CANADIAN APPLIED
MATHEMATICS QUARTERLY
Volume 6, Number 2, Summer 1998

ON THE DYNAMICAL BEHAVIOR OF THE APPROXIMATE EQUIVALENT CONTROL IN SLIDING MANIFOLD SYSTEMS

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ABSTRACT. The aim of this paper is to study the dynamical stability properties of a suitably defined approximation of the equivalent control in sliding manifold control systems. For this we use both the classical theory of singularly perturbed dynamical systems and the theory of twist maps. Examples of control problems which possess dynamical stability properties are presented. This paper continues the investigation of the problem of control design, via singular perturbation theory, for control systems involving sliding manifolds.

1. Introduction.

The present paper is a further contribution to the singular perturbation approach to sliding manifold systems. Such systems are of importance in a wide range of applied problems, see the standard texts [**20, 21**] for an introduction to the subject. The singular perturbation approach was developed in [**6, 8, 13**]. Its goal is to obtain a smooth approximation in the uniform norm to the so-called equivalent control [**2, 20**], which does not exhibit the chattering phenomenon and which has good robustness properties.

Our purpose here is to discuss the dynamical stability properties of what we will call the approximate equivalent control, hereafter abbreviated to *a.e. control.* This is an important question when one tries to realize the a.e. control in applications. See the papers [**5, 7, 9, 10, 11**], where in fact the question of dynamical stability was sidestepped in an effective and elegant way by introducing an adjustment term which depends on the initial state of the dynamical system, in the a.e. control. Here we wish to meet the issue directly, since the initial state is not always accessible. We will discuss simple but fairly realistic examples of control problems which illustrate the following points.

Received by the editors on October 18, 1996.
Key words and phrases. Singularly perturbed systems, sliding manifolds, equivalent control, twist maps.
1991 AMS *Mathematics Subject Classifications.* 54H20, 34D15, 34H05.
Research partially supported by the national research project M.U.R.S.T. "Teoria dei Sistemi e del Controllo."

(i) In certain situations dynamical stability, in a sense which we will define, can be achieved. Roughly speaking, the a.e. control holds initial states near the sliding manifold in a prescribed neighborhood of the sliding manifold, without introducing adjustment terms.

(ii) However, the presence or absence of dynamical stability tends to be sensitively related to the structure of the control process.

We will present our results in detail in the body of the paper. At this point we pause to discuss in more detail some of the terminology used above and to provide some background material which will help put our paper in perspective.

In control problems involving sliding manifolds, a classical controller design is carried out, using feedback switching controls. The controller steers a general state vector to a fixed manifold (the sliding manifold) and holds it there. The sliding manifold is conventionally viewed as a level set of a smooth function. Control systems with a sliding manifold typically possess very good properties such as stable behavior, accurate tracking, robust performance, and insensitivity with respect to disturbances and variation of plant parameters. In regard to the question of steering state vectors to the sliding manifold, the notions of equivalent control and the related "approximability property" have direct physical meaning [**2, 20**].

One of the main drawbacks of the classical sliding mode control design is the chattering phenomenon. Several attempts have been made to eliminate or at least to reduce chattering. See, e.g., [**3, 18, 19**].

In [**13**] a different approach to solve sliding manifold control problems was introduced. This approach is based on the consideration of a class of feedback controls defined using the theory of singularly perturbed ordinary differential equations. These controls are smooth. They depend on a small parameter $\varepsilon > 0$, and are approximations, in the uniform topology, to the equivalent control except for an arbitrarily small transient. (We caution the reader that the approximations to the equivalent control which we are now discussing do not necessarily coincide with the a.e. controls to be discussed later; however, the two concepts of approximation to the equivalent control are closely related.)

Returning to the approximating controls now under discussion, we observe that the corresponding states do not, in general, remain on the sliding manifold even if they initiate there. However, given some finite time interval, we can do the following: for any prescribed neighborhood of the sliding manifold, we can determine values of the parameter ε for which the corresponding trajectories belong to that neighborhood for times in the interval. Furthermore, using such controls eliminates chattering and eliminates the necessity of considering Filippov solutions of the dynamical system.

A drawback to these approximating controls is illustrated in the papers [**5, 7, 8, 11**]. There it is seen that, to render the controls effective in solving linear and nonlinear tracking problems, it is necessary to introduce an additional term in the control design so as to render the linear tracking dynamics exponentially stable. This term has the form

$$e^{Ct}[\hat{x}(0) - x(0)], \quad 0 \leq t < \infty,$$

where C is a stable symmetric matrix and $\hat{x}(0), x(0)$ are initial conditions of the reference model and the state vector, respectively. this means that both the sliding manifold and the resulting approximation to the equivalent control depend on the initial state of the system.

Now, precise knowledge of the initial state is not always available. Thus it may be convenient to guarantee the stability of the tracking error without using any information on the initial state, if that is possible. In this case the sliding manifold will not depend on the initial condition. Indeed, it will coincide with the reference trajectory. Furthermore, the resulting control will not be a high gain control. One of the goals of this paper is to present examples where the stability of the tracking error obtains. In fact, we will use the theory of twist maps to prove the stability of a reference trajectory represented by a closed curve.

We remark parenthetically that such reference trajectories occur in the control of rigid robot manipulators and in the control of satellites. Possible applications of our results to these areas is under study. One possibility is to combine the proposed control technique with other existing techniques. For instance, one could first employ the classical variable structure control design, based on a suitable Liapunov function, in order to bring a general state vector to a convenient neighborhood of the reference model. Then our results can be used to keep the state within this neighborhood.

We finish this introduction by noting that the paper contains four subsequent sections. In Section 2 we describe, in a preliminary fashion, the examples which we will study, introducing basic concepts and definitions as we go along. In the third section we present some material concerning the theory of twist maps. We will prove the existence of invariant curves under an hypothesis slightly weaker than the standard area-preserving condition. Our proof relies on basic methods of K-A-M stability theory, especially on the proof of the invariant curve theorem for monotone twist maps given in Siegel-Moser [**17**]. Then, in Section 4, we discuss in detail the dynamical stability of the a.e. control in a class of examples. Finally, in Section 5, we discuss briefly how to find examples in which the a.e. control is dynamically unstable. As we will see, such examples are easy to construct. The lesson seems to be that, to obtain dynamical stability, one must design the controller rather carefully.

2. Preliminaries. We begin with the nonlinear control system

$$\begin{aligned} x_1' &= f_1(t, x_1, x_2) \\ x_2' &= f_2(t, x_1, x_2) \end{aligned} \tag{2.1}$$

where x_1 and x_2 are scalar state variables, u is a scalar control lying in some control set $U \subset \mathbf{R}$, and the functions f_1, f_2 are analytic in all arguments and are one-periodic with respect to $t : f_i(t+1, \cdot) = f_i(t, \cdot)$, $i = 1, 2$. A *sliding manifold* S for (2.1) is by definition a level set $\{x \in \mathbf{R}^2 \mid s(x) = s_0\}$ of a smooth function $s : \mathbf{R}^2 \to \mathbf{R}$. In our work we will consider the simple case $s(x_1, x_2) = x_2$. Thus, S is simply a line parallel to the x_1-axis. Of course, a smooth curve $S \subset \mathbf{R}^2$ can be locally represented in this way by an appropriate smooth function s, but this fact will not be of importance in what follows.

We remark parenthetically that a sliding manifold S for a general finite-dimensional control process $x' = f(t, x, u)$, $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, is a level set $\{x \in \mathbf{R}^n \mid s(x) = s_0\}$ of a smooth function $s : \mathbf{R}^n \to \mathbf{R}^k$, $k \leq n$. Sometimes it is convenient to let s depend on t as well, so that $S = \{(t, x) \mid s(t, x) = s_0\} \subset \mathbf{R} \times \mathbf{R}^n$. See [**8, 13**].

Returning to equations (2.1), we pose the standard problem of steering a given initial vector $(x_1, x_2) \in \mathbf{R}^2$ to (a neighborhood of) the sliding manifold S and holding it there with a control of feedback type: $u = u(t, x_1, x_2)$. This problem can be confronted as follows [**13**]. Introduce a small parameter ε, and consider the singular perturbation problem

$$
\begin{aligned}
\varepsilon u' &= \operatorname{grad} s \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = f_2(t, x_1, x_2, u) \\
x_1' &= f_1(t, x_1, x_2, u) \\
x_2' &= f_2(t, x_1, x_2, u).
\end{aligned} \tag{2.2}
$$

We suppose that the equation $0 = f_2(t, x_1, x_2, u)$ is uniquely solvable for $u \in U$ as a function $u = u_0(t, x_1, x_2)$ in some domain $\mathbf{R} \times D$, where to be specific we take $D \subset \mathbf{R}^2$ to be a disc centered at $(0, 0)$. We suppose that $u_0(t, x_1, x_2)$ is analytic in (t, x_1, x_2), and that it is one-periodic as a function of t. The function $u_0(t, x_1, x_2)$ is by definition the *equivalent control* relative to which we will study the stability properties of the sliding manifold S.

Let us now impose the *Tychonov condition*; we suppose that

$$
\frac{\partial f_2}{\partial u}(t, x_1, x_2, u_0(t, x_1, x_2)) < 0 \tag{2.3}
$$

for each $(t, x_1, x_2) \in \mathbf{R} \times D$. It is then a standard fact that, for small values of ε, there is a locally invariant manifold M_ε for equations (2.2) which is a graph:

$$
M_\varepsilon = \{(t, x_1, x_2, u_\varepsilon(t, x_1, x_2)) \mid (t, x_1, x_2) \in \mathbf{R} \times D\}.
$$

Our assumptions imply that $(\varepsilon, t, x_1, x_2) \to u_\varepsilon(t, x_1, x_2)$ is of class C^r in some neighborhood $(-\varepsilon_0, \varepsilon_0) \times \mathbf{R} \times D$ of $\{\varepsilon = 0\} \times \mathbf{R} \times D$, see [**4**]. However, this function need not be analytic.

Definition 2.1. The *approximate equivalent control* (a.e. control) corresponding to (2.2) is the function u_ε defined above.

We pause to point out the difference between our approximate equivalent control, or a.e. control, and the approximating controls considered in [**5, 7, 8, 11, 13**]. In those papers, the control $u_\varepsilon(t)$ corresponding to a given state vector $(x_1(0), x_2(0))$ was obtained by solving (2.2) with $\varepsilon \neq 0$ and $u_\varepsilon(0)$ equal to some convenient value u_0. It was not required that $u_0 = u_\varepsilon(0, x_1(0), x_2(0))$, so in general $u_\varepsilon(t) \neq u_\varepsilon(t, x_1(t), x_2(t))$. (In other words, $u_\varepsilon(t)$ does not necessarily lie on the manifold M_ε.)

We introduce the period map P_ε defined by the equations (2.2). Thus, P_ε maps the initial point $(x_1, x_2, u_\varepsilon(0, x_1, x_2))$ into the point $(x_1(1), x_2(1), u_\varepsilon(1, x_1(1), x_2(1)))$, where (x_1, x_2) belongs to a subdomain D_1 of D. Clearly we can associate with P_ε the map

$$g_\varepsilon : D_1 \to D : (x_1, x_2) \to (x_1(1), x_2(1)).$$

Having done this, let $A \subset D_1$ be an annulus centered at the origin with coordinates $(x_1, x_2) = (\theta, r)$. That is, x_1 and x_2 are taken to be polar coordinates with the usual order reversed. We consider the one-parameter family of maps

$$g_\varepsilon : A \longrightarrow D, \quad \varepsilon > 0.$$

Note that the sliding manifold S is a circle $\{(\theta, r) \mid r = r_0, 0 \leq \theta < 2\pi\}$ which we take to lie in A.

The family $\{g_\varepsilon\}$ consists of so-called twist maps if a simple hypothesis is satisfied at $\varepsilon = 0$, of which more below. It is well known that such maps have a very rich structure. They have been studied in classical work by Poincaré [**16**], Birkhoff [**1**] and Siegel-Moser [**17**]. More recently, many authors have contributed to the theory of these maps, e.g., Herman [**12**], Le Calvez [**14**] and Mather [**15**].

Our interest here is not in the detailed structure of such maps. Rather, we will use a small portion of the theory of twist maps to study the dynamical stability of the a.e. control in a class of examples. In particular, we will determine a class of examples in which the dynamical stability is present.

3. Some facts about twist maps.

Let $A \subset \mathbf{R}^2$ be an annulus centered at the origin. We study a one-parameter family of analytic maps $\{g_\varepsilon\}$, where each g_ε maps A into $\mathbf{R}^2$. It is not required that g_ε maps A into itself. (The mappings g_ε introduced in Section 2 are not necessarily analytic, but it will turn out that this fact is of no relevance. We will for the moment identify the maps considered in this section with the maps g_ε constructed in Section 2.)

Return for a moment to the equations (2.2). Note that, at $\varepsilon = 0$, the circles $r =$ const. are invariant under g_0. Thus, A is foliated by invariant circles. This is obviously a degenerate structure. Roughly speaking, there is a well-developed theory of the behavior of the iterates of g_ε for small ε when

(i) g_ε is area-preserving: $m(g_\varepsilon(B)) = m(B)$ for each Borel set $B \subset A$,

(ii) g_ε is dissipative: $m(g_\varepsilon(B)) \leq cm(B)$ for each Borel set $B \subset A$, where $c < 1$.

We have written m for the Lebesgue measure on $\mathbf{R}^2$.

Our main interest will be in a situation where g_ε is "almost" area preserving. It turns out that, in the context of problem (2.2), it is not natural to require that g_ε preserves area in the strict sense. However, we will give examples in which g_ε is area preserving "to order ε^2." The purpose of this section is to develop some implications of this condition.

We begin with the following proposition.

Proposition 3.1. *Let $A \subset \mathbf{R}^2$ be an annulus centered at $r = 0$. Let $g_\varepsilon : A \to \mathbf{R}^2$ be a one-parameter family of analytic maps, defined for $0 \leq \varepsilon \leq \varepsilon_0$ and of class C^2 in ε. Suppose that $\{g_\varepsilon\}$ satisfies the following conditions:*

1. $g_0(r,\theta) = \begin{pmatrix} r \\ \theta+\alpha(r) \end{pmatrix}$ *where $\alpha'(r) > 0$, (monotone twist condition);*

2. *there is a constant $c_0 \geq 0$ such that, for each Borel set $B \subset A$ and each $0 \leq \varepsilon \leq \varepsilon_0$:*

$$m(g_\varepsilon(B)) = m(B)[1 + c(B,\varepsilon)\varepsilon^2], \tag{3.1}$$

where $|c(B,\varepsilon)| \leq c_0$;

3. *if $g_\varepsilon(r,\theta) = \begin{pmatrix} \rho_\varepsilon(r,\theta) \\ \psi_\varepsilon(r,\theta) \end{pmatrix}$, and if*

$$\rho_\varepsilon(r,\theta) = r + \varepsilon\rho_1(r,\theta) + O(\varepsilon^2)$$
$$\psi_\varepsilon(r,\theta) = \theta + \alpha(r) + \varepsilon\psi_1(r,\theta) + O(\varepsilon^2),$$

then the differential relation

$$\rho_1(s,\theta) = s'(\theta + \alpha(s)) \cdot \psi_1(s,\theta) \tag{3.2}$$

admits no 2π-periodic solution $s = R(\theta)$ which satisfies the additional condition:

$$R(\theta) = R(\theta + \alpha(R(\theta))). \tag{3.3}$$

If the above conditions hold and if $C \subset A$ is a simple, closed analytic curve which encircles the origin, parametrizable by θ, then there exists $0 < \varepsilon_1 \leq \varepsilon_0$ such that, if $0 \leq \varepsilon \leq \varepsilon_1$, then

$$g_\varepsilon(C) \cap C \neq \varnothing.$$

That is, C has the intersection property with respect to g_ε for $\varepsilon \in [0, \varepsilon_1]$.

Proof. Let $C : \theta \to R(\theta)$ be an analytic simple closed curve in A which encircles the origin, with $r = R(\theta)$. Suppose, to begin, that, for each $\theta \in [0, 2\pi]$,

$$\rho_\varepsilon(R(\theta), \theta) - R(\psi_\varepsilon(R(\theta), \theta)) = O(\varepsilon^2) \tag{3.4}$$

as $\varepsilon \to 0$. We show that the parametrization $s = R(\theta)$ of C satisfies the relations (3.2) and (3.3).

To see this, we write out the difference:

$$\begin{aligned}
&\rho_\varepsilon(R(\theta), \theta) - R(\psi_\varepsilon(R(\theta), \theta)) \\
&= \{R(\theta) + \varepsilon\rho_1(R(\theta), \theta) + \cdots\} - R(\theta + \alpha(R(\theta)) + \varepsilon\psi_1(R(\theta), \theta) + \cdots) \\
&= R(\theta) + \varepsilon\rho_1(R(\theta), \theta) - R(\theta + \alpha(R(\theta))) - \varepsilon R'(\theta + \alpha(R(\theta))) \cdot \psi_1(R(\theta), \theta) \\
&\quad + \cdots\cdots\cdots .
\end{aligned}$$

Then (3.4) implies that

$$\begin{aligned}
R(\theta) &= R(\theta + \alpha(R(\theta))), \qquad 0 \leq \theta \leq 2\pi, \\
\rho_1(R(\theta), \theta) &= R'(\theta + \alpha(R(\theta))) \cdot \psi_1(R(\theta), \theta).
\end{aligned}$$

These are just the relations (3.2) and (3.3).

We conclude that the relations (3.4) is not possible. It follows that there exists $\varepsilon(C) > 0$ such that, if $0 \leq \varepsilon \leq \varepsilon(C)$, then $g_\varepsilon(C) \cap C \neq \varnothing$. For, suppose that no such number $\varepsilon(C)$ exists. Then there exists a sequence $\varepsilon_n \to 0$ such that $g_{\varepsilon_n}(C) \cap C = \varnothing$. We can suppose without loss of generality that $g_{\varepsilon_n}(C)$ lies outside of C for each n.

Let $\theta_0 \in [0, 2\pi]$ be a point such that either

$$R(\theta_0) - R(\theta_0 + \alpha(R(\theta_0))) > 0$$

or

$$\frac{d}{d\varepsilon}[\rho_\varepsilon(R(\theta_0),\theta_0) - R(\psi_\varepsilon(R(\theta_0),\theta_0))]_{\varepsilon=0} \neq 0.$$

(The derivative is positive if $g_\varepsilon(C)$ lies outside of C.) By continuity of the derivative, there is a sector $\theta_1 < \theta < \theta_2$ on which either the first or the second quantity is bounded away from zero, say by $c_1 > 0$. Using hypothesis 1, we see that $m(\operatorname{int} g_{\varepsilon_n}(C) - \operatorname{int} C)$ is greater than $c_1 \Delta\theta$ or $c_1 \varepsilon_n \Delta\theta$. This contradicts hypothesis 2, so indeed $\varepsilon(C) > 0$ exists as stated. □

For appropriate classes of curves the positive number $\varepsilon(C)$ is independent of C. We consider such a class of curves in the following.

Corollary 3.2. *Assume the hypotheses of Proposition* 3.1. *Let* M *and* r_0 *be positive numbers. Let* $\mathcal{C}$ *be the class of all analytic,* 2π*-periodic curves* $\theta \to R(\theta)$ *such that:*

1. R *admits an analytic extension to the strip* $|\operatorname{Im}\theta| < r_0$;
2. $\sup_\theta |R(\theta)| \leq M$ *for all* θ *with* $|\operatorname{Im}\ \theta| < r_0$.

Then there exists ε_1 *independent of* $C \in \mathcal{C}$ *such that* $g_\varepsilon(C) \cap C \neq \varnothing$ *for all* $C \in \mathcal{C}$ *and* $0 \leq \varepsilon \leq \varepsilon_1$.

Proof. For each $C \in \mathcal{C}$, consider the vector

$$v_C = \left(\sup_{0\leq\theta\leq 2\pi} |R(\theta) - R(\theta + \alpha(R(\theta)))|, \right.$$
$$\left. \sup_{0\leq\theta\leq 2\pi} \left| \frac{d}{d\varepsilon}[\rho_\varepsilon(R(\theta),\theta) - R(\psi_\varepsilon(R(\theta),\theta))]_{\varepsilon=0} \right| \right).$$

This vector is not the zero vector for each $C \in \mathcal{C}$. Suppose for contradiction that there is a sequence C_n of curves in $\mathcal{C}$ such that $v_n = v_{C_n}$ tends to zero in $\mathbf{R}^2$. By the Cauchy theorem, we can assume that the functions R_n converge uniformly together with their derivatives in compact subsets of $|\operatorname{Im}\theta| < r_0$. Let R be the limit function and $C : \theta \to R(\theta)$ the corresponding curve. Clearly, $C \in \mathcal{C}$ and $v_C = 0$, a contradiction. This proves the corollary. □

Next we give a simple criterion which is sufficient to verify hypothesis 3 of Proposition 3.1.

Proposition 3.3. *Suppose that* $g_\varepsilon : A \to \mathbf{R}^2$ *is a one-parameter family of analytic maps, of class* C^2 *in* ε. *Suppose* $\alpha(r) = \alpha_0 + r$ *where* α_0 *is a constant. Suppose further that* $\rho_1(r, \theta)$ *does not reduce to a function of* r *alone in* A. *Then hypothesis* 3 *of Proposition* 3.1 *is satisfied.*

Proof. Suppose that $s = R(\theta)$ is a 2π-periodic solution of class C^1 of the relation (3.3). We then have

$$R(\theta) = R(\theta + \alpha_0 + R(\theta)).$$

Differentiating this relation, we obtain

$$R'(\theta) = R'(\theta + \alpha_0 + R(\theta))[1 + R'(\theta)].$$

Write

$$x = x(\theta) = R'(\theta + \alpha_0 + R(\theta)), \quad \theta \in \mathbf{R}. \tag{3.5}$$

Then

$$R'(\theta) = \frac{x(\theta)}{1 - x(\theta)}. \tag{3.6}$$

Now, the function $x \to x/(1-x)$ takes values in $(-1, \infty)$ for $-\infty < x < 1$, and takes values in $(-\infty, -1)$ for $x > 1$. Also, since $R(\theta)$ is 2π-periodic by assumption, the map $\theta \to \theta + \alpha_0 + R(\theta)$ maps $[0, 2\pi]$ onto an interval of length greater or equal to 2π. Combining these two observations, we see that $x(\theta) < 1$ for $0 \le \theta \le 2\pi$, since otherwise the function $R(\cdot)$ would have derivative less than -1 for all θ, and hence could not be 2π-periodic.

We conclude, then, that $-\infty < x(\theta) < 1$ for all values of θ, and hence by (3.5), $R'(\theta) < 1$ for all $\theta \in \mathbf{R}$. By virtue of (3.6), this implies that

$$\frac{x(\theta)}{1 - x(\theta)} < 1$$

for all values of θ. But then $x(\theta) < 1/2$ for all θ, and so $R'(\theta) < 1/2$ by using (3.5) again.

Now using (3.5) and (3.6) repeatedly and passing to the limit we finally conclude that $x(\theta) \leq 0$ for all $\theta \in \mathbf{R}$. Thus, by (3.5) again, $R'(\theta) \leq 0$ for all $\theta \in \mathbf{R}$. So we must have $s = R(\theta) = \text{const}$.

It follows that, if $s = R(\theta) = R_0$ is a 2π-periodic solution of (3.3) which also satisfies (3.2), then $\rho_1(r,\theta)$ is zero on the circle $r = R_0$. So, by analyticity, ρ_1 is a function of r alone in A. But this contradicts our hypothesis, so hypothesis 3 of Proposition 3.1 holds. □

Remark 3.4. We now relate the above remarks to the invariant curve theorem in [**17**]. The main observation is that the intersection property on the map $M : A \to \mathbf{R}^2$ considered in that proof:

$$MC \cap C \neq \varnothing,$$

need only be verified for a class of curves $\mathcal{C}$ satisfying the hypotheses of Corollary 3.2. This can be seen from [**17**, pages 241–242] and comparing the curves $\eta = \text{const.}$ introduced there with earlier estimates, especially those on pages 234–235. We can thus draw the following conclusion. Let $g_\varepsilon : A \to \mathbf{R}^2$ be a family of maps satisfying the hypotheses of Proposition 3.1 with $\alpha(r) = \alpha_0 + r$, and in addition suppose that $\rho_1(r,\theta)$ does not reduce to a function of r alone in A. Let c_0 and μ be given positive real numbers, and suppose that ω is an irrational number satisfying

$$|m\omega - n| \geq \frac{c_0}{m^\mu}$$

for all pairs of integers m, n. Such an irrational is called *diophantine*. Let $\delta > 0$ be a small constant. Then there exists $\varepsilon_0 = \varepsilon_0(c_0, \mu, \delta) > 0$ such that, if the circle $C : r = \omega - \alpha_0$ lies in A, and if $\operatorname{dist}(C, \partial A) \geq \delta$, then, for each $\varepsilon \in [0, \varepsilon_0]$, there is a curve $C_\varepsilon \subset A$ which is invariant under g_ε : $g_\varepsilon(C_\varepsilon) = C_\varepsilon$ (in the sense of set equality). Moreover, the restriction of g_ε to C_ε is conjugate to the rotation by $\omega : \theta \to \theta + \omega$.

4. Examples. We introduce some control systems for which the hypotheses considered in Section 3 can all be satisfied. Begin with

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= f(t, x_1, x_2, u). \end{aligned} \tag{4.1}$$

Our goal is to determine an analytic, periodic function f for which the period map of the system

$$\begin{aligned} \varepsilon u' &= f(t, x_1, x_2, u), \\ x_1' &= x_2 \\ x_2' &= f(t, x_1, x_2, u) \end{aligned} \tag{4.2}$$

admits a two-dimensional submanifold of the (x_1, x_2, u)-space on which it is area-preserving to second order in ε.

We will look for the function f in the form

$$f(t, x_1, x_2, u) = 1 - B(t, x_1, x_2)u$$

where B is to be determined so as to satisfy the conditions discussed below. We state the first two of these conditions immediately

$$B(t, x_1, x_2) > 0 \tag{4.3}$$

$$B(t + 2\pi, x_1, x_2) = B(t, x_1, x_2) \tag{4.4}$$

for all (t, x_1, x_2) of interest. The condition (4.3) implies that the singular perturbation problem (4.2) satisfies the Tychonov condition.

Now put

$$u_0(t, x_1, x_2) = \frac{1}{B(t, x_1, x_2)},$$

so that $f(t, x_1, x_2, u_0) \equiv 0$. By the Tychonov theorem [**22**], for small $\varepsilon > 0$ equations (4.2) admits a locally invariant manifold

$$\{(t, x_1, x_2, u_\varepsilon(t, x_1, x_2)) : t \in \mathbf{R}, (x_1, x_2) \in A\}$$

where u_ε is 2π-periodic in t and A is an annulus in $\mathbf{R}^2$ centered at the origin. We write

$$u_\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots, \tag{4.5}$$

where $u_i = u_i(t, x_1, x_2)$, $i = 1, 2$.

Next we consider the differential equation (4.1) on the invariant manifold with parameter ε:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= f(t, x_1, x_2, u_\varepsilon(t, x_1, x_2)). \end{aligned} \tag{4.6}$$

We wish to arrange f so that the time-one map $g_\varepsilon = g_\varepsilon(x_1, x_2)$ of the equation (4.6) is area-preserving to order ε^2. To this end, let G be the divergence of the vector field $\binom{x_2}{f}$, thus

$$\begin{aligned}
G &= \left(\frac{\partial f}{\partial x_2} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x_2}\right)(t, x_1, x_2, u_\varepsilon(t, x_1, x_2)) \\
&= \frac{\partial f}{\partial x_2}(t, x_1, x_2, u_0) + \varepsilon\frac{\partial^2 f}{\partial x_2 \partial u}(t, x_1, x_2, u_0)u_1 \\
&\quad + \frac{\partial f}{\partial u}(t, x_1, x_2, u_\varepsilon)\left\{\frac{\partial u_0}{\partial x_2} + \varepsilon\frac{\partial u_1}{\partial x_2}\right\} + O(\varepsilon^2) \\
&= \left\{\frac{\partial f}{\partial x_2}(t, x_1, x_2, u_0) + \frac{\partial f}{\partial u}(t, x_1, x_2, u_0)\frac{\partial u_0}{\partial x_2}\right\} \\
&\quad + \varepsilon\left\{\frac{\partial^2 f}{\partial x_2 \partial u}(t, x_1, x_2, u_0)u_1 + \frac{\partial f}{\partial u}(t, x_1, x_2, u_0)\frac{\partial u_1}{\partial x_2}\right. \\
&\qquad \left. + \frac{\partial^2 f}{\partial u^2}(t, x_1, x_2, u_0)\frac{\partial u_0}{\partial x_2}\right\} + O(\varepsilon^2),
\end{aligned}$$

where the term $(\partial^2 f/\partial u^2)(\partial u_0/\partial x_1) \equiv 0$, because f is linear in u. Of course, the zero-order term vanishes because it is

$$(\partial f/\partial x_2)(t, x_1, x_2, u_0(t, x_1, x_2)) \equiv 0.$$

We proceed to study the coefficient of ε. To this end, we return to the first of equations (4.2):

$$\varepsilon u_\varepsilon' = f(t, x_1, x_2, u_\varepsilon),$$

or equivalently,

$$\varepsilon[u_0' + \varepsilon u_1' + \cdots] = f(t, x_1, x_2, u_0) + \varepsilon\frac{\partial f}{\partial u}(t, x_1, x_2, u_0)u_1 + O(\varepsilon^2).$$

From this,

$$u_0' = \frac{\partial f}{\partial u}(t, x_1, x_2, u_0)u_1,$$

that is,

$$u_1 = \frac{u_0'}{(\partial f/\partial u)(t, x_1, x_2, u_0)}. \tag{4.7}$$

Here we have abused notation slightly: u_0' actually may contain terms of all orders in ε; what is meant in (4.7) is the term of 0th order in ε. In any case, we obtain that the first-order G_1 of $G = \varepsilon G_1 + \varepsilon^2 G_2 + \cdots$ is

$$\begin{aligned}
G_1 &= \frac{\partial^2 f/\partial u\partial x_2}{\partial f/\partial u} u_0' + \frac{\partial f}{\partial u}\frac{\partial}{\partial x_2}\left(\frac{u_0'}{\partial f/\partial u}\right) \\
&= \frac{\partial^2 f/\partial u\partial x_2}{\partial f/\partial u} u_0' + \frac{\partial f}{\partial u} \\
&\quad \cdot \left\{\frac{(\partial f/\partial u)(\partial(u_0')/\partial x_2) - u_0'[\partial^2 f/\partial u\partial x_2 + (\partial^2 f/\partial u^2)(\partial u_0/\partial x_2)]}{[\partial f/\partial u]^2}\right\} \\
&= \frac{\partial(u_0')}{\partial x_2},
\end{aligned}$$

since $\partial^2 f/\partial u^2 \equiv 0$ and the two terms cancel.

We conclude that

$$G_1 = \frac{\partial(u_0')}{\partial x_2} = \frac{\partial^2 u_0}{\partial t\partial x_2} + \frac{\partial u_0}{\partial x_1} + x_2\frac{\partial^2 u_0}{\partial x_1\partial x_2} \tag{4.8}$$

where we used the fact that

$$\begin{aligned}
u_0' &= \frac{\partial u_0}{\partial t} + \frac{\partial u_0}{\partial x_1}x_1' + \frac{\partial u_0}{\partial x_2}x_2' \\
&= \frac{\partial u_0}{\partial t} + \frac{\partial u_0}{\partial x_1}x_2 \\
&\quad + \frac{\partial u_0}{\partial x_2}\left[f(t, x_1, x_2, u_0) + \varepsilon\frac{\partial f}{\partial u}(t, x_1, x_2, u_0)u_1 + \cdots\right]
\end{aligned}$$

and the relation $f(t, x_1, x_2, u_0) \equiv 0$.

As indicated earlier we retain only the terms in u_0' of zero-order with respect to ε. We are led to look for one-periodic solutions of the partial differential equation

$$\frac{\partial^2 u_0}{\partial t\partial x_2} + \frac{\partial u_0}{\partial x_1} + x_2\frac{\partial^2 u_0}{\partial x_1\partial x_2} = 0. \tag{4.9}$$

Clearly any constant is a solution of the linear equation (4.9). So is any function of t alone. We can find more solutions by separating variables. Write

$$u_0 = e^{2\pi i t}v(x_1, x_2)$$

and note that $\operatorname{Re} u_0$, $\operatorname{Im} u_0$ will be solutions of (4.9) if u_0 is a complex solution. We get

$$2\pi i e^{2\pi i t}\frac{\partial v}{\partial x_2} + e^{2\pi i t}\frac{\partial v}{\partial x_1} + x_2 e^{2\pi i t}\frac{\partial^2 v}{\partial x_1 \partial x_2} = 0.$$

or equivalently,

$$2\pi i\frac{\partial v}{\partial x_2} + \frac{\partial v}{\partial x_1} + x_2\frac{\partial v}{\partial x_2} = 0.$$

Writing $v(x_1, x_2) = Q(x_1)R(x_2)$, we get

$$2\pi i\frac{R'}{R} + \frac{Q'}{Q} + x_2\frac{Q'}{Q}\frac{R'}{R} = 0,$$

that is,

$$\frac{Q'}{Q}\left[1 + x_2\frac{R'}{R}\right] = -2\pi i\frac{R'}{R},$$

and

$$\frac{Q'}{Q} = \frac{-2\pi i R'/R}{1 + x_2 R'/R} = \lambda \in \mathbf{C}.$$

So $Q(x_1) = Q_0 e^{\lambda x_1}$, while $-2\pi i(R'/R) = \lambda + \lambda x_2(R'/R)$. Hence $-\lambda = (\lambda x_2 + 2\pi i)(R'/R)$, and finally,

$$R(x_2) = \frac{R_0}{\lambda x_2 + 2\pi i}.$$

Choosing $\lambda = 2\pi i$, we get the relation

$$v(x_1, x_2) = \frac{c e^{2\pi i x_1}}{1 + x_2},$$

and thus

$$u_0(t, x_1, x_2) = \frac{c e^{2\pi i (t + x_1)}}{1 + x_2},$$

or in real form,

$$u_0(t, x_1, x_2) = \frac{a \cos 2\pi(t + x_1 + \varphi)}{1 + x_2} \tag{4.10}$$

where a is an amplitude and φ is a phase. So, adding a positive function of t to these oscillatory quantities, we get a class of functions B for which the divergence G vanishes to second order in ε.

We now discuss the dynamical stability of the a.e. control for the equations (4.2). Write $x_1=\theta/(2\pi)$, $x_2=r$ where θ, r are polar coordinates in $\mathbf{R}^2$. Choose u_0 to be a positive function of t. Plus the expression in (4.10), and let $g_\varepsilon : A \to \mathbf{R}$ be the time-one map of equations (4.2). As before, we take the sliding manifold S to be a circle in $A : S = \{(\theta, r) \mid r = r_0, 0 \le \theta < 2\pi\}$. Our point of departure is the following.

Definition 4.1. Say that S is *dynamically stable* (with respect to the family of controls $\{u_\varepsilon\}$), if for each annular neighborhood $W \subset A$ of S there exists $\varepsilon_0 > 0$ such that, if $0 \le \varepsilon \le \varepsilon_0$ and $p = (\theta, r) \in W$, then the iterates g_ε^k satisfy

$$g_\varepsilon^k(p) \in W, \quad k = 0, 1, 2, \dots .$$

We are now going to argue that this definition is not entirely appropriate from a practical point of view; we will replace it with a modified definition. The main point is that one does not usually have at one's disposition the exact function $u_\varepsilon(t, x_1, x_2)$. Rather, one has a finite-order approximating function:

$$u_{\varepsilon,N}(t, x_1, x_2) = \sum_{k=0}^{N} u_k(t, x_1, x_2)\varepsilon^k.$$

We will redefine the dynamical stability of S using $u_{\varepsilon,N}$.

By center manifold theory [**4**], one has $u_\varepsilon = u_{\varepsilon,N} + O(\varepsilon^{N+1})$. Substituting into the first of equations (4.2), we obtain

$$\begin{aligned} u_1 &= -u_0\left[\frac{\partial u_0}{\partial t} + x_2\frac{\partial u_0}{\partial x_1}\right] \\ u_2 &= -u_0\left[\frac{\partial u_1}{\partial t} + x_2\frac{\partial u_1}{\partial x_2}\right] + u_0\frac{\partial u_0}{\partial x_2} \\ &\vdots \end{aligned}$$

$$u_{k+1} = -u_0\left[\frac{\partial u_k}{\partial t} + x_2\frac{\partial u_k}{\partial x_2}\right] + \sum_{l=0}^{k-1}\frac{\partial u_l}{\partial x_2}u_{k-l-1}$$

$$\vdots$$

From these formulas, it is clear that $u_{\varepsilon,N}$ is analytic in (t, x_1, x_2) and one-periodic in t for each $\varepsilon \geq 0$.

Let P_ε be the period map of equations (4.2), thus

$$P_\varepsilon(x_1, x_2, u) = (x_1(1), x_2(1), u(1)),$$

for each initial condition (x_1, x_2, u). For the initial value $u = u_{\varepsilon,N}(0, x_1, x_2)$, write

$$P_\varepsilon(x_1, x_2, u_{\varepsilon,N}(0, x_1, x_2)) = (\tilde{x}_1, \tilde{x}_2, \tilde{u}),$$

and define

$$g_{\varepsilon,N}(x_1, x_2) = (\tilde{x}_1, \tilde{x}_2).$$

Then $g_{\varepsilon,N}$ is defined on the annulus A. The iterate $g^{k+1}_{\varepsilon,N}$ is obtained from $g^k_{\varepsilon,N}$ via the formula

$$P_\varepsilon(g^k_{\varepsilon,N}(x_1, x_2), u_{\varepsilon,N}(0, g^k_{\varepsilon,N}(x_1, x_2))) = (g^{k+1}_{\varepsilon,N}(x_1, x_2), *),$$

where $*$ stands for the u-coordinate at time 1 of the point on the solution of equations (4.2) whose initial point is

$$(g^k_\varepsilon(x_1, x_2), u_{\varepsilon,N}(0, g^k_{\varepsilon,N}(x_1, x_2))).$$

We see that the iterates of $g_{\varepsilon,N}$ do not correspond to trajectories of (4.2). However, given k and ε, one can determine N so that the points

$$\{P_\varepsilon(g^l_{\varepsilon,N}(x_1, x_2), u_{\varepsilon,N}(0, g^l_{\varepsilon,N}(x_1, x_2))) \mid 0 \leq l \leq k\}$$

lie as closely as desired to the trajectory of (4.2) passing through the point $(x_1, x_2, u_{\varepsilon,N}(0, x_1, x_2))$.

An alternative point of view is that the iterates $g^k_{\varepsilon,N}(x_1, x_2)$ are obtained by successively "updating" the u-coordinate of the argument of

P_ε. Thus, for example, $g^2_{\varepsilon,N}(x_1, x_2)$ is obtained from $(\tilde{x}_1, \tilde{x}_2, \tilde{u})$ by substituting $u_{\varepsilon,N}(0, \tilde{x}_1, \tilde{x}_2)$ for $\tilde{u}$ and calculating $P_\varepsilon(\tilde{x}_1, \tilde{x}_2, u_{\varepsilon,N}(0, \tilde{x}_1, \tilde{x}_2))$.

In any case, we now propose the following reformulation of the concept of dynamical stability.

Definition 4.2. Fix $N \geq 2$. Say that S is *dynamically stable* (with respect to the family of controls $\{u_{\varepsilon,N}\}$) if for each annular neighborhood $W \subset A$ of S, there exists $\varepsilon_0 > 0$ such that, if $0 \leq \varepsilon \leq \varepsilon_0$ and $p = (\theta, r) \in W$, then the iterates $g^k_{\varepsilon,N}$ satisfy

$$g^k_{\varepsilon,N}(p) \in W, \quad k = 0, 1, 2, \dots .$$

We now use Remark 3.4 to draw the following conclusion. Let $W \subset A$ be an annular neighborhood of the sliding manifold $S = \{(\theta, r) \mid r = r_0, 0 \leq \theta < 2\pi\}$. Since diophantine irrationals are dense in $\mathbf{R}$, we can find two diophantine numbers $r_1 < r_0 < r_2$ such that the circles of radius r_1, r_2 lie in W. By Remark 3.4, we have

Theorem 4.3. *Fix $N \geq 2$. If ε is sufficiently small, then there are two simple closed curves C_1 and C_2 in W containing S between them, such that C_1 and C_2 are invariant under $g_{\varepsilon,N} : g_{\varepsilon,N}(C_i) = C_i$, $i = 1, 2$, in the sense of set equality.*

Proof. In fact, we need only verify hypotheses 1 and 2 of Proposition 3.1, and those of Proposition 3.3. First, if $\varepsilon = 0$ one has $g_0\binom{r}{\theta} = \binom{r}{\theta + 2\pi r}$. Hypothesis 2 is satisfied in A because of the divergence condition $G_1 = 0$. Finally, the hypothesis $\rho_1(r, \theta) \neq \rho_1(r)$ of Proposition 3.3 is satisfied because of the relation $\varepsilon u' = r'$ and the form of $u_0(t, \theta, r)$ as the reader may check.

Since the region between C_1 and C_2 is invariant with respect to $g_{\varepsilon,N}$, if C_1 and C_2 are invariant, Theorem 4.3 shows that S is dynamically stable in the sense of Definition 4.2. $\square$

Remark 4.4. It may well be true that, by using the theory of C^r but nonanalytic twist maps, one can prove that S is also dynamically stable in the sense of Definition 4.1.

5. Brief remarks on dynamical instability. It is easy to produce examples, similar in structure to those considered in Section 4, where a natural type of dynamical instability prevails. Let us return to equations (4.2). If the first-order coefficient G_1 of the divergence G of the vector field in (4.6) is strictly negative on an annulus $A \subset \mathbf{R}^2$, then there is a constant $c < 1$ such that the dissipativity condition $m(g_\varepsilon(B)) \leq cm(B)$ is satisfied for each Borel set $B \subset A$. The theory of mappings g_ε with this property was developed by Birkhoff [**1**] and in a modern form by Le Calvez [**14**]. In general, such a mapping admits a unique compact attractor A_0 in the interior of A. The set A_0 may be a circle, or may have a more complex structure; see [**14**] for a careful presentation of various possibilities. The point of interest to us here is that A_0 may very well be disjoint from the sliding manifold S. Since the sequence of iterates $\{g_\varepsilon^k(p)\}$ tends to A_0 as $k \to \infty$ for each $p \in A$, S certainly will not be dynamically stable unless S is contained in A_0.

Having made these considerations, return to quantity

$$G_1 = \frac{\partial^2 u_0}{\partial t \partial x_2} + \frac{\partial u_0}{\partial x_1} + x_2 \frac{\partial^2 u_0}{\partial x_1 \partial x_2},$$

where $x_1 = \theta/(2\pi)$, $x_2 = r$.

We seek a positive function u_0, one-periodic in t and x, such that G_1 is strictly negative on some annulus $0 \leq \theta < 2\pi$, $r_1 < r < r_2$. But it is easy to determine such functions $u_0, \dots$.

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