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A Definition of Asymptotic Spectrum for Multi-Valued Maps in Banach Spaces (**).

Summary. – *The aim of this note is to look for conditions to impose on* $\lambda \in \mathbf{C}$ *and* T *in such a way that the multi-valued equation* $p \in \lambda x - T(x)$, *where* $T: E \multimap E$ *is a quasibounded multi-valued map, and* E *is a Banach space, has a solution* $x \in E$ *for any* $p \in E$. *To this aim we need a suitable definition of asymptotic spectrum for* T *and we obtain results, wich extend to the multi-valued case the ones obtained in* [3] *and* [4] *for single-valued maps.*

Sunto. – *Data l'equazione multivoca* $p \in \lambda x - T(x)$, *dove* $T: E \multimap E$ *è un'applicazione multivoca quasilimitata,* $\lambda \in \mathbf{C}$ *ed* E *è uno spazio di Banach, si cercano condizioni da imporre a* λ *e* T *in modo che la suddetta equazione ammetta soluzioni per ogni* $p \in E$. *A tal fine si dà una conveniente definizione di spettro asintotico per applicazioni multivoche e si ottengono risultati che estendono al caso multivoco quelli ottenuti in* [3] *e* [4] *per applicazioni ad un sol valore.*

1. – Introduction.

Let $T: E \multimap E$ be a quasibounded multi-valued map from a complex Banach space E into itself. Consider the multi-valued equation

$$(*) \qquad p \in \lambda x - T(x)$$

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where λ is a complex number. In this note we look for conditions to impose on λ and T in such a way that the above equation has a solution $x \in E$ for any $p \in E$. To this aim we extend to the context of nonlinear multi-valued maps the concept of asymptotic spectrum for nonlinear single-valued maps introduced by M. Furi and A. Vignoli in [3]. We show that $\Sigma(T)$—the asymptotic spectrum of T—is a compact subset of the complex plane $\boldsymbol{C}$. Further, if $T: E \multimap E$ is a compact, quasibounded acyclic-valued map satisfying property (A) (see below), then the equation $(*)$ has a solution for any $p \in E$ provided that λ belongs to the unbounded component of $\boldsymbol{C} \setminus \Sigma(T)$. Moreover $\Sigma(T) \neq \emptyset$ if either $\dim E = +\infty$ or $\dim E$ is odd. All this results extend to the multi-valued case the ones obtained in [3], [4] for single-valued maps.

2. – Notations and definitions.

Let E and F be two complex Banach spaces and $T: E \multimap F$ be a multi-valued map. We recall that T is *upper-semicontinuous* (u.s.c.) on E if, for every point $x \in E$, $T(x)$ is a nonempty and compact subset of F, and for any open subset V of F containing $T(x)$ there exists an open neighborhood U of x such that $T(U) \subset V$, where $T(U) = \bigcup_{z \in U} T(z)$. An u.s.c. multi-valued map T is said to be *compact* if it sends bounded sets into relatively compact sets.

If, for any $x \in E$, $T(x)$ is acyclic in the Vietoris homology theory with coefficients in $\boldsymbol{Q}$, then T is said to be *acyclic-valued.*

Let $T: E \multimap F$ be an u.s.c. multi-valued map. Following Martelli and Vignoli (see [8]) we define

$$|T| = \limsup_{\|x\| \to \infty} \frac{\Phi(T(x))}{\|x\|}$$

where $\Phi(T(x)) = \sup\{\|y\| : y \in T(x)\}$. If $|T| < \infty$, then T is said to be *quasibounded* and the number $|T|$ is called the *quasinorm* of T. This definition extend the one given by Granas in [5] for the single-valued case.

By $Q(E)$ we denote the set of all the multi-valued quasibounded maps from E into itself. Note that $Q(E)$ is not a vector space, in fact $T - T \neq 0$ unless T is single-valued.

Let $B_r = \{x \in E: \|x\| \leqslant r\}$ be the ball of radius r centered at the origin, S_r denotes the boundary of B_r.

An u.s.c. multi-valued map $\varphi\colon S_1 \multimap E$ is said to be an *admissible vector field* if it satisfies the following conditions

(i) φ is singularity free, i.e. $0 \notin \operatorname{Im}\varphi$;

(ii) φ is a compact vector field, that is $\varphi(x) = x - T(x)$, where $T\colon S_1 \multimap E$ is a compact multi-valued map;

(iii) φ is acyclic-valued.

Two admissible vector fields φ_0 and φ_1 are said to be *homotopic* if there exists an u.s.c. multi-valued map $H\colon S_1 \times [0, 1] \multimap E$ with the following properties

a) $H(\cdot, t)$ is an admissible vector field for every $t \in [0, 1]$;

b) $H(\cdot, 0) = \varphi_0$ and $H(\cdot, 1) = \varphi_1$;

c) I-H defined by $(x, t) \multimap x - H(x, t)$ is a compact multi-valued map.

Such a map is called an *admissible homotopy* joining φ_0 and φ_1.

We say that an u.s.c. multi-valued map $\tilde{\varphi}$ is an *admissible extension* of the admissible vector field φ if $\tilde{\varphi}$ satisfies the following conditions

I) $\tilde{\varphi}$ is singularity free;

II) $\tilde{\varphi}(x) = x - \tilde{T}(x)$, where $\tilde{T}\colon B_1 \multimap E$ is a compact multi-valued map;

III) $\tilde{\varphi}(x)$ is compact and acyclic for every $x \in B_1$;

IV) $\tilde{\varphi}/S_1 = \varphi$.

φ is called *inessential* if it admits an admissible extension and *essential* otherwise.

M. Furi and M. Martelli defined in [2] a function χ (called *characteristic*) from the set of all admissible vector fields into $\mathbf{Z}_2$ with the following properties

1) *Normalization*: $\chi(I) = 1$, where $I\colon S_1 \hookrightarrow E$ is the inclusion.

2) *Homotopy*: if φ_0 and φ_1 are homotopic then $\chi(\varphi_0) = \chi(\varphi_1)$.

3) *Solvability*: if $\chi(\varphi) \neq 0$ then φ is essential.

4) *Antipodality*: if $\varphi(x) = -\varphi(-x)$ for all $x \in S_1$ then $\chi(\varphi) = 1$.

Let A and B be bounded nonempty subsets of Banach space E. Define the *Hausdorff distance* $\delta(A, B)$ among A and B as follows

$$\delta(A, B) = \inf\{t > 0 \colon A \subset B + t\mathring{B}_1,\ B \subset A + t\mathring{B}_1\} .$$

We recall that $\delta(A, B)$ is a semimetric in the family $B(E)$ of all nonempty bounded subsets of E, while it is a metric in the family $C(E)$ of all nonempty bounded and closed subsets of E. We give now some properties regarding the Hausdorff distance we will use below.

(i) $\delta(tA, tB) = t\delta(A, B)\ ,\ t \geq 0 \qquad A, B \in B(E)\ ;$

(ii) $\delta(A + B, A_1 + B_1) \leq \delta(A, A_1) + \delta(B, B_1)\ ,$

$A, A_1, B, B_1 \in B(E)\ ;$

(iii) $\delta(A + C, B + C) = \delta(A, B)\ , \qquad C \in B(E)\ ;\ A, B \in C_0(E)\ ,$

where $C_0(E)$ is the family of all nonempty, bounded, closed and convex subsets of E.

3. – Spectral theory for quasibounded multi-valued maps and surjectivity.

In what follows, unless otherwise stated, E will be a complex Banach space. Let us define in $Q(E)$ a semimetric as follows

$$\Delta(T, S) = \limsup_{\|x\| \to \infty} \frac{\delta(T(x), S(x))}{\|x\|} \qquad \text{whatever } T, S \in Q(E)\ .$$

By the properties of δ it is easy to see that Δ is a semimetric in $Q(E)$; it cannot be a metric since $\Delta(T, S) = 0$ does not imply $T = S$ on E.

Note that $\Delta(T, 0) = |T|$, since $\delta(T(x), 0) = \Phi(T(x))$, therefore it follows that $\Delta(T, S) < +\infty$, whatever $T, S \in Q(E)$. We say that $T, S \in Q(E)$ are *asymptotically equivalent* (notation $T \sim S$) if and only if $\Delta(T, S) = 0$.

It is easy to see that this is an equivalence relation. Consider the space $A(E)$ of all the equivalence classes of quasibounded maps. If we put $\bar{\Delta}(\{T\}, \{S\}) = \Delta(T, S)$ where $\{T\}, \{S\} \in A(E)$ and $T \in \{T\}$, $S \in \{S\}$, the space $(A(E), \bar{\Delta})$ is a metric space.

Let $T \in Q(E)$. Define the asymptotic spectrum of T as follows

$$\Sigma(T) = \left\{\lambda \in \boldsymbol{C} \colon \liminf_{\|x\|\to\infty} \frac{\delta(T(x), \lambda x)}{\|x\|} = 0\right\}$$

and we denote by $r(T)$ the spectral radius of T which is defined as follows

$$r(T) = \sup\left\{|\lambda| \colon \lambda \in \Sigma(T)\right\} .$$

For the sake of simplicity we put as in [3]

$$d(\lambda - T) = \liminf_{\|x\|\to\infty} \frac{\delta(T(x), \lambda x)}{\|x\|}$$

where by $\lambda - T$ we denote the mapping $\lambda I - T$, I the identity on E.

Remark 1. Note that $\delta(T(x), \lambda x) = \Phi(\lambda x - T(x))$, hence if T is a single-valued map then the asymptotic spectrum $\Sigma(T)$ coincides with that defined by Furi-Vignoli in [3].

We have now the following

Proposition 1. *The asymptotic spectrum has the following properties*

(i) *Let* $T, S \in Q(E)$. *If* $T \sim S$ *then* $\Sigma(T) = \Sigma(S)$;

(ii) $r(T) \leqslant |T|$;

(iii) $\Sigma(T)$ *is compact*;

(iv) $\Sigma(\gamma T) = \gamma \Sigma(T)$, $\gamma \in \boldsymbol{C}$;

(v) $\Sigma(\gamma + T) = \gamma + \Sigma(T)$, $\gamma \in \boldsymbol{C}$;

(vi) $\lambda \in \Sigma(T) \Rightarrow d(T) \leqslant |\lambda|$.

Proof.

(i) First assume $\lambda \in \Sigma(T)$; we have to show that $\lambda \in \Sigma(S)$. By our hypothesis we get

$$\liminf_{\|x\|\to\infty} \frac{\delta(S(x), \lambda x)}{\|x\|} \leqslant \limsup_{\|x\|\to\infty} \frac{\delta(S(x), T(x))}{\|x\|} + \liminf_{\|x\|\to\infty} \frac{\delta(T(x), \lambda x)}{\|x\|} = 0 .$$

Hence $\lambda \in \Sigma(S)$. In the same way we can prove that if $\lambda \in \Sigma(S)$ then $\lambda \in \Sigma(T)$.

(ii) Let $\lambda \in \Sigma(T)$, we have

$$\frac{\delta(T(x), \lambda x)}{\|x\|} \geqslant |\lambda| - \frac{\delta(0, T(x))}{\|x\|}$$

and so $|\lambda| \leqslant d(\lambda - T) + |T| = |T|$.

(iii) First let us show that the map $d\colon Q(E) \to \boldsymbol{R}$ is continuous. It is enough to prove that

$$|d(T) - d(S)| \leqslant \Delta(T, S) .$$

We have

$$d(T) = \liminf_{\|x\|\to\infty} \frac{\delta(T(x), 0)}{\|x\|} \leqslant \limsup_{\|x\|\to\infty} \frac{\delta(T(x), S(x))}{\|x\|} + \\ + \liminf_{\|x\|\to\infty} \frac{\delta(S(x), 0)}{\|x\|} = \Delta(T, S) + d(S) .$$

Hence $d(T) - d(S) \leqslant \Delta(T, S)$. In the same way $d(S) - d(T) \leqslant \Delta(S, T) = \Delta(T, S)$. Therefore

$$|d(T) - d(S)| \leqslant \Delta(T, S) .$$

Then the map $\lambda \to d(\lambda - T)$ is continuous, so $\Sigma(T)$ is closed. By (ii) $\Sigma(T)$ is bounded hence it is compact.

(iv) For $\gamma = 0$ it is obvious. For $\gamma \in \boldsymbol{C} \setminus \{0\}$ we have

$$d(\lambda - \gamma T) = |\gamma| d\left(\frac{\lambda}{\gamma} - T\right)$$

hence $\lambda \in \Sigma(\gamma T)$ if and only if $\lambda/\gamma \in \Sigma(T)$.

(v) We have

$$d(\lambda - (\gamma + T)) = d((\lambda - \gamma) - T)$$

hence $\lambda \in \Sigma(\gamma + T)$ if and only if $\lambda - \gamma \in \Sigma(T)$.

(vi) We have

$$d(\lambda - T) = \liminf_{\|x\|\to\infty} \frac{\delta(T(x), \lambda x)}{\|x\|} \geqslant \liminf_{\|x\|\to\infty} \frac{\delta(T(x), 0)}{\|x\|} - \\ - \limsup_{\|x\|\to\infty} \frac{\delta(\lambda x, 0)}{\|x\|} = d(T) - |\lambda| .$$

If $\lambda \in \Sigma(T)$ then $d(T) \leqslant |\lambda|$. Q.E.D.

Let $G\colon E \multimap E$ be a multi-valued map. We say that G *satisfies property* (A) if

$$\lim_{\|x\|\to\infty} \frac{\operatorname{diam} T(x)}{\|x\|} = 0\,,$$

where $\operatorname{diam} T(x)$ denotes the diameter of the set $T(x)$.

Let $T\colon E \multimap E$ be a compact, quasibounded, acyclic-valued map satisfying property (A). Suppose that $d(I-T)>0$. From the definition it follows that there exists $r_0>0$ such that $x\notin T(x)$ for any $x\in E$ with $\|x\|>r_0$. In fact suppose the contrary. Then there exists a sequence $\{x_n\}\subset E$ such that $x_n\in T(x_n)$ and $\|x_n\|>n$, hence in view of the property (A) we get

$$\lim_{n\to\infty} \frac{\delta\big(T(x_n), x_n\big)}{\|x_n\|} = 0$$

and so we have a contradiction.

Now we want to define for the compact vector field $\Phi(x) = x - T(x)$ such that $d(I-T)>0$, a surjectivity degree as in [3]. We denote by Φ_r the restriction of Φ to S_r.

For $r>r_0$ we have $0\notin \Phi_r(S_r)$.

Let us define

$$\chi(\Phi_r) = \chi(\Phi_1^r)$$

where $\Phi_1^r(y) = y - \big(T(ry)\big)/r$ and $y\in S_1$.

For such a map the characteristic function χ is well defined, since from $0\notin\Phi_r(S_r)$ it follows that $0\notin\Phi_1^r(S_1)$ and so Φ_1^r is an admissible vector that field. We set

$$\deg(I-T) = \chi(\Phi_r) \qquad \text{with } r\geqslant r_0\,.$$

This is a well posed definition since we have $\chi(\Phi_r)=\chi(\Phi_s)$ for any $r, s\geqslant r_0$. In fact from the definition we have

$$\chi(\Phi_r) = \chi(\Phi_1^r) \qquad \text{and} \qquad \chi(\Phi_s) = \chi(\Phi_1^s)\,.$$

Assume that $s>r$. If we consider the homotopy

$$H(y,t) = y - \frac{T\big(t(s-r)\,y + ry\big)}{t(s-r)+r}$$

defined on $S_1\times[0,1]$, it easy to see that this is an admissible homotopy joining Φ_1^r and Φ_1^s, hence $\chi(\Phi_1^r) = \chi(\Phi_1^s)$ by the homotopy property of the characteristic function χ.

Let T be as above and $\lambda \in \boldsymbol{C}$. We say that $\lambda - T$ is *admissible for surjectivity* (*s-admissible*) if $\lambda \notin \Sigma(T) \cup \{0\}$.

We define the *surjectivity degree* of $\lambda - T$ as follows

$$\deg(\lambda - T) = \deg(I - \lambda^{-1}T) .$$

An homotopy $H\colon E \times [0,1] \multimap E$ is said to be an *s-homotopy* if the following conditions are satisfied:

(a) $H(x,t) = \lambda(t)x - F(x,t)$ where $\lambda\colon [0,1] \to \boldsymbol{C}$ is continuous and $F\colon E \times [0,1] \multimap E$ is a compact, acyclic-valued map;

(b) $$\lim_{\substack{t \to t_0 \\ \|x\| \to \infty}} \frac{\delta\big(F(x,t), F(x,t_0)\big)}{\|x\|} = 0 ;$$

(c) The map $\lambda(t) - F(\cdot, t)$ is s-admissible for any $t \in [0,1]$;

(d) The map $F(x,t)$ satisfies property (A) uniformly with respect to $t \in [0,1]$.

Two s-admissible maps are said to be *s-homotopic* if there exists an s-homotopy joining them.

We have the following

PROPOSITION 2. *The degree of surjectivity has the following properties.*

(i) *Two s-homotopic maps have the same degree of surjectivity.*

(ii) *Let* $T\colon E \multimap E$ *be compact, quasibounded, with acyclic values. If* $\lambda_1, \lambda_2 \neq 0$ *and* λ_1, λ_2 *belong to the same component of* $\boldsymbol{C} \setminus \Sigma(T)$, *then* $\deg(\lambda_1 - T) = \deg(\lambda_2 - T)$.

(iii) *If* $\deg(\lambda - T) = 1$ *then* $\lambda - T$ *is onto.*

(iv) $\deg(\lambda - 0) = 1$ *for all* $\lambda \in \boldsymbol{C} \setminus \{0\}$.

PROOF.

(i) Let $H(x,t) = \lambda(x)x - F(x,t)$ be an s-homotopy joining $\lambda_0 - T_0$ and $\lambda_1 - T_1$. First we prove that there exists $r_0 > 0$ such that $0 \notin \lambda(t)x - F(x,t)$ for any $x \in E$, with $\|x\| \geq r_0$ and for any $t \in [0,1]$.

Assume the contrary. Then there exist two sequences $\{x_n\}$, $\{t_n\}$ with $\|x_n\| > n$ and

$$0 \in \lambda(t_n)x_n - F(x_n, t_n) .$$

Since $\{t_n\} \subset [0, 1]$, without loss of generality we can assume that $t_n \to \bar{t} \in [0, 1]$. We have

$$\delta(\lambda(\bar{t})x_n, F(x_n, \bar{t})) \leqslant \delta(\lambda(\bar{t})x_n, \lambda(t_n)x_n) + \\ + \delta(\lambda(t_n)x_n, F(x_n, t_n)) + \delta(F(x_n, t_n), F(x_n, \bar{t})) \, .$$

Hence

$$d(\lambda(\bar{t}) - F(\cdot, \bar{t})) \leqslant \limsup_{n\to\infty} \frac{\delta(\lambda(\bar{t})x_n, \lambda(t_n)x_n)}{\|x_n\|} + \\ + \liminf_{n\to\infty} \frac{\delta(\lambda(t_n)x_n, F(x_n, t_n))}{\|x_n\|} + \limsup_{n\to\infty} \frac{\delta(F(x_n, t_n), F(x_n, \bar{t}))}{\|x_n\|} \, .$$

Respectively from the continuity of the function λ and properties (d) and (b) we obtain $d(\lambda(\bar{t}) - F(\cdot, \bar{t})) = 0$, contradicting the property (c) of the s-homotopy. Consider now the homotopy

$$\tilde{H}(y, t) = y - \lambda^{-1}(t) \frac{F(ry, t)}{r}$$

where:

$$\tilde{H}\colon S_1 \times [0, 1] \multimap E \qquad \text{and} \qquad r \geqslant r_0 \, .$$

It is easy so see that $\tilde{H}(y, t)$ is an admissible homotopy in the sense of Furi - Martelli [2].

Therefore, by our definition of the degree of surjectivity, we have

$$\deg(\lambda_0 - T_0) = \deg(\lambda_1 - T_1) \, .$$

(ii) Let Ω be the component of $\boldsymbol{C} \setminus \Sigma(T)$ which contains λ_1 and λ_2. Since Ω is open there exists a path $s\colon [0, 1] \to \Omega \setminus \{0\}$ joining λ_1 and λ_2. Consider the homotopy $H(x, t) = s(t)x - T(x)$ defined on $E \times [0, 1] \multimap E$.

This is an s-homotopy, hence from the property (i), we have the assertion.

(iii) It sufficies to prove that for any $p \in E$, there exists $x \in E$ such that $p \in \lambda x - T(x)$, i.e., $T_p(x) = \lambda^{-1}(T(x) + p)$ has a fixed point. Consider the homotopy:

$$H(x, t) = \lambda x - (T(x) + tp)$$

defined on $E \times [0, 1] \multimap E$.

We have $d(\lambda - T - tp) = d(\lambda - T) > 0$ and so (c) is verified. Hence $H(x, t)$ is an s-homotopy, in fact the other conditions are easily verified.

Furthermore $H(x, 0) = \lambda x - T(x)$ and $H(x, 1) = \lambda x - T(x) - p$, hence $\deg(\lambda - T - p) = 1$. Then from the definition of the degree of surjectivity there exists $r_0 > 0$ such that the restriction of the map

$$y \to y - \frac{\lambda^{-1}(T(ry) + p)}{r} \qquad (r \geqslant r_0)$$

to S_1 is essential. Therefore

$$0 \in y - \frac{\lambda^{-1}(T(ry) + p)}{r}$$

for some y such that $\|y\| \leqslant 1$, i.e. $p \in \lambda x - T(x)$, where $x = ry$.

(iv) We have for any $\lambda \in \boldsymbol{C} \setminus \{0\}$

$$\deg(\lambda - 0) = \chi(I)$$

where I is the identity restricted on S_1, and so by the normalization property of χ we have the assertion. Q.E.D.

We are now in a position of proving the following

THEOREM 1. *Let* $T\colon E \multimap E$ *be a compact, quasibounded, acyclic-valued map satisfying property* (A) *and* $\lambda \neq 0$. *If* λ *belongs to the unbounded component of* $\boldsymbol{C} \setminus \Sigma(T)$ *then* $\lambda - T$ *is onto.*

PROOF. – Let $r > |T|$. Consider the homotopy

$$H(x, t) = rx - tT(x)$$

defined on $E \times [0, 1]$. This is an s-homotopy; in fact the conditions (a), (b) and (d) are easily verified.

Moreover we have

$$0 < r - t|T| \leqslant d(r - tT) \qquad \text{for any } t \in [0, 1]$$

and so the condition (c) is also verified.

By (i) and (iv) of the above theorem we have

$$\deg(r - T) = \deg(r - 0) = 1\,.$$

Hence from (ii) and (iii) we get that $\lambda - T$ is onto for any $\lambda \neq 0$ belonging to the unbounded component of $\boldsymbol{C} \setminus \Sigma(T)$. Q.E.D.

REMARK 2. From the definition of the asymptotic spectrum of a multi-valued quasibounded map $T\colon E \multimap E$, we can easily prove that

$$\Sigma(T) = \bigcap_{f \in \Gamma} \Sigma(f)$$

where Γ is the set of all the non necessarily continuous selections of T and $\Sigma(f)$ is the asymptotic spectrum for a quasibounded map $f\colon E \to E$ introduced in [3]. If T satisfies property (A) we have

$$\Sigma(T) = \Sigma(f) \qquad \text{whatever } f \in \Gamma .$$

In fact in this case, if $f, g \in \Gamma$, we have $|f - g| = 0$ and so $\Sigma(f) = \Sigma(g)$ (see [3]).

Let $T\colon E \multimap E$ be a multi-valued quasibounded map. We say that T is *asymptotically linear* if there exists a bounded linear (single-valued) map T'_∞ such that $T'_\infty \sim T$. This linear map will be called the *asymptotic derivative* of T. The asymptotic derivative of a multi-valued quasibounded map T is unique. In fact if there exist two linear maps S'_∞ and T'_∞ such that $T \sim T'_\infty$ and $T \sim S'_\infty$, then $T'_\infty \sim S'_\infty$. Since T'_∞ and S'_∞ are linear, we have $T'_\infty = S'_\infty$. It is easy to see that if T is asymptotically linear then T satisfies property (A). If T is compact then T'_∞ is compact. The proof of this fact is implicitely contained in [9].

We have the following

COROLLARY 1. *Let $T\colon E \multimap E$ be a compact, quasibounded, acyclic-valued map, and suppose that T is asymptotically linear. If the spectral radius $r(T'_\infty)$ is less than one, then $I - T$ is onto.*

The proof follows from Proposition 1 and Theorem 1.

REMARK 3. Since T'_∞ is linear, $\Sigma(T'_\infty)$ is just the usual approximate point spectrum $\sigma_a(T'_\infty)$ (see [4]), i.e.,

$$\sigma_a(T'_\infty) = \left\{\lambda \in \boldsymbol{C}\colon \inf_{\|x\| = 1} \|\lambda x - T'_\infty(x)\| = 0\right\}.$$

In [4] an example of a continuous quasibounded map $T\colon E \to E$ with empty asymptotic spectrum is given. Moreover it is proved that $\Sigma(T) \neq \emptyset$ in the case where T is compact and $\dim E = +\infty$. The following theorem can be regarded as an extension of the above result.

THEOREM 2. *Let* $T\colon E \multimap E$ *be a compact, quasibounded, acyclic-valued map, satisfying property* (A). *Suppose* $\dim E = +\infty$. *Then* $\Sigma(T) \neq \emptyset$.

PROOF. Consider the sphere S_n. Put $A = \{n \in N\colon 0 \notin \overline{T(S_n)}\}$ and $B = N \setminus A$. Clearly either A or B is infinite. Assume first that A is infinite; then for any $n \in A$ there exists ε_n such that $B_{\varepsilon_n} \cap T(S_n) = \emptyset$. By Birkoff-Kellog theorem for multi-valued maps (see [7]), there exists $x_n \in S_n$ and $\lambda_n > 0$ such that $\lambda_n x_n \in T(x_n)$. From the fact that T is quasibounded it follows that $\{\lambda_n\}$ is bounded hence we may suppose, without loss of generality, that $\lambda_n \to \lambda$. We have

$$\frac{\delta(\lambda x_n, T(x_n))}{\|x_n\|} \leqslant |\lambda_n - \lambda| + \frac{\delta(\lambda_n x_n, T(x_n))}{\|x_n\|} .$$

Hence, by property (A), $\lambda \in \Sigma(T)$. Assume now that B is infinite. Then, there exists $x_n \in S_n$ such that

$$\mu_n = \inf\{\|z\|\colon z \in T(x_n)\} < 1 .$$

We have

$$\frac{\delta(0, T(x_n))}{\|x_n\|} \leqslant \frac{\operatorname{diam} T(x_n)}{\|x_n\|} + \frac{\mu_n}{\|x_n\|} .$$

Thus $0 \in \Sigma(T)$. Q.E.D.

Let $T\colon E \multimap E$ be as in Theorem 2 and $\dim E < +\infty$. Then if n is even $\Sigma(T)$ can be empty (see [4]). For n odd we have the following result

THEOREM 3. *Let* $T\colon E \multimap E$ *be a compact, quasibounded, acyclic-valued map, satisfying property* (A). *Suppose that* E *is a real Banach space and* $\dim E$ *is finite and odd. Then* $\Sigma(T) \neq \emptyset$.

PROOF. Consider the sphere S_n. Put $A = \{n \in N\colon 0 \notin \overline{T(S_n)}\}$ and $B = N \setminus A$. If B is infinite the same proof of Theorem 2, can be used to show that $0 \in \Sigma(T)$. Therefore we may assume that A is infinite. Clearly it sufficies to prove that for any $n \in A$ there exist $\lambda_n \in \mathbf{R}$ and $x_n \in S_n$ such that $\lambda_n x_n \in T(x_n)$. For any $n \in A$ there exist r_n and R_n such that

$$T(x_n) \subset C_n = \{x \in E\colon r_n \leqslant \|x\| \leqslant R_n\} .$$

Consider the map $T_n \circ \Pi \colon C_n \multimap C_n$, where T_n is the restriction of T to S_n and $\Pi \colon C_n \to S_n$ is the radial retraction.

Let W be the graph of $T_n \circ \Pi$ and let $r, s \colon W \to C_n$ be the projections defined by

$$r(x, y) = x , \qquad s(x, y) = y .$$

The maps r and s are continuous, and the sets $r^{-1}(x)$ and $T_n \circ \Pi(x)$ are homeomorphic. It follows that $r^{-1}(x)$ is acyclic for every $x \in C_n$. Hence by Vietoris mapping theorem, the homomorphism $r_k^* \colon H_k(W) \to H_k(C_n)$ induced by r is a isomorphism (see [6]).

If $-x \in T_n \circ \Pi(x)$ for some $x \in C_n$, then clearly $\lambda_n x_n \in T(x_n)$ for some $\lambda_n \in \boldsymbol{R}$ and $x_n \in S_n$.

Assume now that $-x \notin T_n \circ \Pi(x)$ for every $x \in C_n$.

Then r and s are homotopic (see [6]) and so the homomorphism $(T_n \circ \Pi)_k^* = s_k^* r_k^{*-1}$ is the identity.

This implies that $\Lambda(T_n \circ \Pi)$, the Lefschetz number of $T_n \circ \Pi$ (in the sense of Eilenberg-Montgomery), equals the Euler-Poincaré characteristic of C_n. Since S_n is a homotopy retract of C_n it follows that $\Lambda(T_n \circ \Pi) = 2$. By the Eilenberg-Montgomery theorem (see [1]) $T_n \circ \Pi$ has a fixed point. Thus there exist $\lambda_n \in \boldsymbol{R}$ and $x_n \in S_n$ such that $\lambda_n x_n \in T(x_n)$. Q.E.D.

REMARK 4. The following example shows that if T does not satisfy property (A), then $\Sigma(T)$ can be empty, even in the case when $\dim E$ is odd. In fact the multivalued map $T \colon \boldsymbol{R} \multimap \boldsymbol{R}$ defined by

$$T(x) = \{y \in \boldsymbol{R} \colon -|x| \leqslant y \leqslant |x|\}$$

does not satisfy property (A) and its asymptotic spectrum is empty.

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