq F_0(T, w), w)$ is isolated and $|\mathrm{ind}(\hat{w}_0, \Gamma_0)| = 1$.

Proof. Consider a functional $l \in E^*$, where $E = C_{T_0}(H^1(\Omega) \times L^2(\Omega))$, such that $l(e_0) \neq 0$, where e_0 is a vector of unitary length generating the subspace corresponding to the eigenvalue 1 of $(F_0)'_w(T_0, \hat{w}_0)$. Define a functional $T: E \to \mathbf{R}$ as follows:

$$T(w) = T_0 + l(w - \hat{w}_0)$$

Let $B: E \to E$ be the linear operator given by

$$Bk = (F_0)'_T(T_0, \hat{w}_0)\, l(k) + (F_0)'_w(T_0, \hat{w}_0)\, k$$

Now we prove that

$$1 \notin \sigma(B) \tag{19}$$

For this, assume for contradiction that $1 \in \sigma(B)$, then there exists a vector $k \neq 0$ such that $Bk = k$. We have that $l(k) = 0$, in fact if $l(k) \neq 0$ then $w = k/l(k)$ would be a solution of Eq. (18). Hence

$$k = (F_0)'_w(T_0, \hat{w}_0)\, k$$

and by assumption (b) we get $k = \eta e_0$ for some $\eta \neq 0$.

On the other hand $0 = l(k) = \eta l(e_0)$ which is a contradiction. Therefore (19) holds and we have (see, e.g., [9])

$$|\operatorname{ind}(\hat{w}_0, \hat{w}_0 + B(w - \hat{w}_0))| = 1$$

Consider now for $\lambda \in [0, 1]$ the homotopy

$$\begin{aligned} H_\lambda(w) = w - \hat{w}_0 - (F_0)'_w\,(T_0 + \lambda l(w - \hat{w}_0), \hat{w}_0)(w - \hat{w}_0) \\ - (F_0)'_T\,(T_0, \hat{w}_0)\, l(w - \hat{w}_0) - \lambda\omega_1(T(w), w - \hat{w}_0) \\ - \lambda\omega_2(l(w - \hat{w}_0)) \end{aligned}$$

We prove that H_λ is an admissible homotopy between $H_0(w) = w - \hat{w}_0 - B(w - \hat{w}_0)$ and $H_1(w) = w - \Gamma_0(w)$ on and in the sphere $S(\hat{w}_0, \rho) = \{w \in C_{T_0}(H^1(\Omega) \times L^2(\Omega)) \colon \|w - \hat{w}_0\|_{C_{T_0}} = \rho\}$ for ρ sufficiently small.

Since the map $I - H_\lambda$ is compact with respect to both the variables λ, u we have only to prove that there exists $\rho_0 > 0$ such that $H_\lambda(w) \neq 0$ for any $\lambda \in [0, 1]$, $w \in S(\hat{w}_0, \rho)$ and $\rho \in [0, \rho_0]$. Assume the contrary, then there exist sequences $\{\rho_n\}$, $\{\lambda_n\}$, $\{w_n\}$ such that $\rho_n \to 0$, $\lambda_n \in [0, 1]$, $w_n \in S(\hat{w}_0, \rho_n)$ and $H_{\lambda_n}(w_n) = 0$. Let $\zeta_n = w_n - \hat{w}_0$ and $e_n = \zeta_n / \|\zeta_n\|$, we have

$$\begin{aligned} e_n = (F_0)'_w\,(T_0 + \lambda_n l(\zeta_n), \hat{w}_0)\, e_n + (F_0)'_T\,(T_0, \hat{w}_0)\, l(e_n) \\ + \lambda_n(\omega_1(T_0 + \lambda_n l(\zeta_n), \zeta_n)/\|\zeta_n\| + \omega_2(l(\zeta_n))/\|\zeta_n\|) \end{aligned}$$

Since $\{e_n\}$ is compact in $C_{T_0}(H^1(\Omega) \times L^2(\Omega))$, without loss of generality, we can assume that $e_n \to e_0$ and passing to the limit in the previous equation we obtain

$$e_0 = (F_0)'_w\,(T_0, \hat{w}_0)\, e_0 + (F_0)'_T\,(T_0, \hat{w}_0)\, l(e_0)$$

which is a contradiction with (19), since $\|e_0\| = 1$. This concludes the proof with $T(w) = T_0 + l(w - \hat{w}_0)$. □

The following result provides a sufficient condition to ensure (c) of Lemma 1.

Lemma 2. *Assume that the linearized reduced equation*

$$w = (F_0)'_w\,(T_0, \hat{w}_0)\, w \tag{17}$$

does not possess any solution of the form $w = \binom{y}{y_t} + (t/T_0)\binom{z_t}{z_{tt}}$, *where* y *is* T_0*-periodic and* $\binom{z}{z_t} = \hat{w}_0$. *Then assumption (c) of Lemma 1 is verified.*

Proof. We argue by contradiction, therefore let $\tilde{w} = \tilde{w}(t)$ be a solution of the equation

$$(F_0)'_T(T_0, \hat{w}_0) = w - (F_0)'_w(T_0, \hat{w}_0)\, w$$

then $\zeta(T) \triangleq \hat{w}_0 + (T - T_0)\,\tilde{w}$ satisfies the equation

$$\zeta(T) = J_0(T)\, f_0(T, \zeta(T)) + \Delta(T, T_0)$$

where $\Delta(T, T_0)/T - T_0 \to 0$ as $T \to T_0$.

In fact

$$\begin{aligned}\Delta(T, T_0) &= -J_0(T)\, f_0(T, \hat{w}_0 + (T - T_0)\,\tilde{w}) + \hat{w}_0 + (T - T_0)\,\tilde{w} \\ &= -J_0(T)[f_0(T, \hat{w}_0 + (T - T_0)\,\tilde{w}) - f_0(T, \hat{w}_0)] \\ &\quad - J_0(T)[f_0(T, \hat{w}_0) - f_0(T_0, \hat{w}_0)] - [J_0(T) - J_0(T_0)]\, f_0(T_0, \hat{w}_0) \\ &\quad + (T - T_0)\,\tilde{w} \\ &= -(T - T_0)[J_0(T)(f_0)'_w(T, \hat{w}_0)\,\tilde{w} \\ &\quad + J_0(T)(f_0)'_T(T_0, \hat{w}_0) + (J_0)'_T(T_0)\, f_0(T_0, \hat{w}_0) - \tilde{w}] + o(T - T_0)\end{aligned}$$

Consider now a sequence of continuous functions $\{w_n\} \subset D(A_0)$ such that $w_n(t) \rightrightarrows \tilde{w}(t)$ (where the uniformity is in $t \in [0, T_0]$), and where $A_0 = \begin{pmatrix} 0 & -I \\ -\Delta_x + \alpha & \beta \end{pmatrix}$. Then $\hat{w}_0 + (T - T_0)\, w_n$ satisfies the equation

$$\begin{aligned}&\frac{d\hat{w}_0}{dt} + (T - T_0)\frac{dw_n}{dt} + \frac{T}{T_0} A_0 \hat{w}_0 + \frac{T}{T_0}(T - T_0)\, A_0 w_n \\ &\quad = \frac{T}{T_0} f_0(\hat{w}_0 + (T - T_0)\, w_n) + \delta_n(T, T_0)\end{aligned}$$

where δ_n is defined by the above relation. From this we have

$$\hat{w}_0 + (T - T_0)\, w_n = J_0(T)\, f_0(T, \hat{w}_0 + (T - T_0)\, w_n) + J_0(T)\,\delta(T, T_0)$$

Therefore

$$J_0(T)\,\delta_n(T, T_0) \to \Delta(T, T_0) \qquad \text{as} \quad n \to \infty$$

Since

$$\frac{d\hat{w}_0}{dt} + A_0 \hat{w}_0 = f_0(\hat{w}_0)$$

w_n satisfies the equation

$$\frac{dw_n}{dt}+\frac{T}{T_0}A_0w_n$$

$$=-\frac{1}{T_0}A_0\hat{w}_0+\frac{T}{T_0(T-T_0)}(f_0(\hat{w}_0+(T-T_0)w_n)-f_0(\hat{w}_0))$$

$$+\frac{1}{T_0}f_0(\hat{w}_0)+\frac{1}{T-T_0}\delta_n(T,T_0) \tag{20}$$

Thus

$$w_n=J_0(T)\left(\frac{1}{T-T_0}(f_0(T,\hat{w}_0+(T-T_0)w_n)-f_0(T,\hat{w}_0))\right)$$

$$+J_0(T)\frac{1}{T_0}\frac{d\hat{w}_0}{dt}+\frac{1}{T-T_0}J_0(T)\delta_n(T,T_0)$$

By passing to the limit as $n\to\infty$ we obtain

$$\tilde{w}=J_0(T)\left(\frac{1}{T-T_0}(f_0(T,\hat{w}_0+(T-T_0)\tilde{w})-f_0(T,\hat{w}_0))\right)$$

$$+J_0(T)\frac{1}{T_0}\frac{d\hat{w}_0}{dt}+\frac{1}{T-T_0}\Delta(T,T_0)$$

By letting $T\to T_0$ we get

$$\tilde{w}=J_0(T_0)(f_0)'_w(T_0,\hat{w}_0)\,\tilde{w}+J_0(T_0)\frac{1}{T_0}\frac{d\hat{w}_0}{dt}$$

On the other hand $d\hat{w}_0/dt=\hat{w}'_0$ satisfies (17), hence it is easy to show that

$$\frac{t}{T_0}\hat{w}'_0=J_0(T_0)(f_0)'_w(T_0,\hat{w}_0)\left(\frac{t}{T_0}\hat{w}'_0\right)+J_0(T_0)\frac{1}{T_0}\hat{w}'_0$$

and so $w(t)=-\tilde{w}(t)+(t/T_0)\,\hat{w}'_0(t)$ is a solution of (17), contradicting the assumption. □

4. THE MAIN RESULT

We are now in the position to prove the main result.

Theorem 1. *Suppose that the equation*

$$\frac{\partial^2 u}{\partial t^2} = \Delta_x u - \beta \frac{\partial u}{\partial t} - \alpha u + g(x, 0, u)$$

together with

$$\frac{\partial u}{\partial \nu} = 0 \qquad on\ \partial\Omega$$

has a T_0-periodic solution $z = z(t, x)$ in the classical sense such that the linearized equation

$$\frac{\partial^2 \varphi}{\partial t} = \Delta_x \varphi - \beta \frac{\partial \varphi}{\partial t} - \alpha\varphi + g_u(x, 0, z(t, x))\, \varphi \tag{21}$$

has no T_0-periodic solutions which are linearly independent of z_t. Furthermore, we suppose that (21) does not possess any solution of the form:

$$y(t, x) + \frac{t}{T_0}\, z_t(t, x)$$

where y is T_0-periodic with respect to t.

Then there exists $\varepsilon^0 > 0$ such that for all $\varepsilon \in (0, \varepsilon^0)$ problem (1)–(2) has a T_ε-periodic solution u^ε with $T_\varepsilon \to T_0$ and

$$\left\| \begin{pmatrix} \tilde{u}^\varepsilon \\ \tilde{u}^\varepsilon_t \end{pmatrix} - \mathscr{J} \begin{pmatrix} z \\ z_t \end{pmatrix} \right\|_{C_{T_0}(Y^1_\varepsilon)} \to 0 \qquad as \quad \varepsilon \to 0$$

where $\tilde{u}^\varepsilon(t) = u^\varepsilon((T_\varepsilon / T_0)\, t)$.

Proof. The proof is organized in two steps.

I° step. By Lemmas 1 and 2 it follows that $\hat{w}_0 = \binom{z}{z_t}$ is an isolated fixed point in $C_{T_0}(H^1(\Omega) \times L^2(\Omega))$ of the operator $\Gamma_0(w) = F_0(T(w), w)$ with $T(w_0) = T_0$ and topological index different from zero, hence there exists a bounded neighborhood $V \subset C_{T_0}(H^1(\Omega) \times L^2(\Omega))$ of $\hat{w}_0$ such that Γ_0 has no fixed points on the boundary of V. Then it is not hard to prove that the operators $\Gamma_\varepsilon^0(\cdot) = \mathscr{J} J_0(T(\cdot))\, \mathbf{P} f_\varepsilon(T(\cdot), \cdot)$ and $\mathscr{J}\Gamma_0 \mathbf{P}$ are homotopic on $\overline{\mathscr{J}V}$. Therefore for $r > 0$ sufficiently small we have

$$\begin{aligned}&\operatorname{ind}_{\mathscr{J} C_{T_0}(H^1(\Omega) \times L^2(\Omega))}(\Gamma_\varepsilon^0, B_r(\mathscr{J}V) \cap \mathscr{J} C_{T_0}(H^1(\Omega) \times L^2(\Omega)))\\ &\quad = \operatorname{ind}_{\mathscr{J} C_{T_0}(H^1(\Omega) \times L^2(\Omega))}(\mathscr{J}\Gamma_0 \mathbf{P}, \mathscr{J}V)\end{aligned}$$

where $B_r(\mathscr{J}V) = \{z \in Y_\varepsilon^1 : \operatorname{dist}(z, \mathscr{J}V) < r\}$.

II° step. This is the critical step of the proof. It consists in proving the existence of $r_0 > 0$ such that, for any fixed $r \in (0, r_0]$ there corresponds $\varepsilon_r > 0$ with the property that if $0 < \varepsilon \leqslant \varepsilon_r$ then the operators $\Gamma_\varepsilon(\cdot) \triangleq J_\varepsilon(T(\mathbf{P}(\cdot)))\, f_\varepsilon(T(\mathbf{P}(\cdot), \cdot)) = F_\varepsilon(T(\mathbf{P}(\cdot), \cdot))$ and Γ_ε^0 are linearly homotopic on $B_r(\mathscr{J}V)$.

We give a substantive outline of the proof of this part, for the technical details we refer to the proof of Proposition 3 in [8]. First of all we fix $r_0 > 0$ such that the operator Γ_0 has no fixed points on the set:

$$\mathbf{P}[B_r(\mathscr{J}\partial V) \cap \mathscr{J} C_{T_0}(H^1(\Omega) \times L^2(\Omega))]$$

for all $0 \leqslant r \leqslant r_0$. Then we argue by contradiction, that is, for fixed $r \in (0, r_0]$ we suppose that there exist sequences $\{\lambda_n\} \subset [0, 1]$, $\{w_n\} \subset \partial B_r(\mathscr{J}V) \subset Y_\varepsilon^1$ and $\{\varepsilon_n\} \subset \mathbf{R}_+$ such that $\lambda_n \to \lambda_0$, $\varepsilon_n \to 0$ and

$$w_n = (1 - \lambda_n)\, \Gamma_{\varepsilon_n}(w_n) + \lambda_n \Gamma_{\varepsilon_n}^0(w_n) \tag{22}$$

where $w_n(t) = \binom{u_n(t)}{u_{nt}(t)}$.

Now we observe that the sequence u_n is uniformly bounded in $C_{T_0}(X_{\varepsilon_n}^1)$, and so the set $\{u_n(t) : n \in \mathbf{N}, t \in [0, t_0]\}$ lies in a fixed compact subset of $L^p(Q)$, with $p \geqslant 2(\theta + 1)$ if $N = 1$ or $p \in [2(\theta + 1), 2N + 2/N - 1)$ if $N \geqslant 2$. Therefore there is a fixed compact set $K \subset L^2(Q)$ such that

$$\varphi_{\varepsilon_n}(u_n)(t) \in K \tag{23}$$

for all $n \geqslant 1$ and all $0 \leqslant t \leqslant T$, where

$$\varphi_{\varepsilon_n}(u_n)(t)(x, y) = g(x, \varepsilon_n y, u_n(t, x, y))$$

Observe that the sequence $T_n \triangleq T(\mathbf{P}(w_n))$ is bounded in $\mathbf{R}$. Without loss of generality we assume that $T_n \to T^*$. Recall that

$$f_{\varepsilon_n}(t_n, w_n) = \frac{T_n}{T_0} f_{\varepsilon_n}(w_n)$$

where

$$f_{\varepsilon_n} = \begin{pmatrix} 0 \\ \varphi_{\varepsilon_n}(u_n) \end{pmatrix}$$

We introduce now cutoff functions $\chi_m: \mathbf{R} \to \mathbf{R}$, $m \geqslant 1$ such that $\chi_m \in C^\infty(\mathbf{R})$, $\chi'_m(u)$ is uniformly bounded with respect to m and u, $0 \leqslant \chi_m(u) \leqslant 1$ for all $m \geqslant 1$ and $u \in \mathbf{R}$ and

$$\chi_m(u) = \begin{cases} 1, & |u| \leqslant m \\ 0, & |u| \geqslant m+1 \end{cases}$$

For any $m, n \geqslant 1$ we define the operators

$$\varphi^m_{\varepsilon_n}(u)(t)(x, y) = \chi_m(u(t, x, y))\, g(x, \varepsilon_n y, u(t, x, y))$$

$$f^m_{\varepsilon_n} = \begin{pmatrix} 0 \\ \varphi^m_{\varepsilon_n}(u) \end{pmatrix}$$

and functions $w^m_n = \binom{u^m_n}{v^m_n}$ by the relation

$$w^m_n = (1-\lambda_n)\, J_{\varepsilon_n}(T_n)\, f^m_{\varepsilon_n}(T_n, w_n) + \lambda_n \mathscr{J} J_0(T_n)\, \mathbf{P} f^m_{\varepsilon_n}(T_n, w_n) \tag{24}$$

We rewrite (22) in the form

$$\begin{aligned} w^m_n &= (1-\lambda_n)\, J_{\varepsilon_n}(T_n)\, f^m_{\varepsilon_n}(T_n, w_n) + \lambda_n \mathscr{J} J_0(T_n)\, f^m_{\varepsilon_n}(T_n, w_n) \\ &\quad + (1-\lambda_n)\, I^{nm} + \lambda_n I^{nm}_0 + (w^m_n - w_n) \end{aligned} \tag{25}$$

where

$$I^{nm} = J_{\varepsilon_n}(T_n)(f_{\varepsilon_n}(T_n, w_n) - f^m_{\varepsilon_n}(T_n, w_n))$$

$$I^{nm}_0 = \mathscr{J} J_0(T_n)\, \mathbf{P}(f_{\varepsilon_n}(T_n, w_n) - f^m_{\varepsilon_n}(T_n, w_n))$$

If we apply $\mathbf{P}$ and $(I - \mathbf{P})$ to (25) and use the fact that these projectors and the semigroups $U_\varepsilon(t)$, $U_0(t)$ commute we obtain

$$\begin{aligned} \mathbf{P} w^m_n &= J_0(T_n)\, \mathbf{P} f_{\varepsilon_n}(T_n, \mathbf{P} w^m_n) + J_0(T_n)\, \mathbf{P}(f_{\varepsilon_n}(T_n, w^m_n) \\ &\quad - f_{\varepsilon_n}(T_n, \mathbf{P} w^m_n)) + J_0(T_n)\, \mathbf{P}(f^m_{\varepsilon_n}(T_n, w_n) - f_{\varepsilon_n}(T_n, w^m_n)) \\ &\quad + (1-\lambda_n)\, \mathbf{P} I^{nm} + \lambda_n \mathbf{P} I^{nm}_0 - \mathbf{P}(w_n - w^m_n) \end{aligned} \tag{26}$$

and

$$
\begin{aligned}
(I-\mathbf{P})\, w_n^m = {} & (1-\lambda_n)(I-\mathbf{P})\, J_{\varepsilon_n}(T_n)(I-\mathbf{P})\, f_{\varepsilon_n}^m(T_n, \mathbf{P}w_n^m) \\
& + (1-\lambda_n)(I-\mathbf{P})\, J_{\varepsilon_n}(T_n)(I-\mathbf{P})(f_{\varepsilon_n}^m(T_n, w_n^m) - f_{\varepsilon_n}^m(T_n, \mathbf{P}w_n^m)) \\
& + (1-\lambda_n)(I-\mathbf{P})\, J_{\varepsilon_n}(T_n)(I-\mathbf{P})(f_{\varepsilon_n}^m(T_n, w_n) - f_{\varepsilon_n}^m(T_n, w_n^m)) \\
& + (1-\lambda_n)(I-\mathbf{P})\, I^{nm} + (I-\mathbf{P})(w_n^m - w_n) \qquad (27)
\end{aligned}
$$

Now, using (23), it is possible to prove (see [8, Proposition 3]) that

$$\|f_{\varepsilon_n}^m(T_n, w_n) - f_{\varepsilon_n}(T_n, w_n)\|_{C_T(Y_{\varepsilon_n}^1)} \rightrightarrows 0 \qquad (28)$$

$$\|w_n - w_n^m\|_{C_T(Y_{\varepsilon_n}^1)} \rightrightarrows 0 \qquad (29)$$

$$\|f_{\varepsilon_n}^m(T_n, w_n) - f_{\varepsilon_n}^m(T_n, w_n^m)\|_{C_T(Y_{\varepsilon_n}^1)} \rightrightarrows 0 \qquad (30)$$

when $m \to \infty$ uniformly with respect to $n \geqslant 1$.

Moreover we have that (8, Proposition 3])

$$\|f_{\varepsilon_n}(T_n, w_n^m) - f_{\varepsilon_n}(T_n, \mathbf{P}w_n^m)\|_{C_T(Y_{\varepsilon_n}^1)} \rightrightarrows 0 \qquad (31)$$

and

$$\|(I-\mathbf{P})\, f_{\varepsilon_n}^m(T_n, \mathbf{P}w_n^m)\|_{C_T(Y_{\varepsilon_n}^1)} \rightrightarrows 0 \qquad (32)$$

when $n \to \infty$ uniformly with respect to $m \geqslant 1$.

We can now extract a subsequence $\{\tilde{w}_q \mid q \geqslant 1\}$ from $\{w_n^m \mid n \geqslant 1, m \geqslant 1\}$ in the following way. First, fix $n_1 = m_1 = 1$. Assuming that m_q and n_q have been chosen, we pick $m_{q+1} > m_q$ in such a way that

$$\|\mathbf{P}I^{nm_{q+1}}\|, \qquad \|\mathbf{P}I_0^{nm_{q+1}}\|, \qquad \|\mathbf{P}(w_n - w_n^{m_{q+1}})\|$$

$$\|(I-\mathbf{P})\, I^{nm_{q+1}}\|, \qquad \|(I-\mathbf{P})(w_n - w_n^{m_{q+1}})\|$$

$$\|(I-\mathbf{P})\, J_{\varepsilon_n}(T_n)(I-\mathbf{P})(f_{\varepsilon_n}^{m_{q+1}}(T_n, w_n) - f_{\varepsilon_n}^{m_{q+1}}(T_n, w_n^{m_{q+1}}))\|$$

$$\|J_0(T_n)\, \mathbf{P}(f_{\varepsilon_n}(T_n, w_n) - f_{\varepsilon_n}(T_n, w_n^{m_{q+1}}))\|$$

are all less than 2^{-q-1} for all $n \geqslant 1$. Here the norms are in $C_T(H^1(\Omega) \times L^2(\Omega))$ respectively $C_T(Y_\varepsilon^1)$. This number exists because of (8), (28), (29) and (30).

Having fixed m_{q+1} we choose $n_{q+1} > n_q$ in such a way that

$$\|J_0(T_n)\,\mathbf{P}(f_{\varepsilon_{n_{q+1}}}(T_{n_{q+1}}, w^{m_{q+1}}_{n_{q+1}}) - f_{\varepsilon_{n_{q+1}}}(T_{n_{q+1}}, \mathbf{P}w^{m_{q+1}}_{n_{q+1}}))\|,$$

$$\|(I-\mathbf{P})\,J_{\varepsilon_{n_{q+1}}}(T_{n_{q+1}})(I-\mathbf{P})(f^{m_{q+1}}_{\varepsilon_{n_{q+1}}}(T_{n_{q+1}}, w^{m_{q+1}}_{n_{q+1}}) - f^{m_{q+1}}_{\varepsilon_{n_{q+1}}}(T_{n_{q+1}}, \mathbf{P}w^{m_{q+1}}_{n_{q+1}}))\|,$$

$$\|(I-\mathbf{P})\,J_{\varepsilon_{n_{q+1}}}(T_{n_{q+1}})(I-\mathbf{P})\,f^{m_{q+1}}_{\varepsilon_{n_{q+1}}}(T_{n_{q+1}}, \mathbf{P}w^{m_{q+1}}_{n_{q+1}})\|$$

are all less than 2^{-q-1}. Here the norms are in $C_T(H^1(\Omega)\times L^2(\Omega))$ respectively $C_T(Y^1_\varepsilon)$. This choice of n_{q+1} is possible because of (8), (31) and (32). Put $\tilde{w}_q = w^{m_q}_{n_q}$.

Summarizing from (26), (27) we obtain

$$\|(I-\mathbf{P})\,\tilde{w}_q\|_{C_T(Y^1_\varepsilon)} \to 0 \tag{33}$$

$$\|\mathbf{P}\tilde{w}_q - J_0(T_{n_q})\,\mathbf{P}f_{\varepsilon_{n_q}}(T_{n_q}, \mathbf{P}\tilde{w}_q)\|_{C_T(H^1(\Omega)\times L^2(\Omega))} \to 0 \tag{34}$$

when $q\to\infty$.

On the other hand, the set $\{\mathbf{P}\tilde{u}_q \mid q \geqslant 1\}$ belongs to a fixed compact subset of $L^p(\Omega)$. By (24) we have that $\{\mathbf{P}\tilde{w}_q : q\geqslant 1\}$ is relatively compact in $C_T(H^1(\Omega)\times L^2(\Omega))$ and so, by passing to a subsequence if necessary, we obtain

$$\mathbf{P}\tilde{w}_q \to w^*$$

when $q\to\infty$. By (29) we have $\mathbf{P}w_{n_q} \to w^*$ as $q\to\infty$ in $C_T(H^1(\Omega)\times L^2(\Omega))$.

In conclusion, from (33), (34) together the fact that

$$\|\mathbf{P}f_{\varepsilon_{n_q}}(T_{n_q}, \mathbf{P}\tilde{w}_q) - f_0(T^*, w^*)\|_{C_T(H^1(\Omega)\times L^2(\Omega))} \to 0$$

as $q\to\infty$, we get

$$w^* = J_0(T(w^*))\,f_0(T(w^*), w^*)$$

where $w^*\in\partial B_r(V)$ and $T(w^*) = T^*$. This contradicts the choice of r, thus the theorem is proved. □

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