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Differential inclusions on proximate retracts of Hilbert spaces*

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Abstract

The problem of the existence of viable solutions for differential inclusions defined on proximate retracts of Hilbert spaces is considered. Under different assumptions on the right hand side of the differential inclusion we prove the existence of viable solutions and we investigate the topological properties of the set constituted by these solutions. We consider also Lipschitzian proximate retract for which we establish more general results.

Key Words: Proximate Retracts, Differential Inclusions, Tangentiality Conditions, Viable Solutions.

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1 Introduction.

This paper deals with the problem of the existence of viable solutions of a differential inclusion defined on a closed subset K of a Hilbert space H. We consider for K two different cases: when K is a proximate retract and when K is a Lipschitzian proximate retract. We recall that a closed set $K \subset H$ is called a (Lipschitzian) proximate retract if there exists an open neighborhood $\mathcal{U}$ of K and a continuous (Lipschitz) metric retraction $r : \mathcal{U} \to K$ such that $\mathrm{dist}\,(x, K) = \|x - r(x)\|$ for any $x \in \mathcal{U}$.

Specifically, we first give definitions and preliminary results in Section 2. Then under the assumption that K is a proximate retract and under appropriate tangentiality conditions, expressed in term of the Bouligand tangent cone, we prove in Section 3 two different results concerning the existence and the topological structure of the set of viable solutions. The first result (Theorem 3.9) concerns a differential inclusion with a strong Carathéodory (convex compact valued) right hand side φ, i.e. φ is a Carathéodory, sublinear, compact multivalued map with a measurable selection property. The second result (Theorem 3.13) deals with a semilinear differential inclusion with a nonlinear condensing (with respect to the Hausdorff measure of noncompactness) term whose range is contained in the domain of a generator of analytic semigroup.

Theorem 3.9 and 3.13 generalize the respective results given in earlier papers: [3, 5, 8, 9, 10], where the Cauchy problem is considered on the Hilbert space H instead of its closed subset K. An application of Theorems 3.9 and 3.13 to the periodic problem is presented.

In Section 4, we assume a more restrictive condition on K, that is we assume that K is a Lipschitzian proximate retract together the tangentiality condition. This allows us to weaken the assumptions needed on the map φ in order to obtain the existence of viable solutions. In fact, in Theorem 4.1, φ is only a Carathéodory map which satisfies an appropriate inequality in terms of measure of non compactness. Finally, we state an analogous existence result for a smooth closed submanifold of H. In this Section we obtained a direct generalization of the respective results presented in [4, 10].

The technique employed to prove all the existence results consists in extending, via an Uryshon function, the differential inclusion to all of the Hilbert space H, then to show the existence of a solution starting from any point of K and finally to prove that this solution lies on K for any time.

Finally, we would like to point out that proximate retracts contain both closed convex sets and C^2-manifolds (see [7, 9]). Therefore by using the same technique we obtain existence results simultaneously for convex sets and C^2-manifolds.

2 Definitions and preliminary results

Let K be a nonempty closed subset of an Hilbert space H and let $u \in H$. We define:

$$\operatorname{dist}(u, K) = \inf\{\|u - x\|;\ x \in K\}.$$

We recall ([5, 7, 9]) that the subset $T_K(x) \subset H$, $x \in K$ defined by

$$T_K(x) = \{y \in H;\ \liminf_{t \to 0^+} \frac{1}{t} \operatorname{dist}(x + ty, K) = 0\}$$

is called the Bouligand cone to K at x.

A nonempty closed subset K of H is called a *proximate retract* ([7, 9]) if there exists an open neighbourhood $\mathcal{U}$ of K in H and a continuous map $r : \mathcal{U} \to K$ (called *metric retraction*) such that the following two conditions are verified:

(i) $r(x) = x$ for all $x \in K$

(ii) $\|r(u) - u\| = \operatorname{dist}(u, K)$ for all $u \in \mathcal{U}$.

It is well known (see [1]) that any closed convex $K \subset H$ is a proximate retract, and then we can take $\mathcal{U} = H$. Taking as $\mathcal{U}$ a tubular neighbourhood of K one can show (comp. [6, 7, 9]) that any C^2-submanifold K of H is a proximate retract.

It is easy to prove that, for given K and $\mathcal{U}$, if $r : \mathcal{U} \to K$ exists then it is unique. Since one can take a sufficiently small $\mathcal{U}$, for example by restricting $\mathcal{U}$ to $\mathcal{U} \cap \{u \in H; \operatorname{dist}(u, K) < \delta\}$, $\delta > 0$, we may assume that $\|r(u) - u\| \leq \delta$ for a given $\delta > 0$ and $u \in \mathcal{U}$.

Lemma 2.1 *Let K be a proximate retract. Then*

$$T_K(r(x)) \subseteq \{y \in H;\ (y, x - r(x)) \leq 0\}$$

for any $x \in \mathcal{U}$, where $(\cdot, \cdot)$ denotes the inner product in H.

Proof: First observe that if $x \in K$ then the conclusion follows immediately. Hence assume that $x \in \mathcal{U} \setminus K$ and $y \in H$ are such that $(y, x - r(x)) > 0$ then

$$\lim_{t \to 0^+} \frac{\operatorname{dist}(r(x) + ty, H \setminus B(x, \|x - r(x)\|)}{t} > 0,$$

where $B(x, s)$ is the open ball centered at x of radius $s > 0$. In fact,

$$\begin{aligned} & \lim_{t \to 0^+} \frac{\operatorname{dist}(r(x) + ty, \partial B(x, \|x - r(x)\|)}{t} = \\ = \ & \lim_{t \to 0^+} \frac{\|x - r(x)\| - \|x - r(x) - ty\|)}{t} = \\ = \ & \frac{(y, x - r(x))}{\|x - r(x)\|} > 0. \end{aligned}$$

On the other hand $K \subset H \setminus B(x, \|x - r(x)\|)$ and so

$$\operatorname{dist}(r(x) + ty, K) \geq \operatorname{dist}(r(x) + ty, H \setminus B(x, \|x - r(x)\|),$$

in conclusion $y \notin T_K(r(x))$. □

Lemma 2.2 *Let K be a proximate retract, $r : \mathcal{U} \to K$ a metric retraction and $s > 0$ such that $K \cap \overline{B(0, s)} \neq \emptyset$, where $\overline{B(0, s)}$ is the closure of $B(0, s)$ in H. Then there exists $\epsilon_0 > 0$ such that, for any $0 < \epsilon \leq \epsilon_0$ there exist subsets $K \subset K_\epsilon \subset \mathcal{U}_\epsilon \subset \mathcal{U}$ of H, K_ϵ closed and $\mathcal{U}_\epsilon$ open, and a continuous retraction $r_\epsilon : \mathcal{U}_\epsilon \to K_\epsilon$ such that the following condition are satisfied:*

(2.2.1) $\bigcap_{0 < \epsilon \leq \epsilon_0} K_\epsilon = K,$

(2.2.2) $\|r_\epsilon(u) - u\| = \operatorname{dist}(u, K_\epsilon)$ *for all* $u \in \mathcal{U}_\epsilon \cap \overline{B(0, s)},$

(2.2.3) $\{y \in H;\ (y, x - r(x)) \leq 0\} \supseteq T_{K_\epsilon}(x)$ *for all* $x \in K_\epsilon \cap \overline{B(0, s)}.$

Proof: Let $\epsilon_0 < \frac{1}{2}$ be such that $(K \cap \overline{B(0, s + 1)}) + B(0, 2\epsilon_0) \subset \mathcal{U}$. Let $\epsilon \leq \epsilon_0$, and let $\delta_\epsilon : K \to (0, 1]$ be defined by

$$\delta_\epsilon(x) = \begin{cases} \max\{\delta \in [0, 1];\ x \in \overline{B(0, s + 1 + (1 - \delta)\epsilon)} & \text{if } x \in \overline{B(0, s + 1 + \epsilon)} \\ 0, & \text{otherwise.} \end{cases}$$

We define $K_\epsilon = \{x + \delta_\epsilon(x)y;\ x \in K \text{ and } y \in \overline{B(0, s+1+\epsilon)}\}$, $\mathcal{U}_\epsilon = (K + \overline{B(0,2\epsilon)}) \cap \mathcal{U}$ and

$$r_\epsilon(u) = \begin{cases} u & \text{if } u \in r(u) + \delta_\epsilon(r(u)) \cdot \overline{B(0,\epsilon)} \\ r(u) + \epsilon\delta_\epsilon(r(u))\dfrac{u - r(u)}{\|u - r(u)\|}, & \text{otherwise.} \end{cases}$$

A routine verification shows that $K_\epsilon, \mathcal{U}_\epsilon$ and r_ϵ have the required properties. □

Together with proximate retracts we need to consider Lipschitzian proximate retracts. Namely, a proximate retract $K \subset H$ is called *Lipschitzian proximate retract* provided there exists an open neighbourhood $\mathcal{V}$ of K in H, $\mathcal{V} \subset \mathcal{U}$ such that the metric retraction $r : \mathcal{V} \to K$ (restricted to $\mathcal{V}$) is a Lipschitzian map, i.e., there exists $L > 0$ such that:

$$\|r(u) - r(v)\| \le L\|u - v\| \quad \text{for all } u, v \in \mathcal{V}.$$

It is well known that every closed convex $K \subset H$ is a Lipschitzian proximate retract with constant $L = 1$. Unfortunately C^2-submanifolds of H are not Lipschitzian proximate retracts in general but the following proposition holds true.

Proposition 2.3 ([6]). *Let K be a smooth closed submanifold of H and let $r : \mathcal{U} \to K$ be a metric retraction. Then for every $x \in K$ there exists an open ball $B(x, s) \subset \mathcal{U}$ such that:*

(2.3.1) $\|r(u) - r(v)\| \le 2\|u - v\|$ *for all* $u, v \in B(x, s)$.

Remark 2.4 The problem of a topological characterization of Lipschitzian proximate retracts as a subclass of proximate retracts remains open.

We recall also the notion of R_δ-sets (comp. [7]). A compact (metric) nonempty space X is called an R_δ-set if there exists a decreasing sequence of compact nonempty contractible spaces X_n such that:

$$X = \bigcap_n X_n.$$

Note that an intersection of a decreasing sequence of R_δ-sets is also an R_δ-set.

In what follows by χ we shall denote the Hausdorff measure of noncompactness ([2]) in H. Recall that given a bounded set $A \subset H$,

$$\chi(A) = \inf\{\epsilon > 0;\ A \text{ has a finite } \epsilon\text{-net in } H\}.$$

Below we summarize important properties of the measure χ.

Proposition 2.5

2.5.1) $\chi(\overline{\text{co}}A) = \chi(A)$ *for every* $A \subset H$ *where* $\overline{\text{co}}A$ *denotes the closed convex hull of* A *in* H;

2.5.2) *(monotonicity)*
if $A \subset B$, *then* $\chi(A) \leq \chi(B)$;

2.5.3) *(nonsingularity)*
$\chi(\{u\} \cup A) = \chi(A)$ *for every bounded set* $A \subset H$;

2.5.4) *(triviality)*
$\chi(A) = 0$ *iff* $\overline{A}$ *is compact.*

3 Differential inclusions on proximate retracts

In this Section we shall discuss the Cauchy problem for differential inclusions on proximate retracts. Our considerations are based on [3, 5, 6, 8, 9], we give here a natural generalizations of some results contained therein.

Let K be a proximate retract in H. Consider the Cauchy problem:

$$\begin{cases} x'(t) \in \varphi(t, x(t)) \\ x(0) = x_0 \in K, \end{cases} \tag{3.1}$$

where $\varphi : [0, a] \times K \multimap H$ is a multivalued map.

We shall discuss the problem of the existence and of a topological characterization of the set $S(\varphi; x_0)$ of all solutions of (3.1) under suitable assumptions on φ. Here by a solution of the problem (3.1) we mean an absolutely continuous function $x : [0, a] \to K$ such that $x(0) = x_0$ and $x'(t) \in \varphi(t, x(t))$ for almost all (a.a.) $t \in [0, a]$.

Our method to solve (3.1) consists in solving the following associated problem:

$$\begin{cases} x'(t) \in \tilde{\varphi}(t, x(t)) \\ x(0) = x_0 \in K, \end{cases} \tag{3.2}$$

where $\tilde{\varphi} : [0, a] \times H \multimap H$ is defined by means of the given multivalued map $\varphi : [0, a] \times K \multimap K$ as follows

$$\tilde{\varphi}(t, x) = \begin{cases} \alpha(x) \cdot \varphi(t, r(x)) & \text{if } x \in \mathcal{U} \\ 0 & \text{if } x \notin \mathcal{U}, \end{cases} \tag{3.3}$$

where $r : \mathcal{U} \to K$ is the metric retraction and $\alpha : H \to [0, 1]$ is a continuous Uryshon function such that $\alpha|_K \equiv 1$ and $\alpha|_{H \setminus K} \equiv 0$. Obviously, $\tilde{\varphi}$ is unique up to the choice of the Uryshon function α.

Let us define a class of multivalued maps suitable for problem (3.1).

Definition 3.4 A multivalued map $\varphi : [0, a] \times K \multimap H$ with compact convex nonempty values is called a strong Carathéodory map (s.c.-map) on K provided that the following conditions are satisfied:

(3.4.1) the multivalued map $\varphi(t, \cdot) : K \multimap H$ is upper semicontinuous (u.s.c.) for a.a. $t \in [0, a]$;

(3.4.2) the multivalued map $\varphi(\cdot, x) : [0, a] \multimap H$ has a measurable selector for all $x \in K$;

(3.4.3) there exist summable functions $\mu, \gamma : [0, a] \to R_+$ such that:

$$\|y\| \leq \mu(t) + \gamma(t)\|x\|$$

for every $y \in \varphi(t, x)$, $t \in [0, a]$, $x \in K$;

(3.4.4) for every bounded set $A \subset K$ and a.a. $t \in [0, a]$ the set $\varphi(t, A)$ is relatively compact, i.e., $\overline{\varphi(t, A)}$ is compact.

Note that if we assume that $\varphi(\cdot, \cdot)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable then (3.4.2) is automatically satisfied (see [5, 9]). Obviously if K is compact then (3.4.4) is also satisfied.

We have the following.

Proposition 3.5 *If $\varphi : [0,a] \times K \multimap H$ is a strong Carathéodory map in K then $\tilde{\varphi} : [0,a] \times H \multimap H$ is a strong Carathéodory map on H.*

We need the following definition.

Definition 3.6 A map $\varphi : [0,a] \times K \multimap H$ is called *weakly tangent (tangent)* to K, if $\varphi(t,x) \cap T_K(x) \neq \emptyset$, $(\varphi(t,x) \subset T_K(x))$, for $x \in K$ and a.a. $t \in [0,a]$.

The following lemma is crucial in what follows (compare [5, 9]).

Lemma 3.7 *Let $\varphi : [0,a] \times K \multimap H$ be tangent to K, proximate retract in H. If $x : [0,a] \to H$ is an absolutely continuous function such that $x'(t) \in \tilde{\varphi}(t, x(t))$, for* a.a. $t \in [0,a]$ *and $x(0) \in K$, then $x(t) \in K$ for all $t \in [0,a]$.*

Proof: Let $d : [0,a] \to R_+$ be defined by $d(t) = \operatorname{dist}(x(t), K)$, $t \in [0,a]$. We want to show that $d(t) = 0$ for all $t \in [0,a]$. By our assumption $d(0) = 0$. It is easily verified that d is absolutely continuous. Therefore it is sufficient to show that $d'(t) \leq 0$ for a.a. t which implies that d is nonincreasing. Let $t_0 \in (0,a]$ and $0 < h < a - t_0$ be such that $x'(t_0) \neq 0$. Observe that $x'(t_0) \in T_K(r(x(t_0)))$, moreover

$$\begin{aligned} & d(t_0 + h) = \operatorname{dist}(x(t_0) + hx'(t_0) + o(h), K) = \\ = \; & \operatorname{dist}(x(t_0) - r(x(t_0)) + r(x(t_0)) + hx'(t_0) + o(h), K) \leq \\ \leq \; & \operatorname{dist}(r(x(t_0)) + hx'(t_0), K) + \|x(t_0) - r(x(t_0))\| + o(h) \end{aligned}$$

or equivalently

$$\frac{d(t_0+h) - d(t_0)}{h} \leq \frac{\operatorname{dist}(r(x(t_0)) + hx'(t_0), K) + o(h)}{h},$$

for $h \to 0^+$ we obtain $d'(t_0) \leq 0$. □

Remark 3.8 ([5]). The assumption that K is a proximate retract can be weakened in the last lemma as follows. Let K be a nonempty closed subset of H such that $K \cap \overline{B(0,s)} \neq \emptyset$, for some $s > 0$ and such that, for given $\mathcal{U} \supset K$ and continuous $r : \mathcal{U} \to K$, the condition $\|r(u) - u\| = dist(u, K)$ is verified for $u \in \mathcal{U} \cap \overline{B(0,s)}$. If $x : [0,a] \to H$ is as in Lemma 3.7 and, in addition $\|x(t)\| \leq s$ for all t, then the conclusion of Lemma 3.7 holds.

Now, we are able to formulate the following result.

Theorem 3.9 *Assume that $\varphi : [0, a] \times K \multimap H$ is a strong Carathéodory map on K which is weakly tangent to K. Then the solution set $S(\varphi; x_0)$ of problem (3.1) is an R_δ-set.*

Proof: *Step 1:* Assume that $\varphi(t, x) \subset T_K(x)$, i.e. φ is tangent to K. By Lemma 3.7 we have

$$S(\varphi; x_0) = S(\tilde{\varphi}; x_0)$$

but, in view of [11], $S(\tilde{\varphi}; x_0)$ is an R_δ-set and hence our conclusion follows.

Step 2: Assume $\varphi(t, x) \cap T_K(x) \neq \emptyset$. Since our map satisfies (3.4.3), in view of the Gronwall inequality, one can show that there exists $s_0 > 0$ such that $\|x(t)\| \leq s_0$ for every $t \in [0, a]$ and $x \in S(\varphi; x_0)$.

We take $s = s_0 + 1$ and let $\epsilon, K_\epsilon, r_\epsilon : \mathcal{U}_\epsilon \to K_\epsilon$ be given for $\epsilon \leq \epsilon_0$ as in Lemma 2.2. Define $\psi_\epsilon : [0, a] \times K_\epsilon \multimap H$ by

$$\psi_\epsilon(t, x) = (\mu_\epsilon(x) \cdot \varphi(t, r(x))) \cap \{y \in H;\ (y, x - r(x)) \leq 0\},$$

where $\mu_\epsilon : H \to [0, 1]$ is a continuous Uryshon function for $\overline{B(0, s_0)}$ and $B(0, s_0 + \epsilon)$.

Now, we may easily verify that ψ_ϵ is a strong Carathéodory map on K_ϵ. By Lemmas 2.1 and 2.2 we deduce that $\emptyset \neq \psi_\epsilon(t, x) \subset T_{K_\epsilon}(x)$ for all $x \in K_\epsilon$. By Lemma 2.3 and Remark 3.8 we get $S(\psi_\epsilon, x_0) = S(\tilde{\psi}_\epsilon, x_0)$. By arguing as in the first step we conclude that $S(\psi_\epsilon, x_0)$ is an R_δ-set.

Since $S(\varphi; x_0) = \bigcap_n S(\psi_{\epsilon_n}, x_0)$, where $0 < \epsilon_n < \epsilon_0$ is a sequence converging to 0, we deduce that $S(\varphi; x_0)$ is an R_δ-set too and the proof is completed. □

Now, we are going to consider the following Cauchy problem for semilinear differential inclusions, i.e.,

$$\text{(3.10)} \qquad \begin{cases} x'(t) \in Ax(t) + \varphi(t, x(t)) \\ x(0) = x_0 \in K, \end{cases}$$

where A is the infinitesimal generator of the analytic semigroup e^{At} and $\varphi : [0, a] \multimap H$ is a multivalued map.

As above we shall reduce our problem (3.10) to the following one:

$$\text{(3.11)} \qquad \begin{cases} x'(t) \in Ax(t) + \tilde{\varphi}(t, x(t)) \\ x(0) = x_0 \in K, \end{cases}$$

where $\tilde{\varphi} : [0, a] \times H \multimap H$ is the function associated with φ by (3.3). Then we get $S(\varphi; x_0) = S(\tilde{\varphi}; x_0)$ and our result can be deduced from the corresponding result on H.

Below we shall formulate assumptions on φ for which our procedure is applicable. Namely, we shall assume that $\varphi : [0, a] \times K \multimap H$ has compact convex nonempty values and satisfies the following conditions:

(3.12.1) $\varphi(t, K) \subset \mathcal{D}(A)$ for a.a. $t \in [0, a]$, where $\mathcal{D}(A)$ stands for the domain of A;

(3.12.2) for every $y \in C([0, a], H)$ and $f \in L_1([0, a], H)$ such that $f(t) \in \varphi(t, y(t))$ we have $Af(\cdot) \in L_1([0, a], H)$, where $C([0, a], H)$ is the space of continuous maps and $L_1([0, a], H)$ the space of Bochner integrable functions;

(3.12.3) $\chi(\varphi(t, D)) \leq k(t)\chi(D)$ for every bounded set $D \subset K$, where $k : [0, a] \to [0, 1)$ is a continuous function.

Moreover, we need some tangentiality-type condition:

(3.12.4) $Ax(t) + \varphi(t, x) \subset T_K(r(x))$ for a.a. $t \in [0, a]$ and $x \in \mathcal{U}$, where $r : \mathcal{U} \to K$ is the metric retraction.

Now by using our technique and [3] we can prove the following result.

Theorem 3.13 *If $\varphi : [0, a] \times K \multimap H$ satisfies conditions (3.12.1)÷(3.12.4), then the set $S(\varphi, x_0)$ is an R_δ-set.*

The proof follows the lines of that of step 1 of Theorem 3.9.

In both cases (3.1) and (3.11) we can consider the following *periodic problem*

$$(3.14)\quad \begin{cases} x'(t) \in \varphi(t, x(t)) \\ x(0) = x(a) \in K \end{cases} \quad ; \quad \text{resp.} \quad \begin{cases} x'(t) \in Ax(t) + \varphi(t, x(t)) \\ x(0) = x(a) \in K. \end{cases}$$

To solve problem (3.14) we consider the following diagram:

$$K \overset{P}{\multimap} C([0, a], H) \overset{e_a}{\to} K,$$

where $P(x) = S(\varphi; x)$ or $P(x) = S(A + \varphi; x)$ and $e_a = x(a)$. We let $P_a : K \multimap K$, $P_a = e_a \circ P$. As in [7] one can show that P is an u.s.c. map with R_δ-values.

Consider now the multivalued homotopy

$$\eta : K \times [0, 1] \multimap K,$$

defined by:

$$\eta(x, \mu) = e_{\mu a}(S(\varphi; x)).$$

Then $\eta(x, 0) = x$ and $\eta(x, 1) = P_a(x)$.

On the other hand the problem (3.14) has a solution iff P_a has a fixed point. Since our proximate retract K is an absolute neighbourhood retract and we have an homotopy joining $Id|_K$ with P_a the Lefschetz number $\lambda(P_a)$ is equal to the Lefschetz number $\lambda(Id|_K)$ which is equal the Euler characteristic $E(K)$ of K provided K is compact.

Therefore from the Lefschetz fixed point theorem for multivalued maps (see [1, 7]) the following result is proved.

Corollary 3.15 *Assume that K is a compact proximate retract with $E(K) \neq 0$, then problem (3.14) has a solution.*

4 The case of Lipschitzian proximate retracts

In this Section we shall assume that $K \subset H$ is a Lipschitzian proximate retract. This more restrictive assumption on K allows us to consider a larger class of functions φ for which to prove an existence result for problem (3.1).

First, we recall the notion of Kamke function. A function $h : [0, a] \times \mathbf{R}_+ \to \mathbf{R}_+$ is said to be a Kamke function provided:

(i) h is integrably bounded on bounded sets in $[0, a] \times \mathbf{R}_+$;

(ii) $h(\cdot, x)$ is measurable for every $x \in \mathbf{R}_+$ and $h(t, \cdot)$ is continuous for a.a. $t \in [0, a]$;

(iii) $h(t, 0) = 0$ for a.a. $t \in [0, a]$ and the function $y \equiv 0$ is the only absolutely continuous function which is the solution of the problem $y(0) = 0$ and $y'(t) = h(t, y(t))$ for a.a. $t \in [0, a]$.

From now on, we shall consider compact convex nonempty valued vector fields $\varphi : [0,a] \times K \multimap H$ which are tangent to K. Moreover we assume the following conditions on φ:

(4.1.1) $\varphi(\cdot, x)$ is measurable for every $x \in K$;

(4.1.2) $\varphi(t, \cdot)$ is u.s.c. for a.a. $t \in [0,a]$;

(4.1.3) φ is integrably bounded, i.e. there exists a summable function $\mu : [0,a] \to \mathbf{R}_+$ such that

$$\|y\| \leq \mu(t)$$

for a.a. $t \in [0,a]$, every $x \in K$ and $y \in \varphi(t,x)$;

(4.1.4) there exists $g : [0,a] \times \mathbf{R}_+ \to \mathbf{R}_+$ such that g is nondecreasing in the second argument, $h(t,y) = g(t, Ly)$ is a Kamke function and

$$\chi(F(t,D)) \leq g(t, \chi(D))$$

for every bounded set $D \subset K$ and, for a.a. $t \in [0,a]$, where L is any constant for which there exists a L-Lipschitz metric retraction $r : \mathcal{U} \to K$.

We can prove the following.

Theorem 4.1 *Let $\varphi : [0,a] \times K \multimap H$ be a vector field as specified before. Then problem (3.1) has a solution.*

Proof: Let $r : \mathcal{U} \to K$ be a Lipschitzian metric retraction with constant L. As before we define $\tilde{\varphi} : [0,a] \times H \multimap H$ as follows

$$\tilde{\varphi}(t,x) = \begin{cases} \alpha(x) \cdot \varphi(t, r(x)) & \text{for } x \in \mathcal{U},\ t \in [0,a] \\ 0 & \text{for } x \notin \mathcal{U}. \end{cases}$$

Clearly $\tilde{\varphi}$ satisfies (4.1.1), (4.1.2) and (4.1.3). Moreover, for every bounded set $B \subset \mathcal{U}$ we have

$$\begin{aligned} & \chi(\tilde{\varphi}(t,B)) \leq \chi(\overline{\text{co}}(\{0\} \cup \varphi(t, r(B))) = \\ = \;& \chi(\varphi(t, r(B))) \leq g(t, \chi(r(B))) \leq \\ \leq \;& g(t, L\chi(B)) = h(t, \chi(B)) \end{aligned}$$

for a.a. $t \in [0, a]$.

In view of Lemma 3.7 we have $S(\varphi; x_0) = S(\tilde{\varphi}; x_0)$ for every $x_0 \in K$. On the other hand $\tilde{\varphi}$ satisfies the condition of the existence theorem in [10], hence $S(\varphi; x_0) = S(\tilde{\varphi}; x_0) \neq \emptyset$. This concludes the proof. □

By applying Proposition 2.3 and using the above method it is easy to prove the following.

Theorem 4.2 *Assume that M is a smooth closed submanifold of H and let $\varphi : [0, a] \times M \multimap H$ be a vector field as in Theorem 4.1. Then problem (3.1) has a local solution.*

Remark 4.3 Assume that M is a smooth Hilbert manifold isometrically imbedded as a closed set into H. Let TM denote the tangent bundle to M. Assume further that a strong Riemann metric on TM is given. Consider a vector field $\varphi : [0, a] \times M \multimap TM$ satisfying all assumptions of Theorem 4.1, where (4.1.4) is expressed in terms of the internal Hausdorff measure χ_I defined with respect to the internal metric ρ on M. Then by means of the same considerations we conclude that problem (3.1) has a local solution.

Finally, we would like to point out that the results proved in this section are much stronger than Theorem 9.1 in [4]. In fact the main differences are:

(i) we consider (4.1.1) and (4.1.2) instead of the upper semicontinuity of φ;

(ii) we consider a "pointwise" regularity condition (4.1.3) instead of a "local" one;

(iii) our results can be formulated also for abstract manifolds, i.e., (4.1.4) can be expressed, if necessary, in terms of internal measure on noncompactness which is more natural in the problems on manifolds.

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