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PERIODIC SOLUTIONS TO DIFFERENTIAL EQUATIONS WITH DELAY

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ABSTRACT. We describe a method for proving the existence of periodic solutions to n-dimensional systems of the form $z'(t) - Az(t) - Bz(t-\tau) = F[z(t)]$.

1. Introduction. In this paper we describe an approach for proving the existence of periodic solutions to autonomous systems of differential equations with delay, of the form

$$z'(t) - Az(t) - Bz(t-\tau) = F[z(t)], \tag{1}$$

with $z : R \to R^m$, A and B $m \times m$ constant matrices, $\tau > 0$ and $F : R^m \to R^m$ continuous. Our approach avoids the heavy machinery associated with the use of the Poincaré map

$$\mathcal{P} : \varphi(\cdot) \longrightarrow z(T + \cdot; \varphi)$$

which maps the initial segment $\varphi(s)$, $-\tau \leq s \leq 0$, into the terminal segment $z(T+s;\varphi)$, $-\tau \leq s \leq 0$, of the corresponding solution to (1) with initial data $\varphi(\cdot)$. T here is the unknown period.

Our method is based on ideas that go by various names: Cesari's method, Urabe's method (see [8]); the method of harmonic balance, (see [**4**, **5**, **6**]), the theory of reproducing kernels (see [**1**,**2**]). Our method extends easily to problems with several delays; we will sketch this extension via an example.

Systems of the above type occur in the modelling of electronic circuits, ship motion and population growth. We describe two examples.

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Hastings [**3**] has studied a predator-prey model of the form

$$\begin{aligned} H'(t) &= rH(t-T) - dH(t) - f(H(t))P(t), \\ P'(t) &= P(t)f(H(t)) - P(t) \end{aligned}$$

where H is the prey (host) population and P is the predator population.

Minorsky [**7**, p. 534 et seq] has studied the "activated tank method" for stabilizing the rolling motion (yaw) of a ship, which pumps liquid ballast to different holding tanks on a ship in order to counteract the rolling motion induced by sea swells. If $\theta(t)$ is the angle of inclination from a vertical plane through the keel, then this system is modelled by

$$\theta''(t) + k\theta'(t) + K\theta'(t-h) + \omega_0^2 \sin\theta(t) = a \sin \omega t.$$

Here the term $K\theta'(t-h)$ represents the delayed effect of the ballast, which must be pumped. A key question for this system is whether it allows periodic motions when $a \equiv 0$.

2. The operator equation. Our first step is to convert (1) to an integral equation on an appropriate Banach space of functions. Consider (1) as a version of $z'(t) - Az(t) = f(t)$, and add the periodic boundary conditions

$$z(0) = z(T) \quad (T \text{ unknown}). \tag{2}$$

We define $x(t) = z(Tt/2\pi)$, so $x(\cdot)$ is 2π-periodic if and only if $z(\cdot)$ is T-periodic. Then the differential equation (1) with the boundary conditions (2) becomes

$$\begin{aligned} \omega\dot{x}(t) - Ax(t) &= Bx(t-\sigma) + F[x(t)], \\ x(0) &= x(2\pi), \end{aligned} \tag{3}$$

where $\sigma = \omega\tau$, $\omega = 2\pi/T$. Thus we have replaced the unknown period T by the fixed period 2π by introducing the parameter ω into the system. A solution of (3) will not in general yield a periodic solution of (1), because of the delay term; to guarantee a periodic solution we would replace $x(0) = x(2\pi)$ by $x(s) = x(2\pi + s)$, $-\sigma \le s \le 0$.

The Green's matrix for $\omega\dot{x} - Ax = f$, with periodic boundary condition $x(0) = x(2\pi)$, is given by (see [8])

$$G_\omega(t-s) = (1/\omega)[I - e^{2\pi A/\omega}]^{-1}\begin{cases} e^{A(t-s)/\omega} & 0 \le s \le t, \\ e^{A(2\pi+t-s)/\omega} & t < s \le 2\pi, \end{cases} \tag{4}$$

assuming $I - e^{2\pi A/\omega}$ is nonsingular. We will make this assumption, which will allow us to write (3) as an operator equation

$$\begin{aligned} x(t) = \mathcal{T}_\omega[x](t) &\equiv \mathcal{G}_\omega[BS_{-\sigma}[x] + \mathcal{F}[x]](t) \\ &\equiv \int_0^{2\pi} G_\omega(t-s)[Bx(s-\sigma) + F[x(s)]]\,ds, \end{aligned} \tag{5}$$

where $S_{-\sigma}[x](t) = x(t-\sigma)$ is the shift operator and $\mathcal{F}[x]$ is the Nemytskii operator $\mathcal{F}[x](t) = F[x(t)]$.

The appropriate space for (5) depends on the nonlinearity F and on the desired degree of regularity of solutions. The idea is to take an appropriate space X_0 of functions defined on $[0, 2\pi]$, say $C([0, 2\pi], R^m)$ or $L_2((0, 2\pi), R^m)$, and create the space of periodic extensions

$$X = \{x \in X_0 \mid x(t + 2\pi) = x(t) \text{ for all } t \in R\}$$

using the norm from X_0.

For example, if $m = 1$ and $F[x] = x^3$, then $X_0 = L_2(0, 2\pi)$ is not appropriate, while $X_0 = C[0, 2\pi]$ (or $X_0 = L_6(0, 2\pi)$) is suitable, in the sense that $\mathcal{T}_\omega$ is well-defined on X. In fact, we ask more, that

$$\mathcal{T}_\omega : X \to X$$

be compact. For F continuous, this will be the case for $X_0 = C[0, 2\pi]$ since in this case $\mathcal{T}_\omega[x]$ will lie in

$$Z = \{y \mid y(t + 2\pi) = y(t) \text{ for all } t \in R, y \in W^{1,2}((0, 2\pi), R^m)\}.$$

Here $W^{1,2}$ is the Sobolev space of functions with generalized derivative in L_2, which for functions of a single variable coincides with space of absolutely continuous functions with derivative in L_2. The natural embedding of this space into C or L_2 is compact.

Assuming that appropriate spaces X and Z have been defined, with $\mathcal{T}_\omega$ a compact mapping from X into $Z \subseteq X$, we note that for a given $\sigma \in [0, 2\pi)$, any fixed point in X of $\mathcal{T}_\omega$ is a 2π-periodic solution of (3), hence corresponds to a T-periodic solution of (1) with delay σ/ω. Thus the search for periodic solutions of (1) has been reduced to a search for those values of the parameters ω and $\sigma = \omega\tau$, which yield fixed points of $\mathcal{T}_\omega$ different from $x = 0$.

3. The method of harmonic balance/reproducing nonlinearities. One can now use fixed point theorems or more delicate methods involving degree theory on (5), as long as one avoids the trivial solution (see Mees [**6**, pp. 130–131], for a discussion of the advantages of degree theory over fixed-point theorems). Our method to avoid the trivial solution is to use the method of harmonic balance to locate a center for a ball to which we can apply Schauder's theorem. Harmonic balance is better known to mathematicians as the "truncated Galerkin" method. The basis of the method of harmonic balance is simple, but the actual application is computationally demanding and almost always requires considerable assistance from a computer.

We assume that functions in X have Fourier expansions

$$x \in X \Longrightarrow x(t) \longleftrightarrow \sum_{-\infty}^{\infty} a_k e^{ikt}$$

with $a_0 \in \mathbf{R}^n$, $a_k \in \mathbf{C}^n$ for $k \neq 0$, $a_{-k} = \bar{a}_k$ (the conjugate of a_k). Define the standard Fourier projection

$$P_n : x \longmapsto x_n, \qquad x_n(t) = \sum_{-n}^{n} a_k e^{ikt}. \tag{6}$$

Note that $F(P_n[x](t))$ is 2π-periodic; we assume our space X has been defined in such a way that $x_n \in X$ and $F[x_n(t)] \in X$, so $F[x_n(t)]$ also has a Fourier expansion with

$$P_n F[x_n(t)] = \sum_{-n}^{n} f_k e^{ikt}, \qquad f_k = f_k(a_0, a_{\pm 1}, \dots, a_{\pm n}).$$

The method of harmonic balance replaces the operator equation (5) by the following *finite-dimensional* (Galerkin) approximation:

$$x_n(t) = P_n \mathcal{T}_\omega[x_n](t) = \mathcal{G}_\omega\{BS_{-\sigma}[x_n] + P_n \mathcal{F}[x_n]\}(t). \tag{7}$$

Here we have used the facts $P_n \mathcal{G}_\omega = \mathcal{G}_\omega P_n$, $P_n^2 = P_n$, $P_n BS_{-\sigma}[x_n] = BS_{-\sigma}[x_n]$. For a given fixed n, (7) yields (after some easy calculations) a finite system of equations for the (vector) coefficients a_k, $k = 0, \pm 1, \dots, \pm n$:

$$(8) \qquad a_k - (i\omega kI - A)^{-1}[Be^{-ik\sigma}a_k + f_k(a_0, a_{\pm 1}, \dots, a_{\pm n})] = 0.$$

These are called the harmonic balance equations; their derivation requires that $(i\omega kI - A)$ be invertible, which we will assume.

There is one small technical problem here. The original system in (1) is autonomous, hence if $x(t)$ is a periodic solution then so is $x(t - t_0)$. The Fourier coefficients of these two solutions are related by a rotation e^{-ikt_0}. We want only one of the family $\{x(t - t_0) \mid t_0 \in R\}$, so we normalize one component of some fixed a_{i_0}, say $a_{i_0 j_0}$, to have a fixed argument (usually arg $= 0$). This selects a single representative from this family (see [**6**] or [**4**] for a fuller discussion).

Although we shall not use it here, we mention in passing the

Harmonic balance theorem. *Assume that for some n and some $\hat{\omega} >$, the system* (8) *has a nontrivial solution. Then, under appropriate assumptions, the operator equation* (5) *will have a solution* (*in general with a value $\omega \neq \hat{\omega}$*).

For a discussion of the assumptions, see [**4**] or (with slightly different notation but a more comprehensive discussion) [**6**]. Although they can seem to be quite intimidating, they are quite tractable computationally, with the aid of a computer. The important point about this method is that it does not pass to a limit, or estimate errors for large n. On the basis of the existence of a solution for a given n (say $n = 1$ or 2) it asserts the existence of a solution to the original problem.

The harmonic balance equations can be straightforward or difficult to solve, depending on the function F. In a different context, Bazley and collaborators (see [**1, 2**]) have identified an important class of nonlinearities for which Galerkin-type equations (of which (7) is an example) are far more tractable than in general. We briefly outline the connection with harmonic balance.

If H is a separable Hilbert space and if $\mathcal{B} = \{u_j\}$ is a countable subset of orthonormal elements (in practice, either $j = 0, 1, 2, \dots$, or

$j = 0, \pm 1, \pm 2, \dots$), then Bazley defines a nonlinear mapping $F : D_F \subset H \to H$ (D_F = domain of F) to be a *reproducing nonlinearity* for $\mathcal{B}$ if for each natural number n there is a natural number $m(n)$ such that

$$\begin{gathered} F(\operatorname{span}\{u_k\}_0^n) \subseteq \operatorname{span}\{u_k\}_0^{m(n)} \\ (\text{respectively } F(\operatorname{span}\{u_k\}_{-n}^n) \subseteq \operatorname{span}\{u_k\}_{-m(n)}^{m(n)}. \end{gathered} \tag{9}$$

For such nonlinearities, the computations associated with Galerkin approximations are simplified. The connection with harmonic balance is clear: harmonic balance uses $u_k = e^{ikt}$ and goes one step further by truncating $\operatorname{span}\{u_k\}_0^{m(n)}$ to $\operatorname{span}\{u_k\}_0^n$ whenever $m(n) > n$; in fact, Bazley et al. often do the same thing in practice.

When one has $\mathcal{B} = \{e^{ikt}\}_{-\infty}^{\infty}$, and $F[x] = x^p$, p a natural number, an easy computation shows F is reproducing, with for example $m(2) = 2p$.

The connection with harmonic balance is even more transparent when one writes (9), for $\mathcal{B} = \{e^{ikt}\}_{-\infty}^{\infty}$, in the more explicit form

$$F\left(\sum_{-n}^{n} a_k e^{ikt}\right) = \sum_{-m(n)}^{m(n)} f_k(a_0, a_{\pm 1}, \dots, a_{\pm n}) e^{ikt}.$$

4. The existence of periodic solutions.

We now describe our existence theorem for periodic solutions of (1), the proof of which is based on Schauder's fixed point theorem applied to a closed ball centered on a solution $\hat{x}_n$ of the harmonic balance equations (8). For convenience, we choose X to be the periodic extension of the subspace $\{x(t) \mid x(\cdot) \in C[0, 2\pi], x(0) = x(2\pi)\}$. In this theorem we consider the operator equation (5) with σ and ω independent parameters. The operator $\mathcal{T}_{\omega,\sigma}$ only depends on σ through the shift operator $\mathcal{S} - \sigma$ and the solution $\hat{x}_n(t; \sigma)$ of (8). We state the theorem in a way that sets up a procedure which exploits this fact.

Theorem. *Assume that there exists a natural number n such that*

1) *There exists an $\mathcal{S} \subset R_+$, $\mathcal{S} \neq \emptyset$, such that for all $\sigma \in \mathcal{S}$, there exists a $(\tilde{\omega}(\sigma), \hat{x}_n(t; \sigma)$ solving (7), with $\hat{\omega} \neq 0$, $\hat{x}_n(t; \sigma) \not\equiv 0$.*

Let $\operatorname{Proj}_\omega(\mathcal{S}) = \{\omega \mid$ *there exists $\sigma \in \mathcal{S}$ such that $\omega = \hat{\omega}(\sigma)\}$ is the projection of $\mathcal{S}$ onto the ω-axis. Assume that*

2) *There exists an* $\mathcal{I} \subset \mathrm{Proj}_\omega(\mathcal{S})$ *such that for all* $\omega \in \mathcal{I}$:

(a) $I - e^{2\pi A/\omega}$ *and* $i\omega kI - A$ *are invertible for* $k = 0, \pm 1, \dots, \pm n$;

(b) *There exists an* $\mathcal{S}_\omega \subset \mathcal{S}$, $\overline{B}_n(\hat{x}_n, \rho_\omega) \subset X$, *with* $\mathcal{S}_\omega \neq \varnothing$, $0 < \rho_\omega < ||\hat{x}_n||$, *such that for all* $\sigma \in \mathcal{S}_\omega$, $\mathcal{T}_{\omega,\sigma} : \overline{B} \to \overline{B}$ *is a compact map.*

Then for all $\omega \in \mathcal{I}$ *there exists a not-identically-zero periodic solution of* (1) *with period* $2\pi/\omega$, *for every delay*

$$\tau \in \{\sigma/\omega \mid \sigma \in \mathcal{S}_\omega\}.$$

Proof. The result follows immediately from the Schauder theorem. □

Remarks. 1) Although (b) excludes the trivial solution ($0 \notin \overline{B}$), the theorem does not exclude constant solutions. These must be excluded by either dealing directly with the original equation, or by choosing $\overline{B}$ so as to exclude all of the equilibria of (1).

2) The only appearance of σ in (5) is in the shift operator $S_{-\sigma}$, which has no effect on norm estimates. However, the center of $\overline{B}$, $\hat{x}_n(\cdot)$ in general depends on σ.

To develop a general procedure for using the theorem, we consider the operator equation (5) $x = \mathcal{T}_{\omega,\sigma}[x]$. Because the delay only appears in the operator equation as a shift, our estimates are essentially the same as in the case of ordinary differential equations without delay (see [8]; they treat a nonautonomous problem with period given).

We look for a ball $\overline{B}(\hat{x}_n, \rho)$ in X, with $\rho < ||\hat{x}_n||$, with $\omega \in \mathcal{I}$, $\sigma \in \mathcal{S}_\omega$. We require $\mathcal{T}_\omega : \overline{B} \to \overline{B}$, so we write

$$\begin{aligned}
||\mathcal{T}_{\omega,\sigma}[x] - \hat{x}_n|| &= ||\mathcal{G}_\omega[BS_{-\sigma}x + \mathcal{F}[x]] - \hat{x}_n|| \\
&= ||\mathcal{G}_\omega BS_{-\sigma}(x - \hat{x}_n) + \mathcal{G}_\omega \circ \mathcal{F}[x] - \mathcal{G}_\omega \circ \mathcal{F}[\hat{x}_n] \\
&\quad + \mathcal{G}_\omega(BS_{-\sigma}\hat{x}_n + \mathcal{F}[\hat{x}_n]) - \hat{x}_n|| \\
&\leq ||\mathcal{G}_\omega BS_{-\sigma}||\, ||x - \hat{x}_n|| + ||\mathcal{G}_\omega \circ (\mathcal{F}[x] - \mathcal{F}[\hat{x}_n])|| \\
&\quad + ||\mathcal{G}_\omega BS_{-\sigma}\hat{x}_n + \mathcal{G}_\omega \circ \mathcal{F}[\hat{x}_n] - \hat{x}_n||.
\end{aligned}$$

If $\mathcal{G}_\omega \circ \mathcal{F}[\cdot]$ is Lipschitz with constant $\lambda_\omega(\rho)$ on the ball $\overline{B}(\hat{x}_n, \rho)$, then

$$\begin{aligned} ||\mathcal{T}_{\omega,\sigma}[x] - \hat{x}_n|| &\le [||\mathcal{G}_\omega BS_{-\sigma}|| + \lambda_\omega(\rho)]||x - \hat{x}_n|| \\ &\quad + ||\mathcal{G}_\omega BS_{-\sigma}\hat{x}_n + \mathcal{G}_\omega \circ \mathcal{F}[\hat{x}_n] - \hat{x}_n||_\infty \\ &\equiv M_\omega ||x - \hat{x}_n|| + N_\omega. \end{aligned} \tag{11}$$

To guarantee a fixed point in $\overline{B}$ we require

$$M_\omega \rho + N_\omega < \rho < ||\hat{x}_n||. \tag{12}$$

Thus we can guarantee a solution of the operator equation (5) if there exists a $\rho > 0$ such that (ρ may depend on ω)

$$N_\omega/(1 - M_\omega) < \rho < ||\hat{x}_n|| \tag{13}$$

and

$$M_\omega \equiv ||\mathcal{G}_\omega BS_{-\sigma}|| + \lambda_\omega(\rho) < 1.$$

Each fixed point $x(t)$ will yield a periodic solution to (1) with delay $\tau = \sigma/\omega$ and period $T = 2\pi/\omega$.

5. Examples.

Example 1. Consider the scalar equation

$$x'(t) - ax(t) - bx(t - \tau) = c \sin x \tag{14}$$

with a, b, c all positive constants. Then

$$\begin{aligned} \mathcal{G}_\omega[x] = (1/\omega)[1 - e^{2\pi a/\omega}]^{-1}\bigg\{ &\int_0^t e^{a(t-s)/\omega} x(s)\, ds \\ &+ \int_t^{2\pi} e^{a(2\pi + t - s)/\omega} x(s)\, ds \bigg\}. \end{aligned} \tag{15}$$

So

$$\mathcal{G}_\omega[e^{ikt}] = \frac{1}{i\omega k - a} e^{ikt}, \tag{16}$$

$$\mathcal{G}_\omega[S_{-\sigma} e^{ikt}] = \frac{1}{i\omega k - a} e^{ik(t-\sigma)}. \tag{17}$$

This shows that $\mathcal{G}_\omega$ and $\mathcal{G}_\omega \circ S_{-\sigma}$ are very simple reproducing operators for $\{e^{ikt}\}$.

If $x_n(t) = \sum_{-n}^{n} a_k e^{ikt}$, and if formally $F(x_n) = \sum_{-\infty}^{+\infty} f_k e^{ikt}$, then formally

$$\begin{aligned} \mathcal{T}_\omega[x] &= \mathcal{G}_\omega\left[bS_{-\sigma}[x] + \sum_{-\infty}^{+\infty} f_k e^{ikt}\right] \\ &= \sum_{-n}^{n} \frac{ba_k e^{ik(t-\sigma)}}{i\omega k - a} + \sum_{-\infty}^{+\infty} \frac{f_k}{i\omega k - a} e^{ikt}. \end{aligned} \tag{18}$$

The harmonic balance equations become

$$a_k\left(1 - \frac{be^{ik\sigma}}{i\omega k - a}\right) = \frac{f_k(a_0, a_{\pm 1}, \dots, a_{\pm n})}{i\omega k - a}, \quad k = 0, \pm 1, \dots, \pm n. \tag{19}$$

We take $n = 1$ and $a_0 = 0$. Then $x_1(t) = a_1 e^{it} + \bar{a}_1 e^{-it}$, and a straightforward computation shows that for $F(x) = c\sin x$ we get

$$P_1 \sin x_1(t) = \frac{1}{2|a_1|} J_1(2|a_1|)(a_1 e^{it} + \bar{a}_1 e^{-it}),$$

where J_1 is the usual Bessel function. The harmonic balance equations are

$$\begin{aligned} a_0\left(1 + \frac{b}{a}\right) &= -(c/a) f_0 = 0, \quad \text{(automatic since } a_0 = 0\text{)} \\ a_1\left(1 - \frac{be^{-i\sigma}}{i\omega - a}\right) &= \frac{c}{2|a_1|} J_1(2|a_1|) a_1. \end{aligned} \tag{20}$$

We normalize $x_1(t)$ by requiring $a_1 \in R_+$, then (20) becomes:

$$\begin{aligned} &\text{(i)} \quad \tan(\sigma) = \omega/a \\ &\text{(ii)} \quad cJ_1(2a_1) = 2a_1\left(1 + \frac{b}{(a^2 + \omega^2)^{1/2}}\right). \end{aligned} \tag{21}$$

If $\mathcal{S} = \cup_{j=0}^{\infty}(j\pi, (2j+1)\pi/2)$, then for $\sigma \in \mathcal{S}$ (i) has a unique solution in R_+, $\hat{\omega} = a\tan\sigma$. If we think of (ii) as representing the intersection

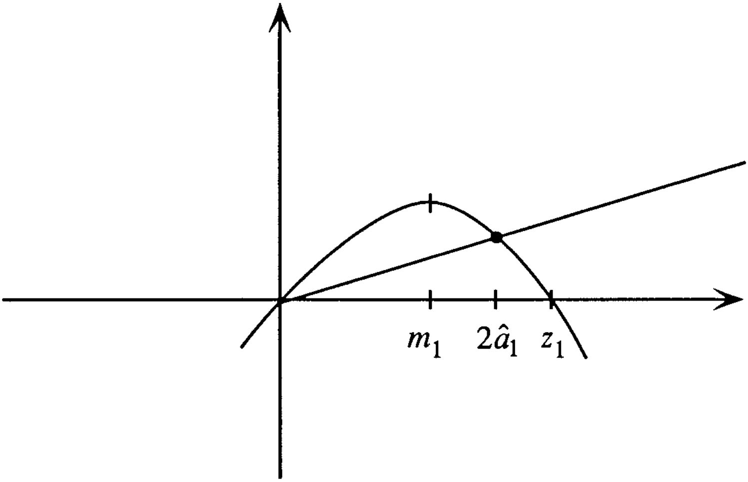

FIGURE 1.

of the curve $cJ_1(x)$ with the straight line $(1+b(a^2+\hat{\omega}^2)^{-1/2})x$, then it is clear that (ii) has a solution $\hat{a}_1$ if and only if

$$\frac{c}{2} \equiv cJ_1'(0) > 1 + \frac{b}{(a^2+\hat{\omega}^2)^{1/2}}. \tag{22}$$

Below we will in fact need (∗) $m_1 < 2\hat{a}_1 < z_1$ where $z_1 = 3.817\ldots$ is the first positive zero of $J_1(z)$ and $m_1 = 1.8\ldots$ is the point at which the first maximum is attained (see Figure 1). Now $J(m_1) = 0.58\ldots$, so what we require is

$$cJ_1(m_1)/m_1 > 1 + \frac{b}{(a^2+\hat{\omega}^2)^{1/2}}; \tag{22$'$}$$

we can replace the left side by the smaller number $c(0.58)/(1.9) \cong (0.30)c$ for a cruder but simpler condition. Thus, we obtain a solution of the harmonic balance equation, $\hat{x}_1(t) = 2\hat{a}_1 \sin t$, belonging to X. Now from (11),

$$||\mathcal{G}_\omega[x]|| \le \left| \frac{(1/\omega)}{1-e^{2\pi a/\omega}} [(\omega/a)(e^{2\pi a/\omega}-1)] \right| ||x||_\infty = \frac{1}{a}||x||.$$

We note that

$$||\mathcal{F}[x] - \mathcal{F}[y]|| \leq c||x - y||,$$

so

$$||\mathcal{G}_\omega \circ \mathcal{F}[x] - \mathcal{G}_\omega \circ \mathcal{F}[y]|| \leq \frac{c}{a}||x - y||.$$

Thus, from (15) we have

$$||\mathcal{G}_\omega[x] - \hat{x}_1|| \leq \frac{b+c}{a}||x - \hat{x}_1|| + \frac{b(2\hat{a}_1) + c}{a}.$$

Therefore, the inequality (13) becomes

$$\frac{(b(2\hat{a}_1) + c)/a}{1 - (b+c)/a} < \rho < 2\hat{a}_1. \tag{23}$$

Now $m_1 < 2\hat{a}_1 < z_1$, so we can find a $\rho > 0$ satisfying (23) if

$$\frac{b(2\hat{a}_1) + c}{a - b - c} < 2\hat{a}_1, \quad \text{i.e.} \quad 2\hat{a}_1 > \frac{c}{a - 2b - c} > 0, \tag{24}$$

or, more crudely,

$$\frac{c}{a - 2b - c} < m_1. \tag{25}$$

For example, (24) is clearly valid if a is sufficiently large, for given b, c and σ, and the cruder condition (25) does not depend on σ.

To recap, the existence of a nontrivial solution of (5) is implied by:

(a) $\sigma \in \mathcal{S} = \cup_{j=0}^{\infty}(j\pi, (2j+1)\pi/2)$

(b) $(0.3)c > 1 + b/(a^2 + \hat{\omega}^2)^{1/2}$ where $\hat{\omega} = a \tan \sigma$,

(c) $0 < c/(a - 2b - c) \leq m_1 = 1.8\ldots$.

Let $\omega \in R_+$, and $\sigma \in \mathcal{S}$, and let b, c be fixed; then, for $a > \alpha$, α defined by the two requirements $0 < c(\alpha - 2b - c)^{-1} \leq m_1$, $(0.3)c > 1 + b/\alpha$, we can conclude that (5) has a nontrivial solution. Thus, for any $\omega \in \mathbf{R}_+$, the original equation (14) will have a (perhaps constant) periodic solution of period ω for any delay $\tau \in \{\sigma/\omega \mid \sigma \in S\} = \cup_{j=0}^{\infty}(j\pi/\omega, (2j+1)\pi/2\omega)$.

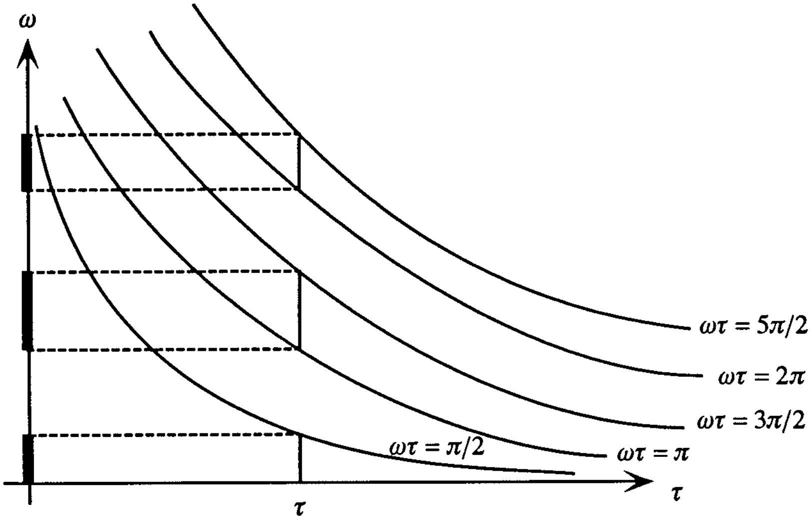

FIGURE 2.

For a given delay $\tau > 0$, the original equation (14) will have a periodic solution of period ω for any ω satisfying (see Figure 2)

$$\omega \in \bigcup_{j=0}^{\infty} \left(\frac{j\pi}{\tau}, \frac{(2j+1)\pi}{2\tau} \right).$$

Notice that we can therefore conclude that, for any delay $\tau > 0$, (14) has solutions of arbitrarily small period $T = 2\pi/\omega$. Note also that we have not identified the minimal period of our periodic solutions; we may be counting a given solution more than once.

Finally, suppose that $x(t) \equiv k$ solves (14), so

$$(-a - b)k = c \sin k.$$

A simple sketch shows that this will have no solutions for $a > \alpha$ if we require

$$\frac{3\pi}{2}(\alpha + b) > c.$$

Example 2. We consider the equation

$$x'(t) - ax(t) - b_1x(t-\tau_1) - b_2x(t-\tau_2) = c\sin[x(t)],$$

with $a, b_1, b_2, \tau_1, \tau_2, c$ given real nonzero constants. Direct calculation shows that $\mathcal{G}_\omega[e^{ikt}] = (i\omega k - a)^{-1}e^{ikt}$, $\mathcal{G}_\omega[\mathcal{S}_{-\sigma_j}e^{ikt}] = (i\omega k - a)^{-1}e^{ik(t-\sigma_j)}$,

$$\begin{aligned}\mathcal{P}_n\mathcal{T}_\omega[x_n] &= \mathcal{G}_\omega\left(b_1\mathcal{S}_{-\sigma_1}[x_n] + b_2\mathcal{S}_{-\sigma_2}[x_n] + \sum_{-n}^{n} f_k e^{ikt}\right)\\ &= \sum_{-n}^{n}((b_1e^{-ik\sigma_1} + b_2e^{-ik\sigma_2})a_k + f_k)(ik\omega - a)^{-1}e^{ikt}.\end{aligned}$$

For the simplest approximation, $n = 1$, we have

$$(26)\qquad \begin{aligned}&\text{(i)}\quad a_0(1 + b/a) = -(c/a)f_0 = 0 \quad (\text{since } f_0 = 0),\\ &\text{(ii)}\quad a_1(1 - [b_1e^{-i\sigma_1} + b_2e^{-i\sigma_2}](i\omega k - a)^{-1}) = cJ_1(2|a_1|)a_1,\end{aligned}$$

where J_1 is the Bessel function of the first kind. As in Example 1, we set $a_0 = 0$, and take $a_1 > 0$ real. The real and imaginary parts of (2) have a solution if we set

$$(27)\qquad \omega_0 = a[b_1\sin\sigma_1 + b_2\sin\sigma_2][b_1\cos\sigma_1 + b_2\cos\sigma_2]^{-1}$$

and require

$$(28)\quad c > (\omega^2 + a^2)^{-1}(a[b_1\cos\sigma_1 + b_2\cos\sigma_2] + \omega[b_1\sin\sigma_1 + b_2\sin\sigma_2]).$$

Equation (27) is easily seen to make sense for $(\sigma_1, \sigma_2) \in S$, the complement of a union of discrete points. An analysis similar to that in Example 1 shows that Equation (28) has solutions for a, b_1, b_2, c satisfying certain restrictions. In order to guarantee the existence of an appropriate ball, we do a norm analysis of the operator $\mathcal{T}_\omega$, exactly as in Example 1, and obtain the sufficient condition $(b_1 + b_2)z_1 + c < m_1(a - b_1 - b_2 - c)$, where z_1, respectively m_1, is the first zero, respectively maximum, of J_1. It is easily seen that there are large classes of equations meeting all of the above restrictions, and we again obtain periodic solutions of arbitrarily small period.

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